Transverse Momentum Dependent parton distribution functions (TMDs)

Jianwei Qiu
Brookhaven National Laboratory

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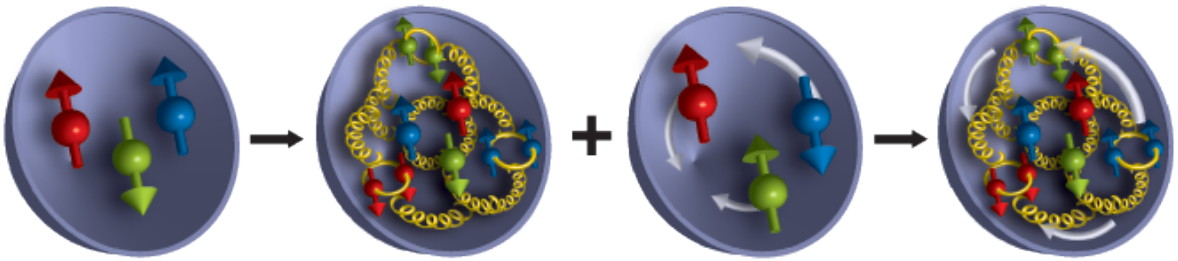
How to "see" and quantify hadron structure?

See also talks by H. Gao, A. Deshpande

- ❑ Our understanding of hadron evolves



1970s 1980s/2000s Now

Nucleon is a strongly interacting, relativistic bound state of quarks and gluons

- ❑ Challenge:

No modern detector can see quarks and gluons in isolation!

- ❑ Question:

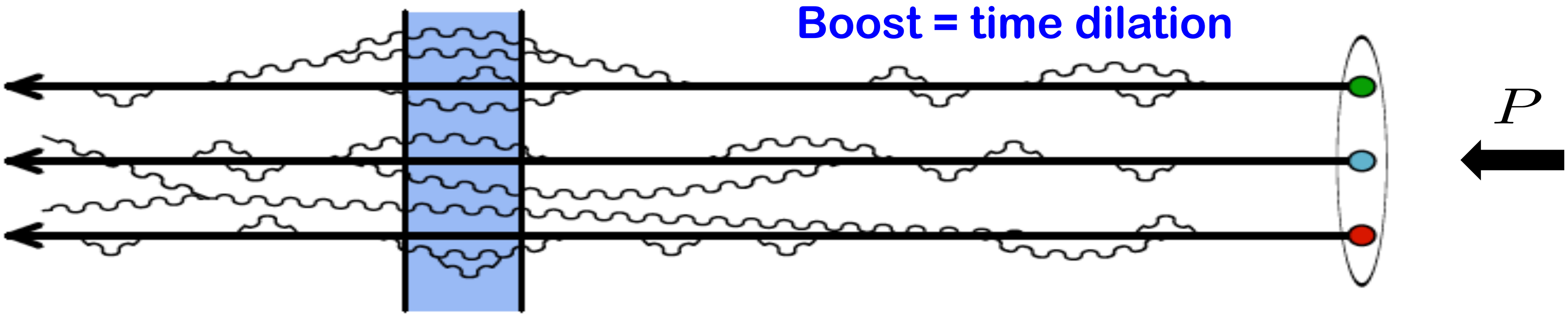
How to quantify the hadron structure if we cannot see quarks and gluons? *We need the probe!*

- ❑ Answer:

QCD factorization! *Not exact, but, controllable approximation!*

The hadron structure we can "see"

❑ High energy probes "see" the boosted partonic structure:



Hard probe ($t \sim 1/Q <$ fm): *Catches the quantum fluctuation!*

- ✧ Longitudinal momentum fraction – *x* : $xP \sim Q$
- ✧ Transverse momentum – confined motion: $1/R \sim \Lambda_{\rm QCD} \ll Q$

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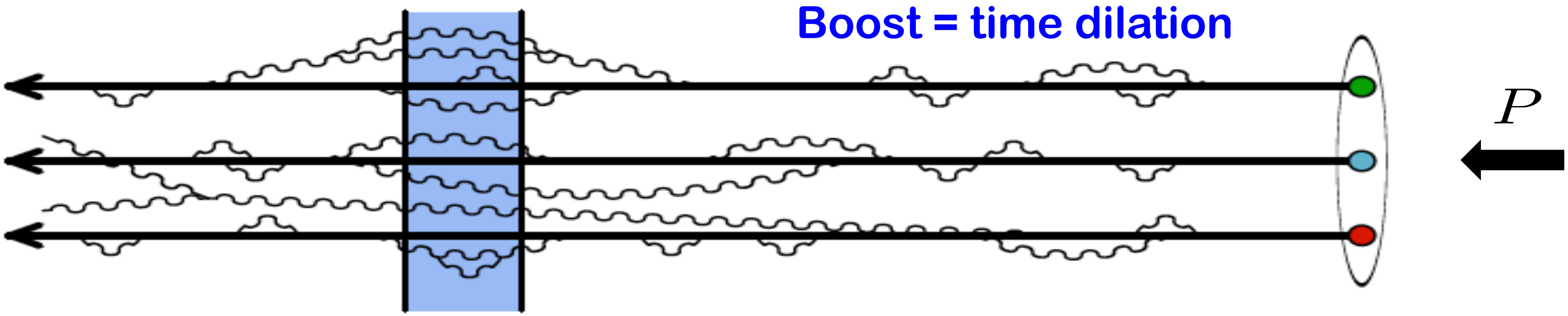

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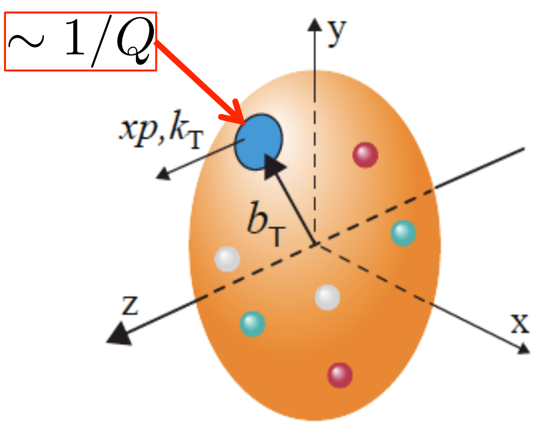

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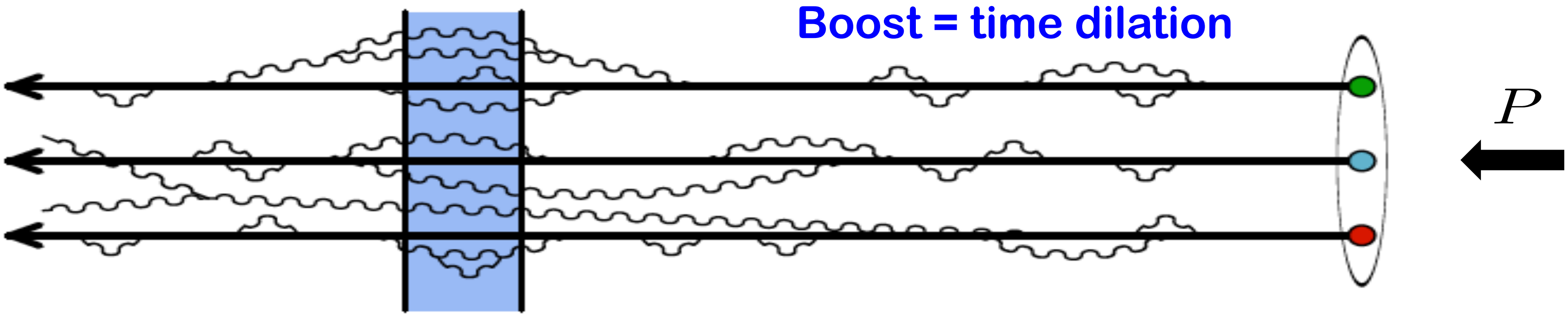

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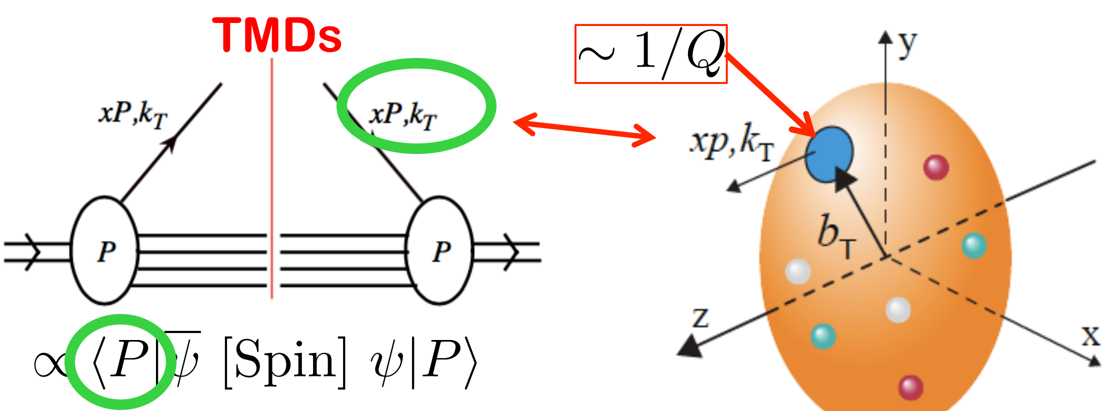

Cross sections

The hadron structure we can "see"

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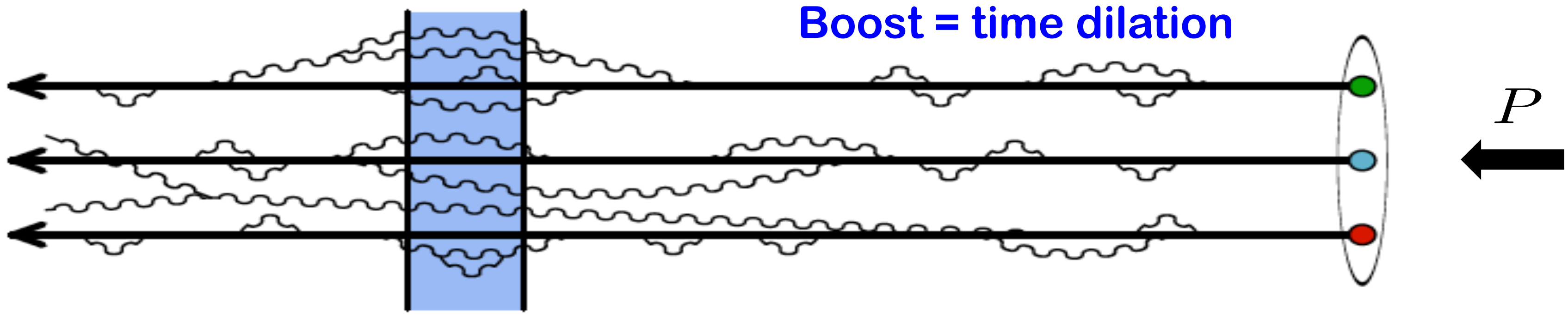

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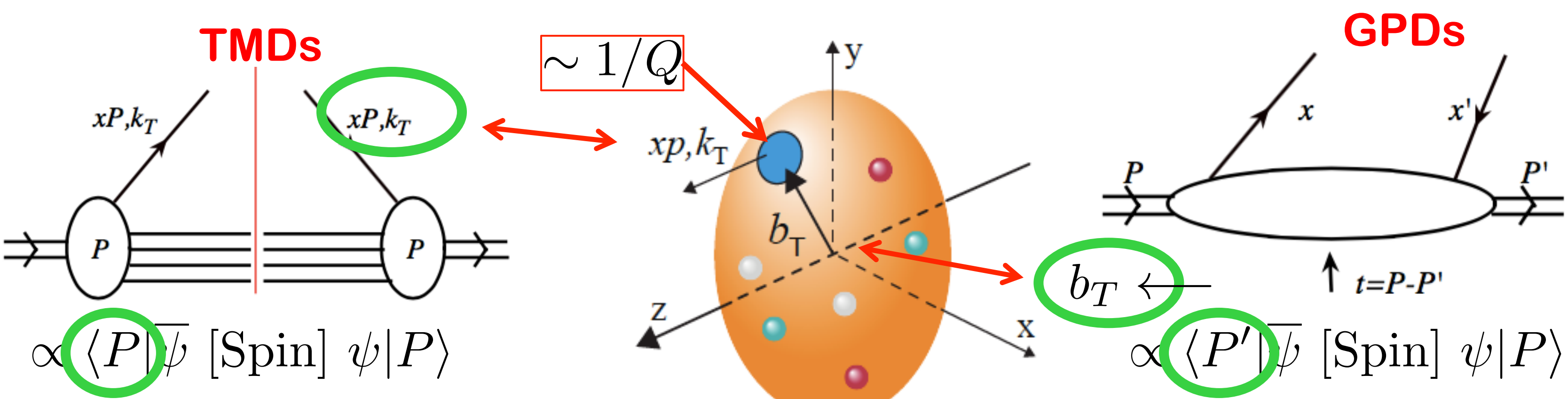


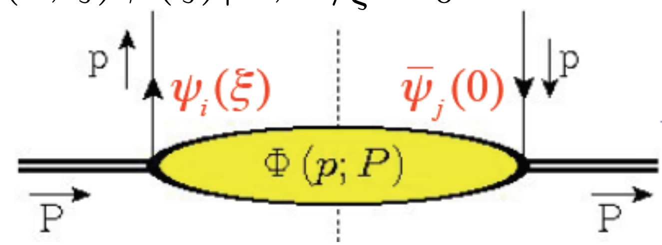
Cross sections *Amplitudes*

Definition of TMDs

❑ Non-perturbative definition:

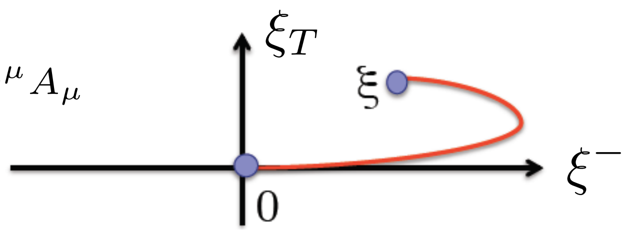
✧ **In terms of matrix elements of parton correlators:**

$$\Phi^{[U]}(x,p_T;n) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3}\, e^{i\,p\cdot\xi}\, \langle P,S|\overline{\psi}(0)U(0,\xi)\psi(\xi)|P,S\rangle_{\xi^+=0}$$



✧ **Depends on the choice of the gauge link:**

$$U(0,\xi) = e^{-ig\int_0^\xi ds^\mu A_\mu}$$



✧ **Decomposes into a list of TMDs:**

$$\Phi^{[U]}(x,p_T;n) = \Big\{ f_1^{[U]}(x,p_T^2) - f_{1T}^{\perp[U]}(x,p_T^2)\,\frac{\epsilon_T^{p_T S_T}}{M} + g_{1s}^{[U]}(x,p_T)\gamma_5 + h_{1T}^{[U]}(x,p_T^2)\,\gamma_5\,\not{S}_T + h_{1s}^{\perp[U]}(x,p_T)\,\frac{\gamma_5\,\not{p}_T}{M} + i h_1^{\perp[U]}(x,p_T^2)\,\frac{\not{p}_T}{M}\Big\}\frac{\not{P}}{2},$$

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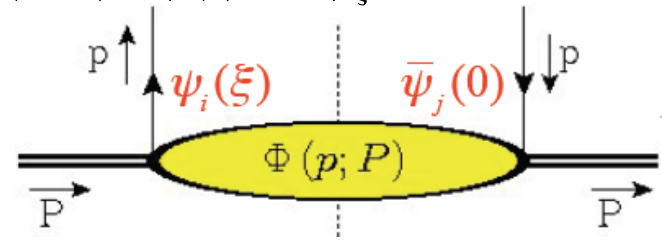

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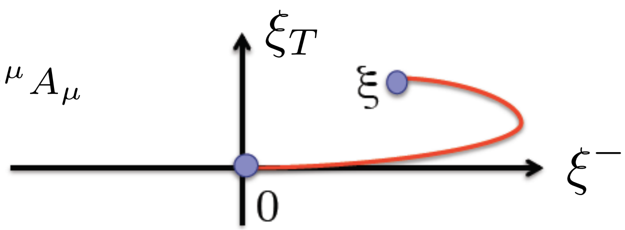

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See also Cloet’s talk

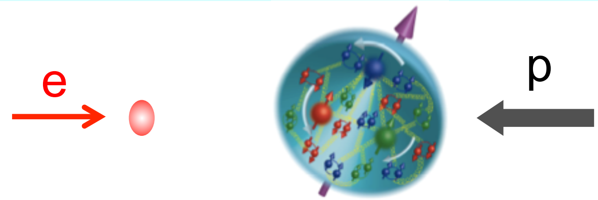
❑ Gives “unique” TMDs, IF we knew proton wave function!

But, we do NOT know proton wave function (calculate it approximately?)

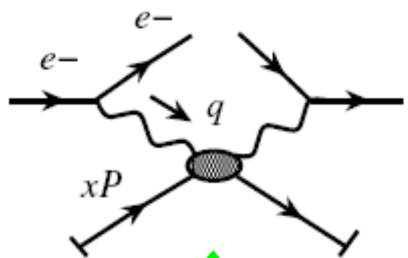
TMDs defined in this way are NOT direct physical observables!

QCD Factorization: connecting parton to hadron

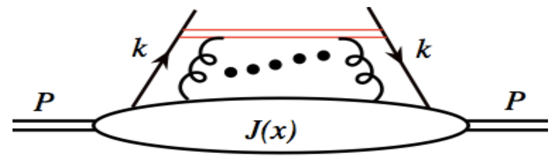
❑ One hadron:



$$\sigma_{\text{tot}}^{\text{DIS}} \sim$$



$$\otimes$$



$$+\, O\left(\frac{1}{QR}\right)$$

Hard-part
Probe

Parton-distribution
Structure

Power corrections
Approximation

QCD Factorization: connecting parton to hadron

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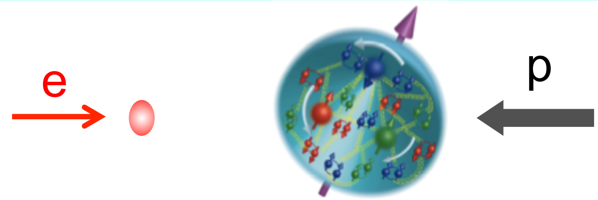
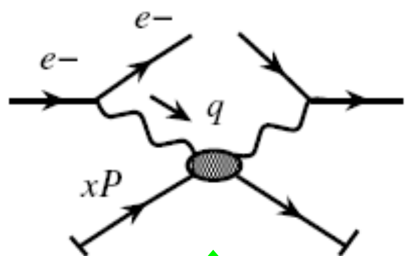


$$\sigma_{tot}^{DIS} \sim$$



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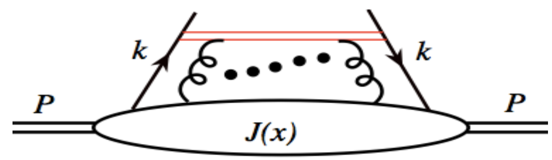

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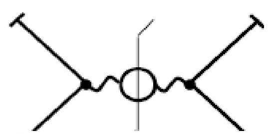
Hard-part Probe	Parton-distribution Structure	Power corrections Approximation

- **Two hadrons:**

$$\sigma_{tot}^{DY} \sim$$



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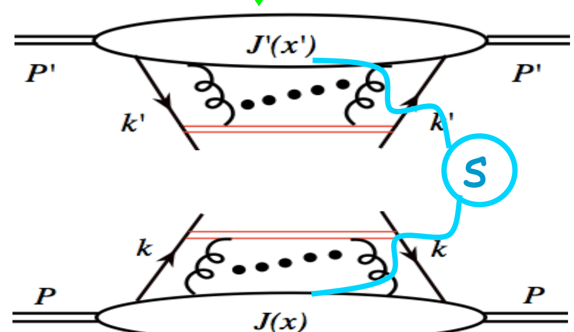

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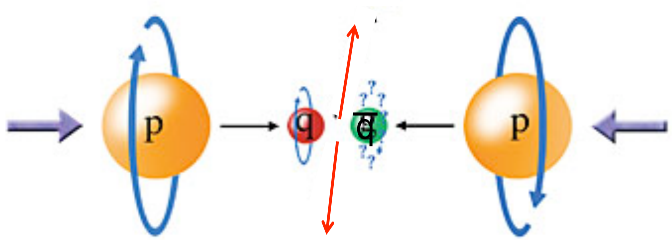
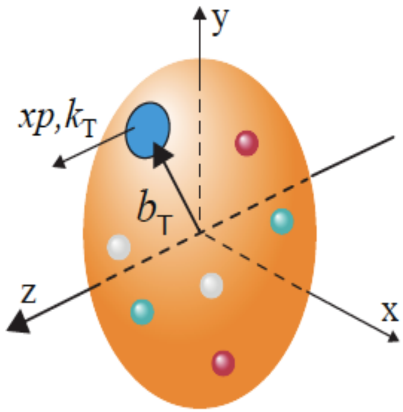


Predictive power:
Universal nonperturbative distributions

Need two-scale observables to "see" TMDs

- **Cross sections with two-momentum scales observed:**

$$Q_1 \gg Q_2 \sim 1/R \sim \Lambda_{\rm QCD}$$



✧ **Hard scale:** Q_1 **localizes the probe to see the quark or gluon d.o.f.**

✧ **"Soft" scale:** Q_2 **could be more sensitive to hadron structure, e.g., confined motion,** k_T

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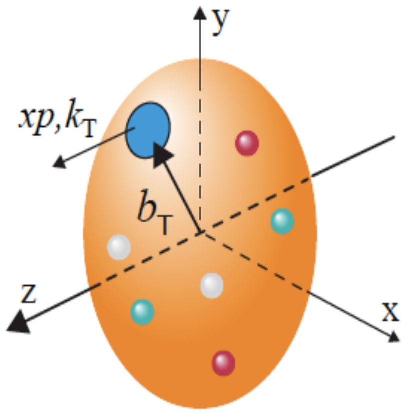

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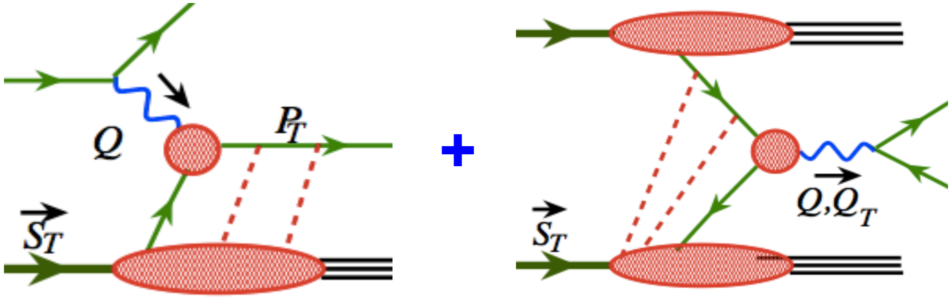
- **Two-scale observables with the hadron broken:**

See also talk by H. Gao



SIDIS: $Q \gg P_T$ + DY: $Q \gg P_T$ + Two-jet momentum imbalance in SIDIS, …

⟹

✧ Natural observables with TWO very different scales

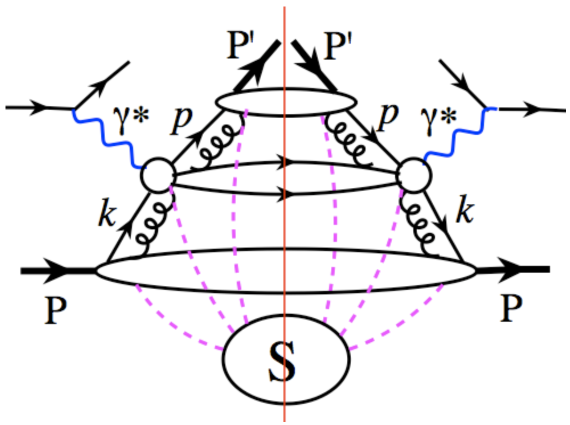
✧ TMD factorization: partons’ confined motion is encoded into TMDs

Factorization for TMDs

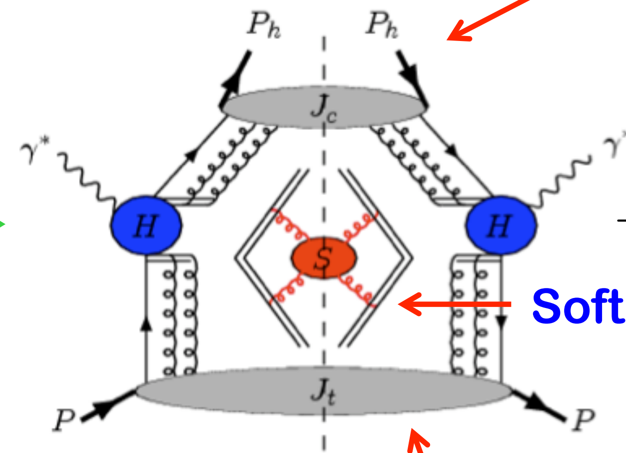
❑ Perturbative definition – in terms of TMD factorization:

SIDIS as an example:

See also talks by R. Ent, H. Matevosyan



TMD fragmentation



$$+\,\mathcal{O}\left(\frac{\langle k^2\rangle}{Q^2}, \frac{\langle p^2\rangle}{Q^2}\right)$$

Soft factors

TMD parton distribution

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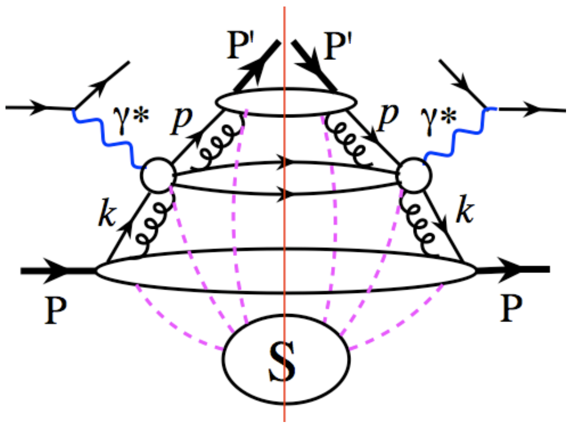

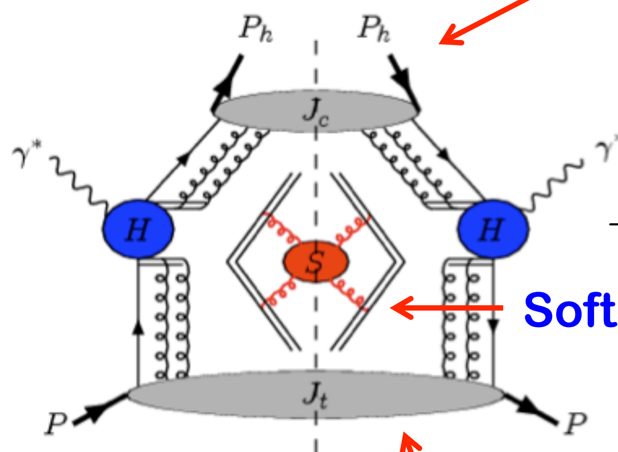

TMD fragmentation

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Soft factors

TMD parton distribution

❑ Extraction of TMDs:

$$\sigma_{\rm SIDIS}(Q, P_{h\perp}, x_B, z_h) = \hat{H}(Q) \otimes \Phi_f(x, k_\perp) \otimes \mathcal{D}_{f\to h}(z, p_\perp) \otimes \mathcal{S}(k_{s\perp}) + \mathcal{O}\left[\frac{P_{h\perp}}{Q}\right]$$

TMDs are extracted by fitting DATA using the factorization formula

Approximation – depending on the perturbatively calculated $\hat{H}(Q;\mu)$

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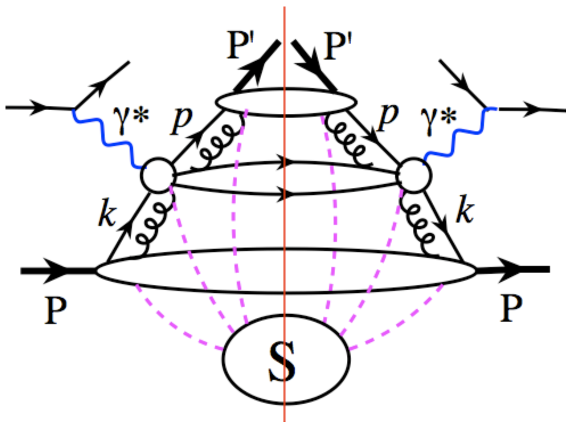

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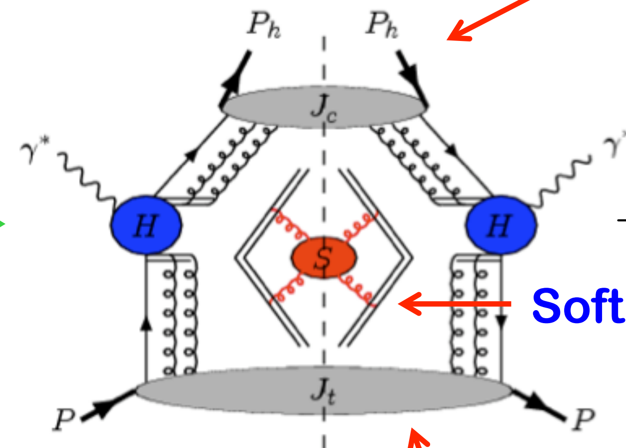

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TMDs are extracted by fitting DATA using the factorization formula

Approximation – depending on the perturbatively calculated $\hat{H}(Q;\mu)$

- ✧ **Extracted TMDs are valid only when** $\langle k^2\rangle \sim \langle p^2\rangle \ll Q^2$
- ✧ **Extracted TMDs:** $\displaystyle\int^{\mu^2} d^2k_\perp \Phi_f(x, k_\perp) \neq \phi_f(x, \mu^2)$

Wrong scale Dependence!

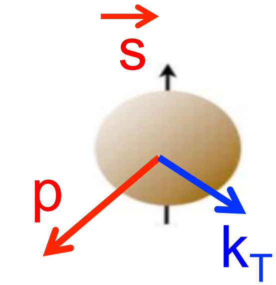
TMDs: confined motion, its spin correlation

- Power of spin – many more correlations:

Nucleon Polarization	Quark Polarization: Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
U	f_1 =		$h_1^\perp$ = − Boer-Mulders
L		g_{1L} = − Helicity	$h_{1L}^\perp$ = −
T	$f_{1T}^\perp$ = − Sivers	$g_{1T}^\perp$ = −	h_1 = − Transversity $h_{1T}^\perp$ = −

Nucleon Spin Quark Spin

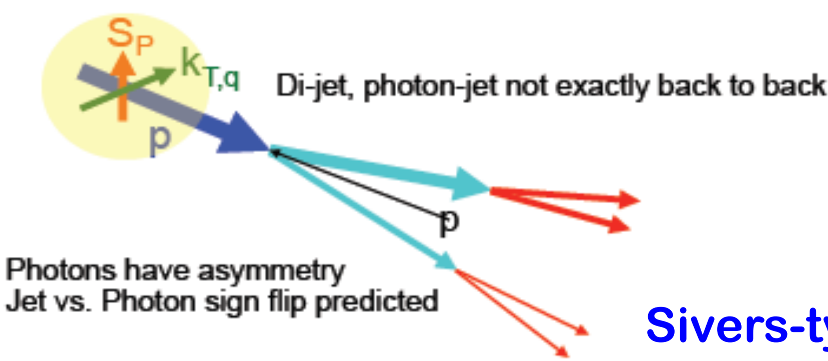
Similar for gluons



Require two Physical scales

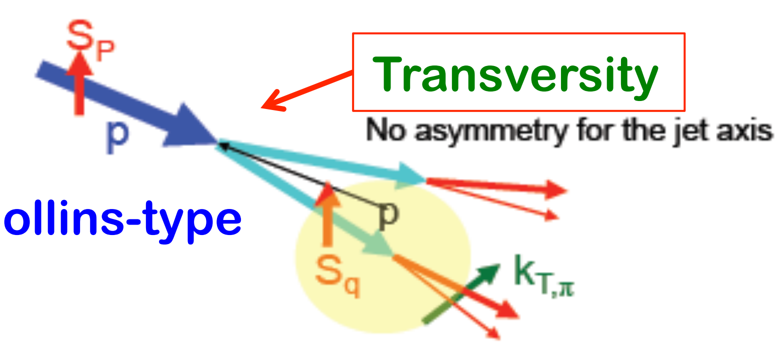
More than one TMD contribute to the same observable!

- A_N – single hadron production:



Sivers-type

Collins-type



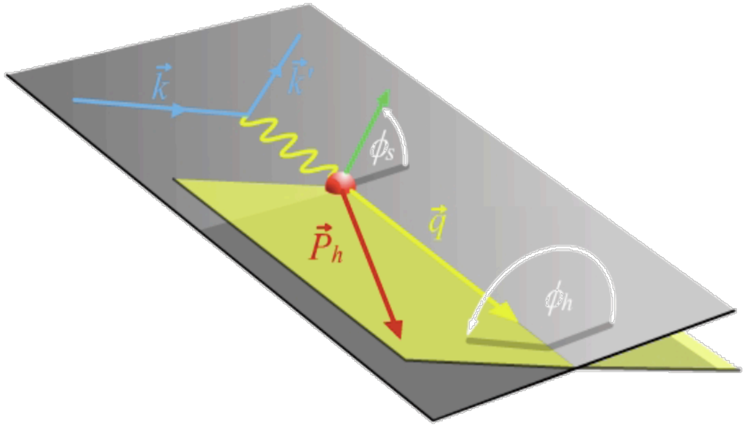
SIDIS is the best for probing TMDs

❑ Naturally, two scales & two planes:

$$A_{UT}(\varphi_h^l, \varphi_S^l) = \frac{1}{P}\frac{N^\uparrow - N^\downarrow}{N^\uparrow + N^\downarrow}$$

$$= A_{UT}^{Collins}\sin(\phi_h + \phi_S) + A_{UT}^{Sivers}\sin(\phi_h - \phi_S)$$

$$+ A_{UT}^{Pretzelosity}\sin(3\phi_h - \phi_S)$$



❑ Separation of TMDs:

$$A_{UT}^{Collins} \propto \langle \sin(\phi_h + \phi_S) \rangle_{UT} \propto h_1 \otimes H_1^\perp$$

$$A_{UT}^{Sivers} \propto \langle \sin(\phi_h - \phi_S) \rangle_{UT} \propto f_{1T}^\perp \otimes D_1$$

$$A_{UT}^{Pretzelosity} \propto \langle \sin(3\phi_h - \phi_S) \rangle_{UT} \propto h_{1T}^\perp \otimes H_1^\perp$$

Collins frag. Func. from e^+e^- collisions

Hard, if not impossible, to separate TMDs in hadronic collisions

Using a combination of different observables (not the same observable): jet, identified hadron, photon, …

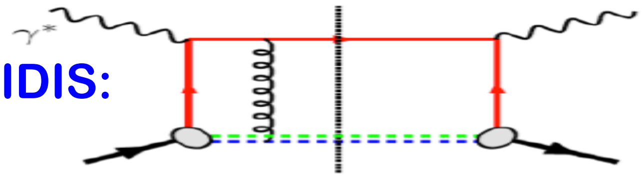
Modified universality for TMDs

❑ Definition:

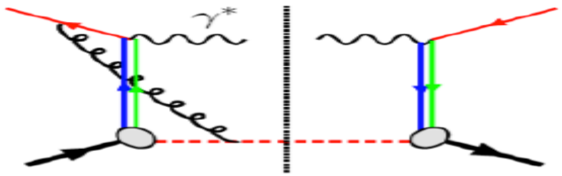
$$f_{q/h^\uparrow}(x,\mathbf{k}_\perp,\vec{S}) = \int \frac{dy^- d^2y_\perp}{(2\pi)^3}\, e^{ixp^+y^- - i\,\mathbf{k}_\perp\cdot\mathbf{y}_\perp}\langle p,\vec{S}|\overline{\psi}(0^-,\mathbf{0}_\perp)\,\boxed{\text{Gauge link}}\,\frac{\gamma^+}{2}\psi(y^-,\mathbf{y}_\perp)|p,\vec{S}\rangle$$

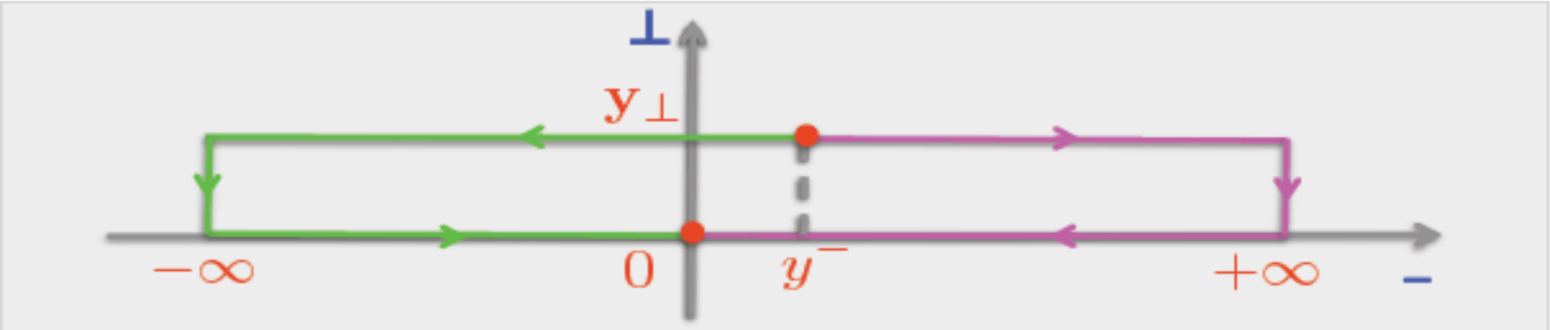
❑ Gauge links:

SIDIS:



DY:





❑ Process dependence:

$$f^{\text{SIDIS}}_{q/h^\uparrow}(x,\mathbf{k}_\perp,\vec{S}) \neq f^{\text{DY}}_{q/h^\uparrow}(x,\mathbf{k}_\perp,\vec{S})$$

Collinear factorized PDFs are process independent

Critical test of TMD factorization

- Parity – Time reversal invariance:

$$f_{q/h^\uparrow}^{\text{SIDIS}}(x, \mathbf{k}_\perp, \vec{S}) = f_{q/h^\uparrow}^{\text{DY}}(x, \mathbf{k}_\perp, -\vec{S})$$

- Definition of Sivers function:

$$f_{q/h^\uparrow}(x, \mathbf{k}_\perp, \vec{S}) \equiv f_{q/h}(x, k_\perp) + \frac{1}{2}\Delta^N f_{q/h^\uparrow}(x, k_\perp)\, \vec{S} \cdot \hat{p} \times \hat{\mathbf{k}}_\perp$$

- Modified universality:

$$\Delta^N f_{q/h^\uparrow}^{\text{SIDIS}}(x, k_\perp) = -\Delta^N f_{q/h^\uparrow}^{\text{DY}}(x, k_\perp)$$

The spin-averaged part of this TMD is process independent,

but, spin-averaged Boer-Mulder's TMD requires the sign change!

Same PT symmetry examination needs for TMD gluon distributions!

Global QCD analysis: extraction of TMDs

❑ QCD TMD factorization:

– Connect cross sections, asymmetries to TMDs

✧ Factorization is known or expected to be valid for SIDIS, Drell-Yan (Υ^*, W/Z, H^0,…), 2-Jet imbalance in DIS, …

✧ *Same level of reliability as collinear factorization for PDFs, up to the sign change*

❑ QCD evolution of TMDs:

– TMDs evolve when probed at different momentum scales

✧ Evolution equations are for TMDs in b_T-space (Fourier Conjugate of k_T)

FACT: QCD evolution does NOT fully fix TMDs in momentum space, even with TMDs fixed at low Q – large b_T-input!!!

✧ *Very different from DGLAP evolution of PDFs – in momentum space*

FACT: QCD evolution uniquely fix PDFs at large Q, once the PDFs is determined at lower Q – linear evolution in momentum space

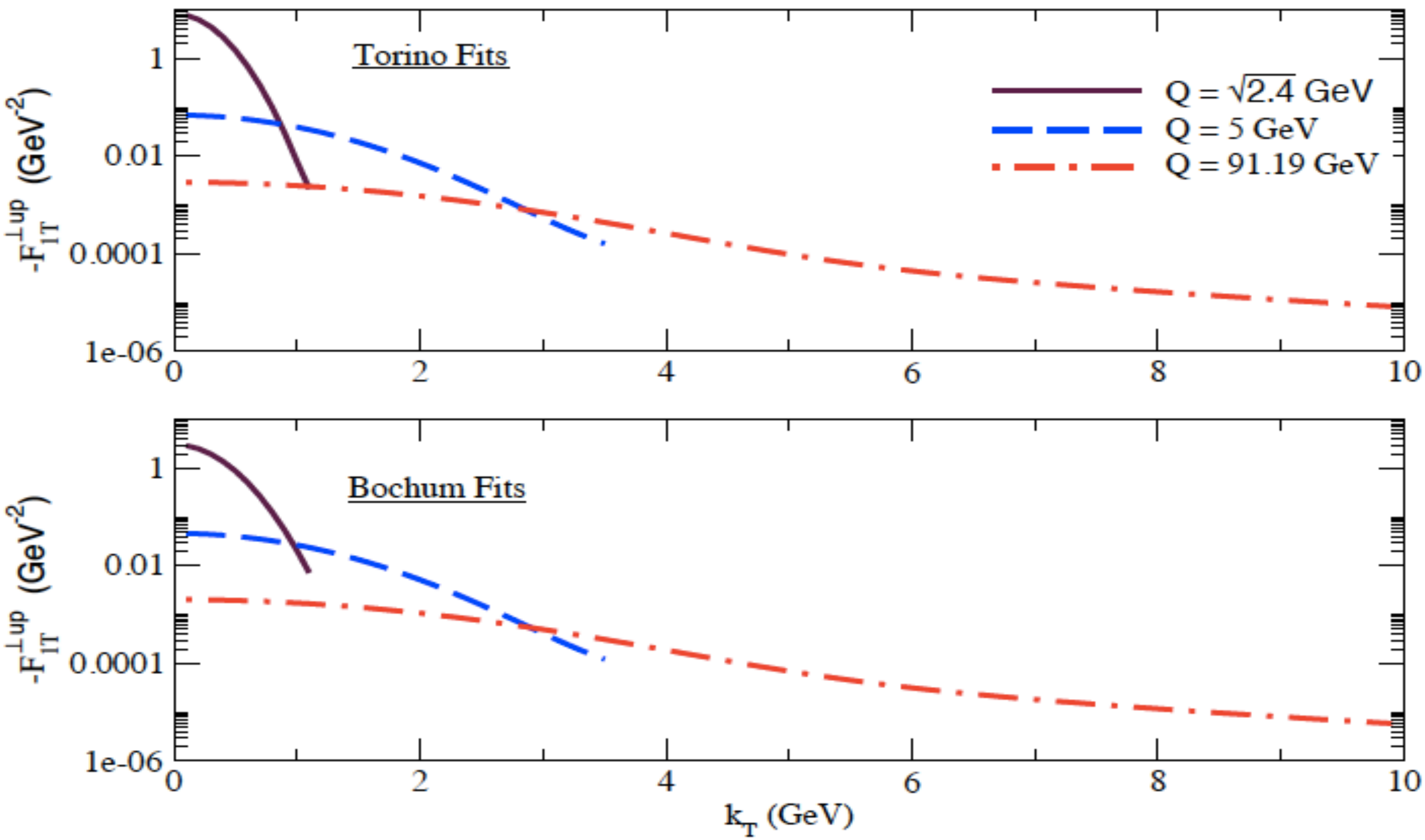
❑ Challenges:

Predictive power, extraction of hadron structure, …

Evolution of Sivers function

Aybat, Collins, Qiu, Rogers, 2011

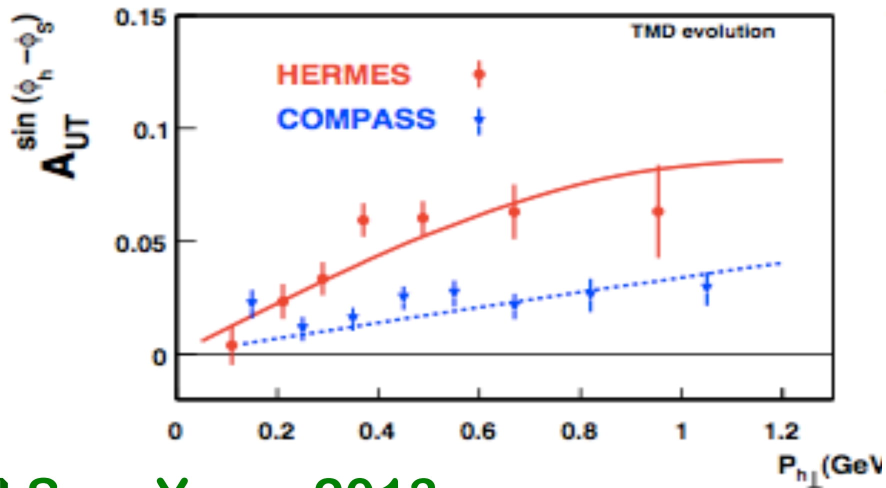
- Up quark Sivers function:

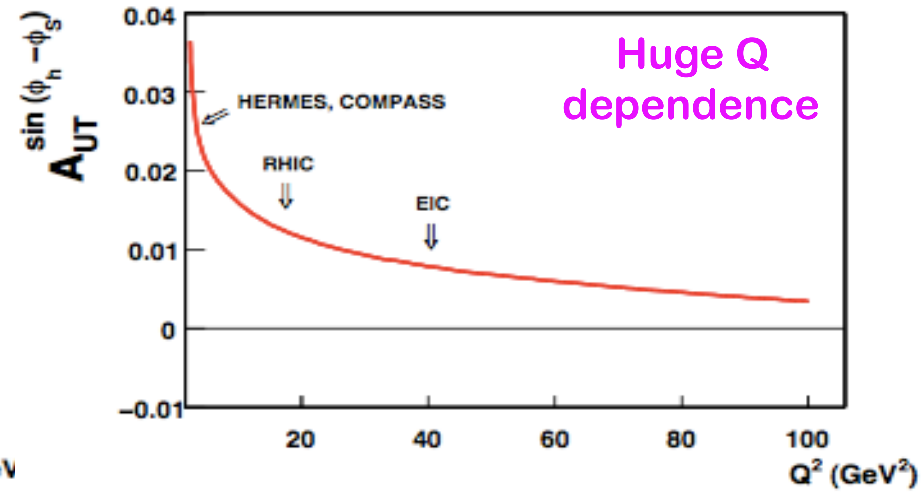


Very significant growth in the width of transverse momentum – shower!

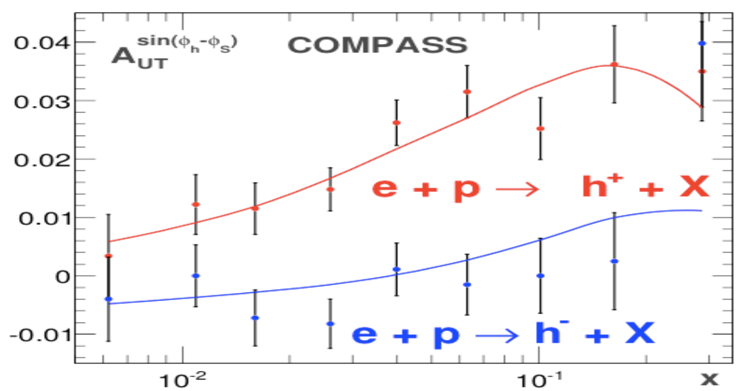
Different fits – different Q-dependence

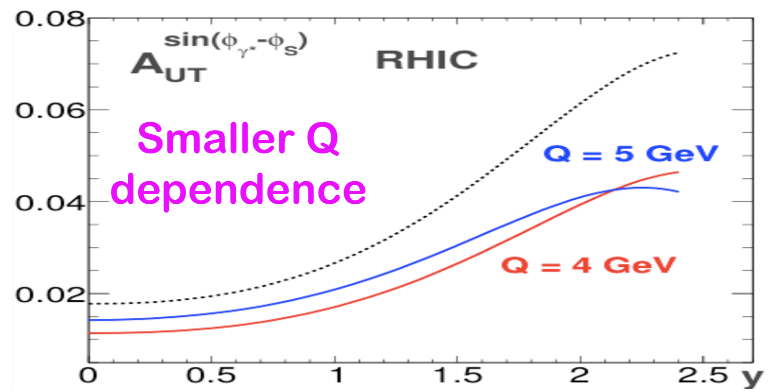
- Aybat, Prokudin, Rogers, 2012:





- Sun, Yuan, 2013:





No disagreement on evolution equations!

Issues: Extrapolation to non-perturbative large b-region
Choice of the Q-dependent "form factor"

What happened?

- **Sivers function:**

Differ from PDFs!

$$F_{1T}^{\perp f}(x, k_T; \mu, \zeta_F) = \frac{-1}{2\pi k_T} \int_0^\infty db_T\, b_T\, J_1(k_T b_T) \tilde{F}_{1T}'^{\perp f}(x, b_T; \mu, \zeta_F)$$

Need non-perturbative large b_T information for any value of Q! $Q = \mu$

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Need non-perturbative large b_T information for any value of Q! $Q=\mu$

- **What is the "correct" Q-dependence of the large b_T tail?**

$$\tilde{F}_{f/P}(x,\mathrm{b}_T;Q,Q^2) = \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}}\tilde{C}_{f/j}(x/\hat{x},b_*;\mu_b^2,\mu_b,g(\mu_b))f_{j/P}(\hat{x},\mu_b)}^{\mathrm{AA}}$$

$$\times \overbrace{\times\exp\left\{\ln\frac{Q}{\mu_b}\tilde{K}(b_*;\mu_b)+\int_{\mu_b}^{Q}\frac{d\mu'}{\mu'}\left[\gamma_F(g(\mu');1)-\ln\frac{Q}{\mu'}\gamma_K(g(\mu'))\right]\right\}}^{\mathrm{BB}}$$

$$\times \boxed{\overbrace{\exp\left\{g_{f/P}(x,b_T)+g_K(b_T)\ln\frac{Q}{Q_0}\right\}}^{\mathrm{CC}}}$$

Nonperturbative "form factor"

$$g_{f/P}(x,b_T)+g_K(b_T)\ln\frac{Q}{Q_0} \equiv -\left[g_1+g_2\ln\frac{Q}{2Q_0}+g_1 g_3\ln(10x)\right]b_T^2$$

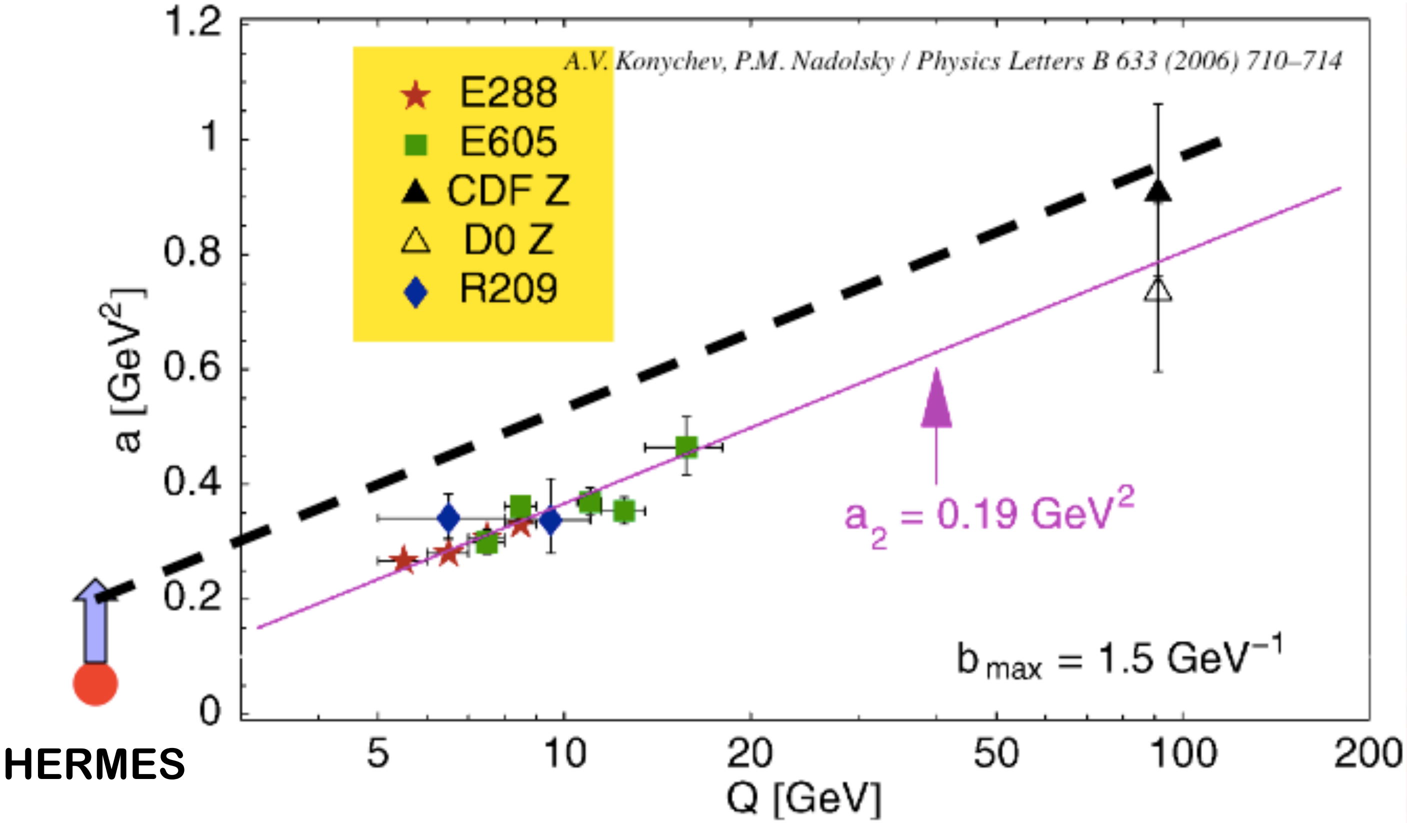
Is the log(Q) dependence sufficient?   Choice of g_2 & b_* affects Q-dep.

The "form factor" and b_* change perturbative results at small b_T!

Q-dependence of the "form" factor

Konychev, Nadolsky, 2006

- Q-dependence of the "form factor" :



$$\mathcal{F}^{\rm NP}(b,Q) = a(Q^2)\, b^2$$

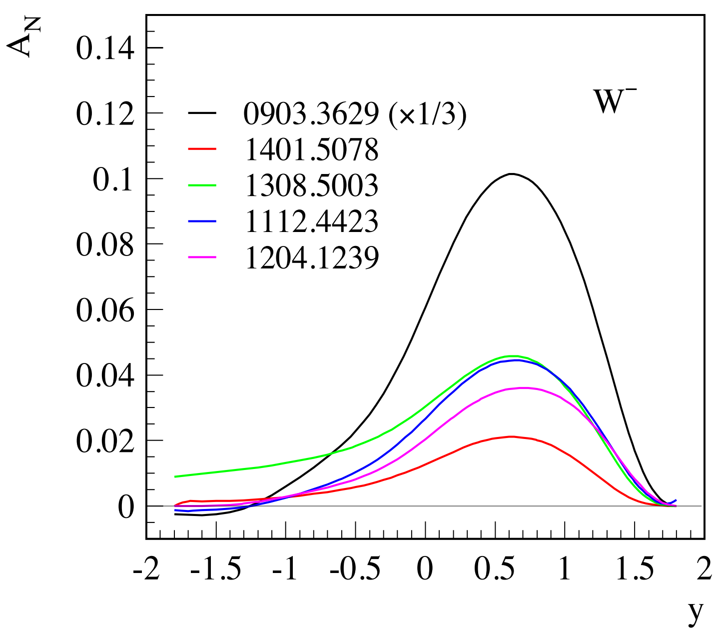
At Q ~ 1 GeV, ln(Q/Q_0) term may not be the dominant one!

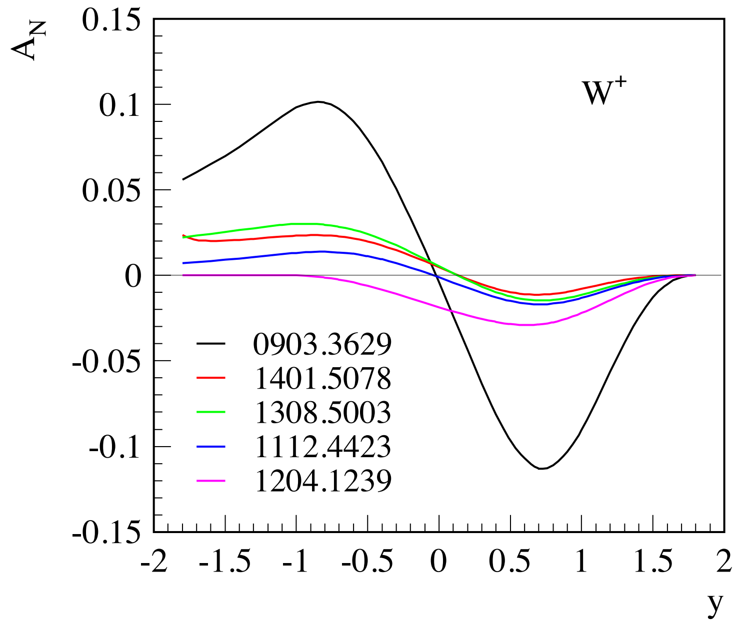
$$\mathcal{F}^{\rm NP} \approx b^2(a_1 + a_2 \ln(Q/Q_0) + a_3 \ln(x_A x_B) + \ldots) + \ldots$$

Power correction? **($Q_0/Q)^n$-term?** ***Better fits for HERMES data?***

“Predictions” for A_N of W-production at RHIC?

- **Sivers Effect:**
  - ✧ Quantum correlation between the spin direction of colliding hadron and the preference of motion direction of its confined partons
  - ✧ QCD Prediction: Sign change of Sivers function from SIDIS and DY
- **Current “prediction” and uncertainty of QCD evolution:**

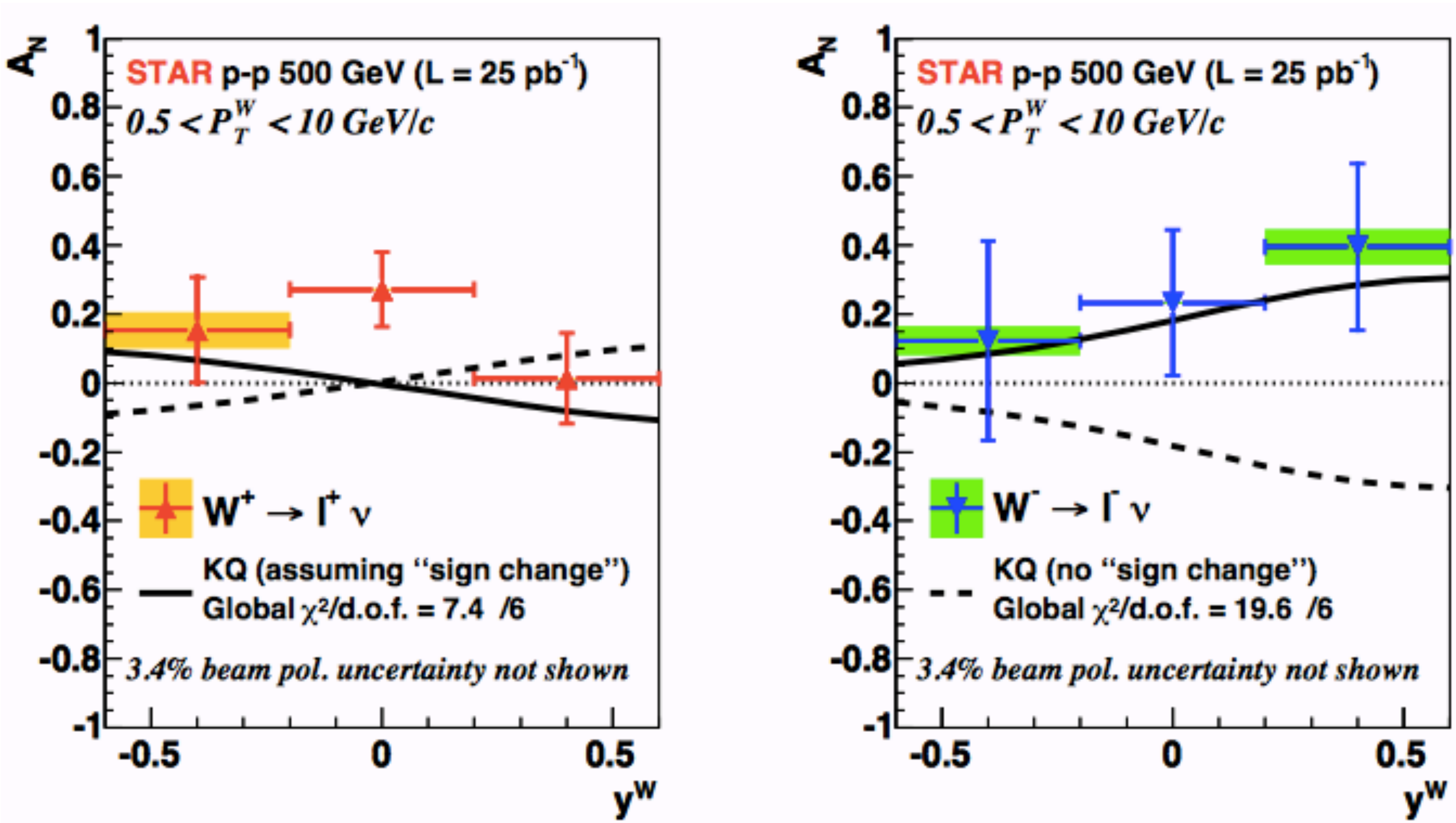




RHIC is the excellent and unique facility to test this (W/Z – DY)!

But, should not be easy! Shower could dilute the spin correlation!

Hint of the sign change: A_N of W production

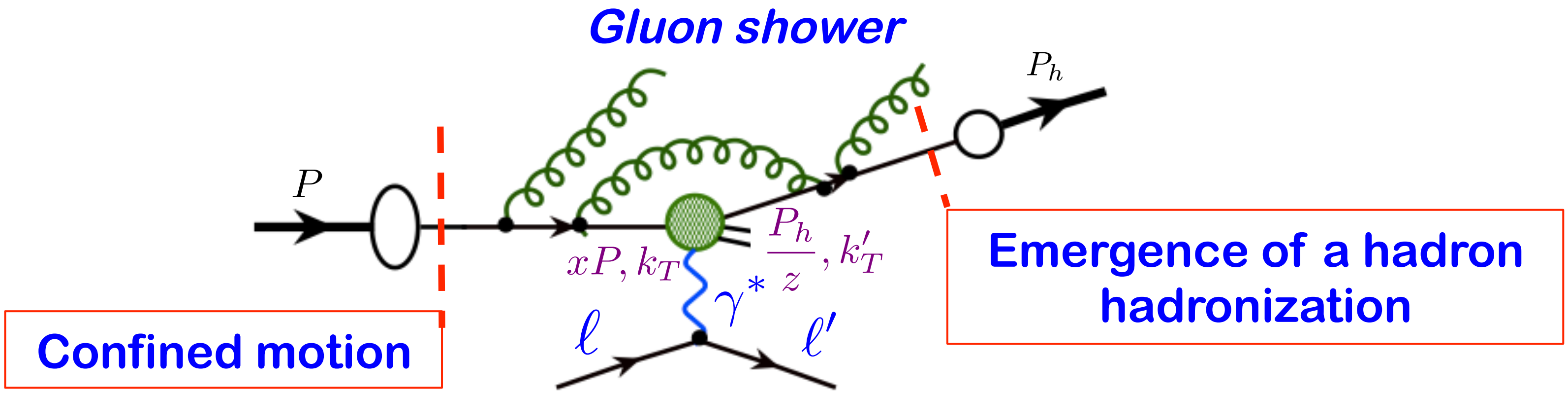


Data from STAR collaboration on A_N for W-production are consistent with a sign change between SIDIS and DY

STAR Collab. Phys. Rev. Lett. 116, 132301 (2016)

Parton k_T at the hard collision

- Sources of parton k_T at the hard collision:



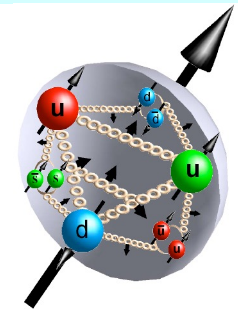
- Large k_T generated by the shower (caused by the collision):
  - ✧ Q^2-dependence – linear evolution equation of TMDs in b-space
  - ✧ The evolution kernels are perturbative at small b, but, not large b

  ➡ The nonperturbative inputs at large b could impact TMDs at all Q^2

- Challenge: to extract the "true" parton's confined motion:
  - ✧ Separation of perturbative shower contribution from nonperturbative hadron structure – not as simple as PDFs
  - ✧ Role of lattice QCD?

Task of the DOE supported TMD collaboration

PDFs, TMDs, GPDs, and hadron structure



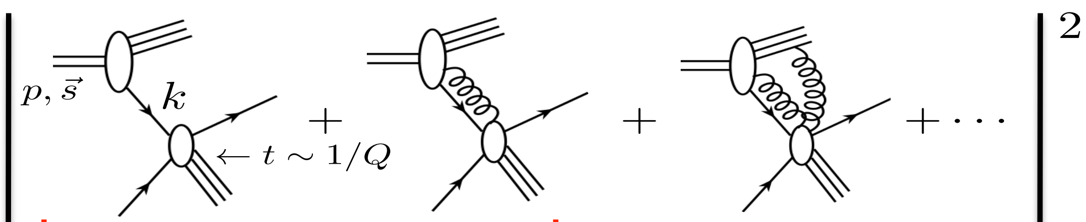
- **What do we need to know for full hadron structure?**
  - **In theory:** $\langle P,S|\mathcal{O}(\overline{\psi},\psi,A^{\mu})|P,S\rangle$ – **Hadronic matrix elements**

    with ALL possible operators: $\mathcal{O}(\overline{\psi},\psi,A^{\mu})$
  - **In fact:** ***None of these matrix elements is a direct physical observable in QCD – color confinement! – need probes!!!***
  - **In practice:** **Accessible hadron structure**
    = hadron matrix elements of quarks and gluons, which

    1) **can be related to physical cross sections of hadrons and leptons with controllable approximation – factorization;**
    2) **can be calculated in lattice QCD**

- **Multi-parton correlations – beyond single parton distributions:**

$\sigma(Q,\vec{s}) \propto$



$\left(\frac{\langle k_\perp\rangle}{Q}\right)^n$ – **Expansion**

See talk by Y. Koike

Quantum interference → **3-parton matrix element – not a probability!**

Summary

- ❑ TMDs, like PDFs, are NOT direct physical observables
  – could be defined differently

- ❑ Knowledge of nonperturbative inputs at large b_T is crucial
  in determining the TMDs from fitting the data

- ❑ QCD factorization is necessary for any controllable “probe”
  for hadron’s quark-gluon structure!

- ❑ JLab12, COMPASS, RHIC spin, … will provide rich information
  on hadron structure via TMDs in years to come!

- ❑ EIC is a ultimate QCD machine, and will provide answers to
  many of our questions on hadron structure, in particular, the
  confined transverse motions (TMDs), spatial distributions
  (GPDs), and multi-parton correlations, …

See also talk by A. Deshpande

Thank you!

Backup slides

Evolution equations for TMDs

J.C. Collins, in his book on QCD

- **TMDs in the b-space:**

$$\tilde{F}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu;\zeta_F)=\tilde{F}^{\mathrm{unsub}}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu;y_P-(-\infty))\sqrt{\frac{\tilde{S}_{(0)}(\mathrm{b}_\mathrm{T};+\infty,y_s)}{\tilde{S}_{(0)}(\mathrm{b}_\mathrm{T};+\infty,-\infty)\tilde{S}_{(0)}(\mathrm{b}_\mathrm{T};y_s,-\infty)}}Z_F\,Z_2$$

- **Collins-Soper equation:**

Renormalization of the soft-factor

$$\frac{\partial\tilde{F}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu;\zeta_F)}{\partial\ln\sqrt{\zeta_F}}=\tilde{K}(b_T;\mu)\tilde{F}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu;\zeta_F)$$

$$\zeta_F=M_P^2x^2e^{2(y_P-y_s)}$$

Introduced to regulate the rapidity divergence of TMDs

$$\tilde{K}(b_T;\mu)=\frac{1}{2}\frac{\partial}{\partial y_s}\ln\left(\frac{\tilde{S}(b_T;y_s,-\infty)}{\tilde{S}(b_T;+\infty,y_s)}\right)$$

- **RG equations:**

Wave function Renormalization

$$\frac{d\tilde{K}(b_T;\mu)}{d\ln\mu}=-\gamma_K(g(\mu))$$

Evolution equations are only valid when $b_T << 1/\Lambda_{QCD}$!

$$\frac{d\tilde{F}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu;\zeta_F)}{d\ln\mu}=\gamma_F(g(\mu);\zeta_F/\mu^2)\tilde{F}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu;\zeta_F).$$

- **Momentum space TMDs:**

Need information at large b_T for all scale μ!

$$F_{f/P^\uparrow}(x,\mathrm{k}_\mathrm{T},S;\mu,\zeta_F)=\frac{1}{(2\pi)^2}\int d^2\mathrm{b}_T\,e^{i\mathrm{k}_T\cdot\mathrm{b}_T}\,\tilde{F}_{f/P^\uparrow}(x,\mathrm{b}_\mathrm{T},S;\mu,\zeta_F)$$

Evolution equations for Sivers function

Aybat, Collins, Qiu, Rogers, 2011

- Sivers function:

$$F_{f/P^\uparrow}(x, k_T, S; \mu, \zeta_F) = F_{f/P}(x, k_T; \mu, \zeta_F) - F_{1T}^{\perp f}(x, k_T; \mu, \zeta_F) \frac{\epsilon_{ij} k_T^i S^j}{M_p}$$

- Collins-Soper equation:

Its derivative obeys the CS equation

$$\frac{\partial \ln \tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F)}{\partial \ln \sqrt{\zeta_F}} = \tilde{K}(b_T; \mu)$$

$$\tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F) \equiv \frac{\partial \tilde{F}^{\perp f}_{1T}(x, b_T; \mu, \zeta_F)}{\partial b_T}$$

- RG equations:

$$\frac{d\tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F)}{d \ln \mu} = \gamma_F(g(\mu); \zeta_F/\mu^2) \tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F)$$

$$\frac{d\tilde{K}(b_T; \mu)}{d \ln \mu} = -\gamma_K(g(\mu)) \quad \Rightarrow \quad \frac{\partial \gamma_F(g(\mu); \zeta_F/\mu^2)}{\partial \ln \sqrt{\zeta_F}} = -\gamma_K(g(\mu)),$$

- Sivers function in momentum space:

$$F_{1T}^{\perp f}(x, k_T; \mu, \zeta_F) = \frac{-1}{2\pi k_T} \int_0^\infty db_T\, b_T\, J_1(k_T b_T) \tilde{F}'^{\perp f}_{1T}(x, b_T; \mu, \zeta_F)$$

JI, Ma, Yuan, 2004
Idilbi, et al, 2004,
Boer, 2001, 2009,
Kang, Xiao, Yuan, 2011
Aybat, Prokudin, Rogers, 2012
Idilbi, et al, 2012,
Sun, Yuan 2013, …

Extrapolation to large b_T

❑ CSS b*-prescription:

Aybat and Rogers, arXiv:1101.5057
Collins and Rogers, arXiv:1412.3820

$$\tilde{F}_{f/P}(x, b_T; Q, Q^2) = \overbrace{\sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P}(\hat{x}, \mu_b)}^{\text{AA}}$$

$$\times \overbrace{\times \exp\left\{ \ln\frac{Q}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{Q} \frac{d\mu'}{\mu'} \left[\gamma_F(g(\mu'); 1) - \ln\frac{Q}{\mu'} \gamma_K(g(\mu')) \right] \right\}}^{\text{BB}}$$

$$\times \boxed{\overbrace{\exp\left\{ g_{f/P}(x, b_T) + g_K(b_T) \ln\frac{Q}{Q_0} \right\}}^{\text{CC}}}$$

Nonperturbative
"form factor"

$$b_* = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}} \qquad \text{with } b_{\max} \sim 1/2 \text{ GeV}^{-1}$$

❑ Nonperturbative fitting functions

Various fits correspond to different choices for $g_{f/P}(x, b_T)$ and $g_K(b_T)$
e.g.

$$g_{f/P}(x, b_T) + g_K(b_T) \ln\frac{Q}{Q_0} \equiv -\left[g_1 + g_2 \ln\frac{Q}{2Q_0} + g_1 g_3 \ln(10x) \right] b_T^2$$

Different choice of g_2 & b_* could lead to different over all Q-dependence!