Lattice QCD and transport coefficients

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Plan

Strongly interacting matter at temperatures $T = 100 - 500$ MeV

- probed in heavy-ion collisions: hadrons $\rightarrow$ quark-gluon plasma
- state of matter for the first microsecond after Big Bang

Thermal physics: $\beta = 1/(kT)$,

$$\langle A \rangle = \frac{1}{Z} \text{Tr} \{ e^{-\beta H} A \}, \quad Z = \text{Tr} \{ e^{-\beta H} \}$$

- Overview of equilibrium properties from lattice QCD

- Near-equilibrium (real-time) properties
  - formalism
  - vector channel for light quarks: dilepton rate, diffusion coefficient
  - heavy-quark momentum diffusion coefficient
  - pion quasiparticle in the hadronic phase
QCD phase diagram

- at $\mu_B = 0$: $T_{\text{transition}} = 155 \pm 8$ MeV from lattice simulations (crossover)
- e.g. from chiral susceptibility $\int d^4x \langle \bar{\psi}(x) \psi(x) \bar{\psi}(0) \psi(0) \rangle$
  [see e.g. review Soltz et al. 1502.02296].

Fig. from Braun-Munzinger, Koch, Schäfer, Stachel, Phys.Rept. 621 (2016) 76
Thermodynamic potentials

Fig. from review by Soltz et al. 1502.02296

- at $T = 260\text{MeV}$, $p \approx 1/2 p_{SB}$: far from weakly interacting quarks and gluons;
- hadron resonance gas (HRG) model works well up to $T = 160 \text{MeV}$;
- HRG also describes well the fluctuations of conserved charges, e.g. $\frac{1}{V} \times \langle Q^2 \rangle$, $\langle B^2 \rangle$ and $\langle S^2 \rangle$. 
Regularization of QCD on a lattice

Gluons: $U_\mu(x) = e^{i\alpha g_0 A_\mu(x)} \in SU(3)$
‘link variables’

Quarks: $\psi(x)$ ‘on site’, Grassmann

Gauge-invariance exactly preserved; no gauge-fixing required.

Imaginary-time path-integral representation of QFT (Matsubara formalism):

Formally: Lattice QCD = 4d statistical mechanics system

Starting point for Monte-Carlo simulations using importance sampling
Near-equilibrium properties

Typical questions:

- What quasiparticles are there in the system?

- How fast does an external perturbation dissipate in the system? For long wavelength perturbations, the rate is parametrized by transport coefficients (shear/bulk viscosity, diffusion coefficients, . . .)

\[
\langle J_0(t, k) \rangle \theta(-t) \mu(k) \propto e^{-Dk^2t}
\]

- What is the production rate of photons or dileptons?
Formalism

Relation between the correlator computed on the lattice and the spectral function:

\[ G(x_0, p) \equiv \int d^3 x \ e^{-ip \cdot x} \langle J(x)^\dagger J(0) \rangle = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, p) \frac{\cosh[\omega(\beta/2 - x_0)]}{\sinh[\omega\beta/2]} \]

- in the low-\(T\) phase, \(J^\text{em}_i\) can excite e.g. an \(\omega\)-meson-like quasiparticle.

- for \(J = J^\text{em}_i\) electromagnetic current, \(\rho(\omega, 0) \xrightarrow{\omega \to 0} 6\chi_s D\omega\)

\[ \chi_s = \int d^4 x \langle J^\text{em}_0(x) J^\text{em}_0(0) \rangle = \text{static susceptibility of electric charge} \]

\(D=\text{diffusion coefficient}\)

- photon rate:

\[ \frac{d\Gamma}{d^3 k} = \frac{e^2}{2(2\pi)^3} \sum_f Q^2_f \frac{\rho(k, \mathbf{k})}{k e^{\beta^k} - 1} \]

\(\ast\) numerically ill-posed inverse problem for \(\rho(\omega, \mathbf{p}) \geq 0; \ 0 \leq x_0 < \beta.\)
The inverse problem has many faces: here is one of them

Linearity: \[ \sum_{i=1}^{n} c_i(\bar{\omega}) G(t_i) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega) \sum_{i=1}^{n} c_i(\bar{\omega}) \frac{\cosh[\omega(\beta/2 - t_i)]}{\sinh[\omega \beta/2]} \delta(\bar{\omega}, \omega) \]

- choose the coefficients \( c_i(\bar{\omega}) \) so that the ‘resolution function’ \( \hat{\delta}(\bar{\omega}, \omega) \) is as narrowly peaked around a given frequency \( \bar{\omega} \) as possible

(idea behind the Backus-Gilbert method, [used in Robaina et al. PRD 92 (2015) 094510.])

Resolution function at \( \bar{\omega} = 4T \) for \( N_t = 24, \ t_i/a = 5, \ldots 12. \)

- Resolution only improves slowly with increasing \( n \)
- Large, sign-alternating coefficients \( \Rightarrow \) need for ultra-precise input data.
Expected thermal changes in spectral functions

Isoscalar vector channel: spectral fct. of \( J_i = \frac{1}{\sqrt{2}}(\bar{u}\gamma_iu + \bar{d}\gamma_id) \)

![Graph showing \( \tanh(\omega/2T) \rho_{ii}(\omega,T)/\omega^2 \) vs. \( \omega/\text{GeV} \), with intercept at 3 \( \chi_s D / T \).]

- presence of weakly coupled quasiparticles ⇒ transport peak at \( \omega = 0 \);
  is it really there at \( T \approx 260\text{MeV} \)?

SND hep-ex/0305049

\( D = \) diffusion coefficient; \( \chi_s = \) static susceptibility.
The isovector vector channel at $p = 0$

Lattice QCD correlators

$(T = 0.8, 1.0, 1.25, 1.67 \times T_c)$

Spectral functions

- fit ansatz method: simultaneous fit to all temperatures, because $\rho(\omega) \xrightarrow{\omega \to \infty} (\# + \# \frac{\alpha_s}{\pi} + \ldots) \omega^2$ with $T$-independent coefficients;
- exact sum rule: $\int_0^\infty \frac{d\omega}{\omega} [\rho(\omega, T') - \rho(\omega, T')] = 0 \xrightarrow{\text{constraint}}$

Francis et al. ($N_f = 2$, $m_\pi|_{T=0} = 267\text{MeV}$, $T_c = 203\text{MeV}$), PRD93 (2016) 054510
Model-dependence of the spectral function

Alternative ansatz at $T = T_c$

- fit ansatz method: simultaneous fit to all temperatures, because
  $\rho(\omega) \stackrel{\omega \to \infty}{\sim} (# + # \frac{\alpha_s}{\pi} + \ldots) \omega^2$ with $T$-independent coefficients;
- exact sum rule: $\int_0^\infty \frac{d\omega}{\omega} [\rho(\omega, T') - \rho(\omega, T')] = 0 \sim \text{constraint}$;

Francis et al. ($N_f = 2$, $m_\pi|_{T=0} = 267\text{MeV}$, $T_c = 203\text{MeV}$), PRD93 (2016) 054510
Lattice QCD vs. phenomenology

\[ \Delta \rho(\omega, T) : \text{lattice vs. pheno} \]

- compare \( \frac{\Delta \rho(\omega, T)}{\tanh(\omega \beta/2)} \equiv \int_0^\infty d\omega' \delta(\omega, \omega') \frac{\Delta \rho(\omega', T)}{\tanh(\omega' \beta/2)} \): lattice result is model-independent.

- shift of spectral weight from the \( \rho \) to low frequency region as \( T \) increases.

Francis et al. PRD93 (2016) 054510; Rapp & Hohler, PLB731, 103 (2014).
Calculations of the diffusion coefficient

- inverse problem treated with the Maximum Entropy Method;
- $D \propto \rho(\omega)/(\chi_s \omega)|_{\omega=0}$ comes out very small;
- stability of the results tested under variations in the procedure.

$N_f = 2 + 1$ simulations, $m_\pi|_{T=0} = 384\text{MeV}$, Aarts et al. JHEP 1502 (2015) 186.
See also $N_f = 0$ continuum calculation using fit ansätze Ding, Kaczmarek, F. Meyer PRD94 (2016) 034504
Selected recent results for the light-quark diffusion coefficient $D$

- lattice calculations yield very low values, $D \approx 1/(\pi T)$;
- however, all results assume that no narrow transport peak is present: these methods would fail at very high temperatures.
- Except green point!
$T > T_c$: Heavy-quark momentum diffusion coefficient $\kappa$

\[
G(\tau) = \left\langle \text{Re} \text{Tr} \left( U(\beta, \tau) gE_k(\tau, 0) U(t, 0) gE_k(0, 0) \right) \right\rangle = \frac{1}{-3 \left\langle \text{Re} \text{Tr} U(\beta, 0) \right\rangle} \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega) \frac{\cosh[\omega(\beta/2 - \tau)]}{\sinh[\omega\beta/2]}
\]

- color parallel transporters $U(t_2, t_1)$ are propagators of static quarks
- (Lorentz) force-force correlator on the worldline of the quark.

\[
\kappa = \lim_{\omega \to 0} \frac{T}{\omega} \rho(\omega), \quad D = 2T^2 / \kappa.
\]

NNLO calculation available:
\[
\rho(\omega) = \text{smooth function} \quad \omega \to \infty \sim g^2 \omega^3.
\]

Result: $2\pi TD = 3.7 \ldots 6.9$

Francis, Kaczmarek, Laine, Neuhaus
PRD92 (2015) 116003
Spectral function on the light-cone $\rightsquigarrow$ photon rate $\frac{d\Gamma_\gamma}{d^3k}$

\[
D_{\text{eff}}(k) = \begin{cases} 
\frac{\rho_V(k,k)}{2X_qk} & k > 0 \\
\lim_{\omega \to 0^+} \frac{\rho^{ii}(\omega,0)}{3X_q} & k = 0
\end{cases}
\]

\[
\frac{d\Gamma_\gamma}{d^3k} = \frac{2\alpha_{em}X_q}{3\pi^2} \frac{D_{\text{eff}}(k)}{e^{\beta k} - 1}.
\]

- at $k = 0$, a narrow transport peak cannot be excluded $\Rightarrow$ large uncertainty on result for $D$.
- for $k \approx 2T$, a more reliable result for $D_{\text{eff}}(k)$ is possible: spectral function expected to be smooth;
- fit ansatz: polynom up to $\omega = \sqrt{k^2 + \pi^2T^2}$, perturbation theory beyond.

$N_f = 0$, analysis in the continuum; Ghiglieri, Kaczmarek, Laine, F. Meyer PRD 94, 016005 (2016)
The pion quasiparticle in the low-temperature phase
Chiral symmetry is spontaneously broken for $T < T_c$: $-\langle \bar{\psi}\psi \rangle > 0$.

Goldstone theorem $\Rightarrow$ a divergent spatial correlation length $m_{\pi}^{-1}$ exists in the limit $m_{u,d} \rightarrow 0$.

also: a massless real-time excitation exists: the pion quasiparticle.

dispersion relation: [Son and Stephanov, PRD 66, 076011 (2002)]

$$\omega_p = u(T)\sqrt{m_{\pi}^2(T) + p^2} + \ldots$$

$T \lesssim 100 \text{MeV}$: Two-loop chiral perturbation theory prediction for the pion quasiparticle mass $u(T)m_{\pi}(T)$ [D. Toublan, PRD 56 5629 (1997)]

Quasiparticle mass $\frac{M_{\pi}^2(T)}{M_{\pi}^2}$

full: physical quark mass
dashed: massless case.
key point: pion dominates parametrically the Euclidean two-point function of the axial charge density \( \int d^3x \ e^{ip \cdot x} \bar{\psi} \gamma_0 \gamma_5 \frac{\tau_a}{2} \psi \) and its second derivative at \( x_0 = \beta/2 \approx 0.6 \text{fm} \) and \( |p| \lesssim 300 \text{MeV} \).

inverse problem can be solved via the ansatz

\[
\rho_A(\omega, p, T) = f_\pi^2(T) \left( m_\pi^2(T) + p^2 \right) \delta(\omega^2 - u^2(T)(m_\pi^2(T) + p^2))
\]

here \( m_\pi(T) \) and \( f_\pi(T) \) are determined from screening (=static) correlation functions; from time-dependent correlator: \( u = 0.75(2) \) and

\[
T = 0 : \quad \text{pion mass} = 267(2) \text{MeV}
\]

\[
T = 169 \text{MeV} : \quad \text{quasiparticle mass} = 223(4) \text{MeV} \quad \text{screening mass} = 303(4) \text{MeV}.
\]

Simulation details: \( N_f = 2 \) (no strange quark); \( 24 \times 64^3 \) lattice;

Transition temperature \( T_c \approx 203 \text{MeV} \).

How does this fit in with the success of the hadron-resonance gas model?

Conclusion

Significant progress in lattice QCD on near-equilibrium quantities:

- few-permille precision on correlation functions at small lattice spacings, even continuum in ‘quenched’ approximation

- advanced weak-coupling calculations, effective field theories, exact sum rules, . . . provide crucial prior information on spectral function.

- many channels not discussed here: fate of quarkonium in the quark-gluon plasma, open-charm spectral functions, shear/sound channels, . . .
Backup slides
Thermal fluctuations and correlations

Fig. from S. Borsanyi et al. 1112.4416

- Light-quark number susceptibility: suggests that deconfinement occurs practically at the same temperature as chiral restoration.
- Successful predictions of the HRG.

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Lattice QCD and transport coefficients
Pion channel, continued: description of the lattice data

\[
\frac{1}{3} \int d^3 x \ e^{i \mathbf{p} \cdot \mathbf{x}} \langle A_0^a(x) A_0^a(0) \rangle = \int_0^\infty \frac{d\omega}{2\pi} \rho^A(\omega, \mathbf{p}) \frac{\cosh[\omega(\beta/2 - x_0)]}{\sinh[\omega\beta/2]}.
\]

Ansatz: \[ \rho^A(\omega, \mathbf{p}) = a_1(\mathbf{p}) \delta(\omega - \omega_\mathbf{p}) + a_2(\mathbf{p})(1 - e^{-\omega\beta}) \theta(\omega - c). \]