New Results on the Structure of Baryons and their Excitations from Lattice QCD

Derek Leinweber
In collaboration with: Jonathan Hall, Waseem Kamleh, Adrian Kiratidis, Zhan-Wei Liu, Ben Menadue, Ben Owen, Tony Thomas, Jia-Jun Wu, Ross Young
Outline

Lattice QCD Baryon Spectra at light quark masses
\( \Lambda(1405), N^*(1535), N^*(1650) \) and \( N'(1440) \) Resonances

Wave Functions of Nucleon Excitations

Isolation of the \( \Lambda(1405) \) in Lattice QCD

Evidence the \( \Lambda(1405) \) is a \( \bar{K}N \) molecule

Hamiltonian Effective Field Theory Description of Spectra

The nature of the low-lying \( N \) Spectrum

Conclusions
Closely Related INPC Talks

- Adrian Kiratidis on Five-Quark Operators: Tuesday, R6, 2:40
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- Ryan Bignell on Nucleon Polarizabilities: Friday, C1, 11:10
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- Jonathan Hall on the magnetic moment of the $\Lambda(1405)$, Friday, C1, 12:10
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- Zhan-Wei Liu on Hamiltonian Effective Field Theory: Friday, C1, 11:25
- Jonathan Hall on the magnetic moment of the $\Lambda(1405)$, Friday, C1, 12:10
- Finn Stokes on Excited State Form Factors: Immediately following this talk.
CSSM Simulation Details

Based on the PACS-CS (2 + 1)-flavour ensembles, available through the ILDG.
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- Lattice size of $32^3 \times 64$ with $\beta = 1.90$. $L \simeq 3$ fm.
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- Lattice size of $32^3 \times 64$ with $\beta = 1.90$. $L \simeq 3$ fm.
- 5 pion masses, ranging from 640 MeV down to 156 MeV.
CSSM Simulation Details

Based on the PACS-CS \((2 + 1)\)-flavour ensembles, available through the ILDG.


- Lattice size of \(32^3 \times 64\) with \(\beta = 1.90\). \(L \simeq 3\) fm.
- 5 pion masses, ranging from 640 MeV down to 156 MeV.
- The strange quark \(\kappa_s\) is held fixed as the light quark masses vary.
  - Changes in the strange quark contributions are environmental effects.
Positive Parity Nucleon Spectrum: CSSM

![Graph showing the positive parity nucleon spectrum with markers and lines indicating P-wave and S-wave contributions.]

- **P-wave N+π**
- **S-wave N+π+π**
Comparison: Hadron Spectrum Collaboration (HSC)

- “Excited state baryon spectroscopy from lattice QCD,”
  R. G. Edwards, J. J. Dudek, D. G. Richards and S. J. Wallace,
CSSM & HSC Comparison: Positive Parity

![Graph showing the comparison between CSSM and HSC for Positive Parity](image-url)
CSSM & HSC Comparison: Negative Parity CSSM
CSSM & HSC Comparison: Negative Parity
Further Information

• “Roper Resonance in 2+1 Flavor QCD,”
  M. S. Mahbub, et al. [CSSM],
  arXiv:1011.5724 [hep-lat],

• “Low-lying Odd-parity States of the Nucleon in Lattice QCD,”
  M. Selim Mahbub, et al. [CSSM],
  Phys. Rev. D Rapid Comm. 87 (2013) 011501,
  arXiv:1209.0240 [hep-lat]

• “Structure and Flow of the Nucleon Eigenstates in Lattice QCD,”
  M. S. Mahbub, et al. [CSSM],
  Phys. Rev. D 87 (2013) 9, 094506
  arXiv:1302.2987 [hep-lat].
Wave Functions of Positive-Parity Nucleons
$d$-quark probability density in ground state proton (CSSM)
d-quark probability density in 1st excited state of proton (CSSM)
$d$-quark probability density in $N = 3$ excited state of proton (CSSM)
Comparison with the Simple Quark Model - CSSM

![Graphs showing probability distribution vs. r/a](image-url)
Finite-Volume Effects in Wave Functions

\[ m^2_{\pi} \text{ (GeV)} \]

\[ M \text{ (GeV)} \]

\[ \text{P-wave } N+\pi \]

\[ \text{S-wave } N+\pi+\pi \]
Finite-Volume Effect in $N = 2$ excited state: $m_\pi = 702$ MeV
Finite-Volume Effect in $N = 2$ excited state: $m_\pi = 570$ MeV
Finite-Volume Effect in $N = 2$ excited state: $m_\pi = 411$ MeV
Finite-Volume Effect in $N = 2$ excited state: $m_\pi = 296$ MeV
Finite-Volume Effect in $N = 2$ excited state: $m_\pi = 156$ MeV
$d$-quark probability density in 1st excited state of proton (CSSM)
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$d$-quark probability density in 1st excited state of proton (CSSM)
$d$-quark probability density in 4th excited state of proton (CSSM)
Our 2012 work successfully isolated three low-lying odd-parity spin-1/2 states.

- An extrapolation of the trend of the lowest state reproduces the mass of the \( \Lambda(1405) \).
- Subsequent studies have confirmed these results.


$\Lambda(1405)$ and Baryon Octet dominated states
Operators Used in Λ(1405) Analysis

We consider local three-quark operators with the correct quantum numbers for the Λ channel, including

- **Flavour-octet operators**

  \[ \chi_1^8 = \frac{1}{\sqrt{6}} \varepsilon^{abc} \left( 2(u^a C \gamma_5 d^b)s^c + (u^a C \gamma_5 s^b)d^c - (d^a C \gamma_5 s^b)u^c \right) \]

  \[ \chi_2^8 = \frac{1}{\sqrt{6}} \varepsilon^{abc} \left( 2(u^a C d^b)\gamma_5 s^c + (u^a C s^b)\gamma_5 d^c - (d^a C s^b)\gamma_5 u^c \right) \]

- **Flavour-singlet operator**

  \[ \chi^1 = 2 \varepsilon^{abc} \left( (u^a C \gamma_5 d^b)s^c - (u^a C \gamma_5 s^b)d^c + (d^a C \gamma_5 s^b)u^c \right) \]

- **Consideration of 16 and 100 sweeps of gauge-invariant Gaussian smearing provides a 6 × 6 correlation matrix.**
Flavour structure of the $\Lambda(1405)$
The importance of eigenstate isolation (red)
Probing with the electromagnetic current

\[ \ln(G) \]
Only the projected correlator has acceptable $\chi^2$/dof.
Is the $\Lambda(1405)$ really exotic?
Strange Magnetic Form Factor of the $\Lambda(1405)$

- Provides direct insight into the possible dominance of a molecular $\bar{K}N$ bound state.
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- Provides direct insight into the possible dominance of a molecular $\overline{K}N$ bound state.
- In forming such a molecular state, the $\Lambda(u, d, s)$ valence quark configuration is complemented by
  - A $u, \bar{u}$ pair making a $K^{-}(s, \bar{u})$ - proton $(u, u, d)$ bound state, or
  - A $d, \bar{d}$ pair making a $\overline{K}^{0}(s, \bar{d})$ - neutron $(d, d, u)$ bound state.
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- In both cases the strange quark is confined within a spin-0 kaon and has no preferred spin orientation.
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  - A $d, \bar{d}$ pair making a $\bar{K}^0(s, \bar{d})$ - neutron $(d, d, u)$ bound state.
- In both cases the strange quark is confined within a spin-0 kaon and has no preferred spin orientation.
- To conserve parity, the kaon has zero orbital angular momentum.
- Thus, the strange quark does not contribute to the magnetic form factor of the $\Lambda(1405)$ when it is dominated by a $\bar{K}N$ molecule.
\( G_M \) for the \( \Lambda(1405) \) at \( Q^2 \sim 0.16 \text{ GeV}^2 \)
$G_M$ for the $\Lambda(1405)$ at $Q^2 \sim 0.16 \text{ GeV}^2$
Hamiltonian Effective Field Theory

- J. M. M. Hall, *et al.* [CSSM]
  "Lattice QCD Evidence that the Λ(1405) Resonance is an Antikaon Nucleon Molecule"
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  "Structure of the Λ(1405) from Hamiltonian effective field theory"
  arXiv:1607.05856 [nucl-th]

- Z. W. Liu, W. Kamleh, DBL, F. M. Stokes, A. W. Thomas and J. J. Wu,
  "Hamiltonian EFT study of the N*(1535) resonance in lattice QCD,"
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- Z. W. Liu, W. Kamleh, DBL, F. M. Stokes, A. W. Thomas and J. J. Wu,
  “Hamiltonian EFT study of the $N^*(1440)$ resonance in lattice QCD,”
  arXiv:1607.04536 [nucl-th]
Hamiltonian Effective Field Theory Model

- Consider the $\Lambda(1405)$. 
Consider the $\Lambda(1405)$. The four octet meson-baryon interaction channels of the $\Lambda(1405)$ are considered: $\pi \Sigma$, $\bar{K}N$, $K\Xi$ and $\eta\Lambda$. 
• Consider the $\Lambda(1405)$.
• The four octet meson-baryon interaction channels of the $\Lambda(1405)$ are considered: $\pi \Sigma, \Omega N, K \Xi$ and $\eta \Lambda$.
• A single-particle state with bare mass, $m_0 + \alpha_0 m_{\pi}^2$ is also included.
Hamiltonian Effective Field Theory Model

- Consider the $\Lambda(1405)$.
- The four octet meson-baryon interaction channels of the $\Lambda(1405)$ are considered: $\pi\Sigma$, $\bar{K}N$, $KN$ and $\eta\Lambda$.
- A single-particle state with bare mass, $m_0 + \alpha_0 m^2_\pi$ is also included.
- In a finite periodic volume, momentum is quantised to $n(2\pi/L)$. 
Consider the $\Lambda(1405)$. The four octet meson-baryon interaction channels of the $\Lambda(1405)$ are considered: $\pi\Sigma$, $KN$, $K\Xi$ and $\eta\Lambda$. A single-particle state with bare mass, $m_0 + \alpha_0 m^2_\pi$ is also included. In a finite periodic volume, momentum is quantised to $n \left(2\pi/L\right)$. Working on a cubic volume of extent $L$ on each side, it is convenient to define the momentum magnitudes

$$k_n = \sqrt{n_x^2 + n_y^2 + n_z^2} \frac{2\pi}{L},$$

with $n_i = 0, 1, 2, \ldots$ and integer $n = n_x^2 + n_y^2 + n_z^2$. 
Hamiltonian model, $H_0$

Denoting each meson-baryon energy by $\omega_{MB}(k_n) = \omega_M(k_n) + \omega_B(k_n)$, with $\omega_A(k_n) \equiv \sqrt{k_n^2 + m_A^2}$, the non-interacting Hamiltonian takes the form

$$H_0 = \begin{pmatrix}
 m_0 + \alpha_0 m_\pi^2 & \omega_{\pi\Sigma}(k_0) & 0 & 0 & \cdots \\
 0 & \omega_{\pi\Sigma}(k_0) & \omega_{\eta\Lambda}(k_0) & 0 & \cdots \\
 0 & 0 & \omega_{\pi\Sigma}(k_1) & \omega_{\eta\Lambda}(k_1) & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \ddots \\
 \vdots & \vdots & \vdots & \ddots & \ddots 
\end{pmatrix}.$$
Hamiltonian model, $H_I$

- Interaction entries describe the coupling of the single-particle state to the two-particle meson-baryon states.
Hamiltonian model, $H_I$

- Interaction entries describe the coupling of the single-particle state to the two-particle meson-baryon states.
- Each entry represents the $S$-wave interaction energy of the $\Lambda(1405)$ with one of the four channels at a certain value for $k_n$.

$$H_I = \begin{pmatrix}
0 & g_{\pi \Sigma}(k_0) & \cdots & g_{\eta \Lambda}(k_0) & g_{\pi \Sigma}(k_1) & \cdots & g_{\eta \Lambda}(k_1) \\
g_{\pi \Sigma}(k_0) & 0 & \cdots & 0 & g_{\pi \Sigma}(k_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
g_{\eta \Lambda}(k_0) & g_{\pi \Sigma}(k_1) & \cdots & 0 & \vdots & \ddots & \vdots \\
g_{\pi \Sigma}(k_1) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
g_{\eta \Lambda}(k_1) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}.$$
The eigenvalue equation corresponding to our Hamiltonian model is

\[ \lambda = m_0 + \alpha_0 m_\pi^2 - \sum_{M,B} \sum_{n=0}^{\infty} \frac{g_{MB}^2(k_n)}{\omega_{MB}(k_n) - \lambda}. \]

with \( \lambda \) denoting the energy eigenvalue.
The eigenvalue equation corresponding to our Hamiltonian model is

$$\lambda = m_0 + \alpha_0 m^2_\pi - \sum_{M,B} \sum_{n=0}^{\infty} \frac{g^2_{MB}(k_n)}{\omega_{MB}(k_n) - \lambda}.$$ 

with $\lambda$ denoting the energy eigenvalue.

- As $\lambda$ is finite, the pole in the denominator of the right-hand side is never accessed.
- The bare mass and the free meson-baryon energies encounter self-energy corrections that lead to avoided level-crossings in the finite-volume energy eigenstates.
The eigenvalue equation corresponding to our Hamiltonian model is

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- As \( \lambda \) is finite, the pole in the denominator of the right-hand side is never accessed.
- The bare mass and the free meson-baryon energies encounter self-energy corrections that lead to avoided level-crossings in the finite-volume energy eigenstates.
- Reference to chiral effective field theory provides the form of \( g_{MB}(k_n) \).
Hamiltonian model solution and fit

- Flavour-singlet couplings between the bare state and the meson-baryon states are constrained by $SU(3)$-flavour symmetry and the width of the $\Lambda(1405)$ resonance.
Hamiltonian model solution and fit

- Flavour-singlet couplings between the bare state and the meson-baryon states are constrained by $SU(3)$-flavour symmetry and the width of the $\Lambda(1405)$ resonance.
- The eigenvalues and eigenvectors of $H$ are obtained via the LAPACK software library.
Hamiltonian model solution and fit

- Flavour-singlet couplings between the bare state and the meson-baryon states are constrained by $SU(3)$-flavour symmetry and the width of the $\Lambda(1405)$ resonance.
- The eigenvalues and eigenvectors of $H$ are obtained via the LAPACK software library.
- The bare mass parameters $m_0$ and $\alpha_0$ are determined by a fit to the lattice QCD results.
Hamiltonian model fit

$E$ (GeV)

$m_{\pi}^2$ (GeV$^2$)

- matrix Hamiltonian model
- non-int. $\pi\Sigma$ energy
- non-int. KN energy
- non-int. $K\Xi$ energy
- non-int. $\eta\Lambda$ energy
- $\Lambda(1405)$ Lattice results
Avoided Level Crossing

![Graph showing avoided level crossing with energy E (GeV) on the y-axis and \( m_\pi^2 \) (GeV^2) on the x-axis. The graph includes lines for matrix Hamiltonian model, non-int. \( \pi\Sigma \) energy, non-int. KN energy, and \( \Lambda(1405) \) Lattice results.]
Energy eigenstate, $|E\rangle$, basis $|state\rangle$ composition

$|\langle state|E\rangle|^2$

$\bar{K}N$

$K\Sigma$

$m_0$

$\pi$ $\Sigma$

$m_0$

$\pi$ $\Sigma$

$m_0$

$\pi$ $\Sigma$

$m_0$

$\pi$ $\Sigma$

$m_\pi$ (MeV)
Hamiltonian model, $H_I$

- Our approach included a bare state dressed by flavour-singlet coupled meson-baryon channels

$$H_I = \begin{pmatrix}
0 & g_{\pi\Sigma}(k_0) & \cdots & g_{\eta\Lambda}(k_0) & g_{\pi\Sigma}(k_1) & \cdots & g_{\eta\Lambda}(k_1) \\
g_{\pi\Sigma}(k_0) & 0 & \cdots & \vdots & \vdots & \vdots & 0 \\
\vdots & \vdots & \ddots & & & & \\
g_{\eta\Lambda}(k_0) & & & 0 & & & \\
g_{\pi\Sigma}(k_1) & & & & 0 & & \\
\vdots & & & & & \ddots & \\
g_{\eta\Lambda}(k_1) & & & & & & 0 \\
\vdots & & & & & & \vdots
\end{pmatrix}.$$
Hamiltonian model, \( H_I \)

- Our approach included a bare state dressed by flavour-singlet coupled meson-baryon channels

\[
H_I = \begin{pmatrix}
0 & g_{\pi\Sigma}(k_0) & \cdots & g_{\eta\Lambda}(k_0) & g_{\pi\Sigma}(k_1) & \cdots & g_{\eta\Lambda}(k_1) & \cdots \\
g_{\pi\Sigma}(k_0) & 0 & \cdots & 0 & g_{\eta\Lambda}(k_1) & \cdots & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
g_{\eta\Lambda}(k_0) & g_{\pi\Sigma}(k_1) & \cdots & 0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
g_{\eta\Lambda}(k_1) & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\end{pmatrix}
\]

- Ironically, most analyses omit these interactions and instead include only the direct meson-baryon to meson-baryon interactions
  - Weinberg-Tomozawa terms
The two-pole description of the $\Lambda(1405)$

Direct two-to-two particle interactions

- We use the potential derived from Weinberg-Tomozawa term

\[
V^{I}_{\alpha,\beta}(k, k') = \frac{g^{I}_{\alpha,\beta}}{8\pi^2 f_{\pi}^2} \frac{\omega_{\alpha M}(k) + \omega_{\beta M}(k')}{\sqrt{2 \omega_{\alpha M}(k)} \sqrt{2 \omega_{\beta M}(k')}} u(k) u(k').
\]
Direct two-to-two particle interactions

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\[ V_{\alpha,\beta}(k, k') = \frac{g^I_{\alpha,\beta}}{8\pi^2 f^2_\pi} \frac{\omega_{\alpha M}(k) + \omega_{\beta M}(k')}{\sqrt{2\omega_{\alpha M}(k)} \sqrt{2\omega_{\beta M}(k')}} \ u(k) \ u(k') . \]

- Dipole regulator functions \( u(k) \) have a fixed scale of \( \Lambda = 1 \text{ GeV} \).
Direct two-to-two particle interactions

- We use the potential derived from Weinberg-Tomozawa term

\[
V^{I}_{\alpha,\beta}(k, k') = \frac{g^{I}_{\alpha,\beta}}{8\pi^2 f^2_{\pi}} \frac{\omega_{\alpha M}(k) + \omega_{\beta M}(k')}{\sqrt{2\omega_{\alpha M}(k)} \sqrt{2\omega_{\beta M}(k')}} \ u(k) \ u(k') .
\]

- Dipole regulator functions \( u(k) \) have a fixed scale of \( \Lambda = 1 \text{ GeV} \).
- Couplings vanishing in the \( SU(3) \)-flavour symmetry limit are not considered.
Direct two-to-two particle interactions

- Eight non-trivial couplings are constrained by experimental data in infinite volume.

\[
g_{\pi\Sigma,\pi\Sigma}^0, \ g_{K^0N,\bar{K}^0N}^0, \ g_{K^0N,\bar{K}^0N}^1, \ g_{H^0,\pi\Sigma}^0, \ g_{K^0N,\bar{K}^0N}^1, \ g_{K^0N,\pi\Sigma}^1, \ g_{K^0N,\pi\Lambda}^1,
\]

with SU(3) flavour symmetry constraints for the heavier $\eta\Lambda$ and $K\Xi$ channels

\[
g_{K^0N,\eta\Lambda}^0 = -3/\sqrt{2} \ g_H^0, \quad g_{\pi\Sigma,K\Xi}^0 = -\sqrt{3/2} \ g_H^0,
\]

\[
g_{\eta\Lambda,K\Xi}^0 = 3/\sqrt{2} \ g_H^0, \quad g_{K\Xi,K\Xi}^0 = -3 \ g_H^0.
\]

(1)
Direct two-to-two particle interactions

- Eight non-trivial couplings are constrained by experimental data in infinite volume.

\[
g^0_{\pi\Sigma,\pi\Sigma}, \ g^0_{K_N\bar{K}_N}, \ g^0_{K_N,\pi\Sigma}, \ g^0_{H}, \ g^1_{\pi\Sigma,\pi\Sigma}, \ g^1_{K_N,\bar{K}_N}, \ g^1_{K_N,\pi\Sigma}, \ g^1_{K_N,\pi\Lambda},
\]

with $SU(3)$ flavour symmetry constraints for the heavier $\eta\Lambda$ and $K\Xi$ channels

\[
g^0_{K_N,\eta\Lambda} = -3/\sqrt{2} \ g^0_H, \quad g^0_{\pi\Sigma,\Xi} = -\sqrt{3/2} \ g^0_H, \\
g^0_{\eta\Lambda,\Xi} = 3/\sqrt{2} \ g^0_H, \quad g^0_{K,\Xi,\Xi} = -3 \ g^0_H.
\]  

(1)

- Finite volume spectrum is then a prediction.
Couplings Constrained by Experiment

(a) $K^- p \rightarrow K^- p$

(b) $K^- p \rightarrow \bar{K}^0 n$

(c) $K^- p \rightarrow \pi^- \Sigma^+$

(d) $K^- p \rightarrow \pi^0 \Sigma^0$

(e) $K^- p \rightarrow \pi^+ \Sigma^-$

(f) $K^- p \rightarrow \pi^0 \Lambda$
Finite Volume $\Lambda$ Spectrum for $L = 3$ fm
Comparison with $U\chi$PT

Direct two-to-two particle interactions & bare state

• Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1$ GeV.
• Same eight two-to-two particle couplings are considered

$$g^{0}_{\pi\Sigma,\pi\Sigma}, \ g^{0}_{\bar{K}N,\bar{K}N}, \ g^{0}_{\bar{K}N,\pi\Sigma}, \ g^{0}_{H}, \ g^{1}_{\pi\Sigma,\pi\Sigma}, \ g^{1}_{\bar{K}N,\bar{K}N}, \ g^{1}_{\bar{K}N,\pi\Sigma}, \ g^{1}_{\bar{K}N,\pi\Lambda};$$
Direct two-to-two particle interactions & bare state

- Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1$ GeV.
- Same eight two-to-two particle couplings are considered:
  \[ g_{\pi \Sigma, \pi \Sigma}^0, g_{\bar{K}N, \bar{K}N}^0, g_{\bar{K}N, \pi \Sigma}^0, g_{\bar{K}N, \pi \Sigma}^1, g_{\bar{K}N, \bar{K}N}^0, g_{\bar{K}N, \pi \Sigma}^0, g_{\bar{K}N, \pi \Lambda}^1, \]
- Five new parameters describing bare to two-particle interactions are introduced:
  \[ m_B^0, g_{\pi \Sigma, B_0}^0, g_{\bar{K}N, B_0}^0, g_{\eta \Lambda, B_0}^0, g_{K \Xi, B_0}^0, \]
Direct two-to-two particle interactions & bare state

- Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1$ GeV.
- Same eight two-to-two particle couplings are considered:
  \[ g_{\pi\Sigma,\pi\Sigma}^0, g_{\bar{K}N,\bar{K}N}^0, g_{\bar{K}N,\pi\Sigma}^0, g_{H}, g_{\pi\Sigma,\pi\Sigma}^1, g_{\bar{K}N,\bar{K}N}^1, g_{\bar{K}N,\pi\Sigma}^1, g_{\bar{K}N,\pi\Lambda}^1, \]

- Five new parameters describing bare to two-particle interactions are introduced:
  \[ m_B^0, g_{\pi\Sigma,B_0}^0, g_{\bar{K}N,B_0}^0, g_{\eta\Lambda,B_0}^0, g_{K\Xi,B_0}^0, \]

- These 13 parameters are constrained by experimental data.
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• Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1$ GeV.

• Same eight two-to-two particle couplings are considered

$$g_{\pi\Sigma,\pi\Sigma}^0, g_{\bar{K}N,\bar{K}N}^0, g_{\bar{K}N,\pi\Sigma}^0, g_{H}, g_{\pi\Sigma,\pi\Sigma}^1, g_{\bar{K}N,\bar{K}N}^1, g_{\bar{K}N,\pi\Sigma}^1, g_{\bar{K}N,\pi\Lambda}^1$$

• Five new parameters describing bare to two-particle interactions are introduced

$$m_B^0, g_{\pi\Sigma,B_0}^0, g_{\bar{K}N,B_0}^0, g_{\eta\Lambda,B_0}^0, g_{K\Xi,B_0}^0$$

• These 13 parameters are constrained by experimental data.

• A linear quark mass dependence for the bare mass is constrained by the lattice results.
Couplings and $m^0_B$ Constrained by Experiment

(a) $K^- p \rightarrow K^- p$  
(b) $K^- p \rightarrow \bar{K}^0 n$  
(c) $K^- p \rightarrow \pi^- \Sigma^+$  
(d) $K^- p \rightarrow \pi^0 \Sigma^0$  
(e) $K^- p \rightarrow \pi^+ \Sigma^-$  
(f) $K^- p \rightarrow \pi^0 \Lambda$
Couplings and $m_B^0$ Constrained by Experiment

(a) $K^- p \rightarrow K^- p$

(b) $K^- p \rightarrow \bar{K^0} n$

(c) $K^- p \rightarrow \pi^- \Sigma^+$

(d) $K^- p \rightarrow \pi^0 \Sigma^0$

(e) $K^- p \rightarrow \pi^+ \Sigma^-$

(f) $K^- p \rightarrow \pi^0 \Lambda$
<table>
<thead>
<tr>
<th>$I = 0$ Parameters</th>
<th>No Bare State</th>
<th>With Bare State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{\pi \Sigma, \pi \Sigma}^0$</td>
<td>-1.77</td>
<td>-1.11</td>
</tr>
<tr>
<td>$g_{\bar{K}N, \bar{K}N}^0$</td>
<td>-2.14</td>
<td>-1.74</td>
</tr>
<tr>
<td>$g_{\bar{K}N, \pi \Sigma}^0$</td>
<td>0.78</td>
<td>1.26</td>
</tr>
<tr>
<td>$g_{H}^0$</td>
<td>0.20</td>
<td>0.46</td>
</tr>
<tr>
<td>$g_{\pi \Sigma, B_0}^0$</td>
<td>-</td>
<td>0.11</td>
</tr>
<tr>
<td>$g_{\bar{K}N, B_0}^0$</td>
<td>-</td>
<td>0.15</td>
</tr>
<tr>
<td>$g_{\eta \Lambda, B_0}^0$</td>
<td>-</td>
<td>-0.17</td>
</tr>
<tr>
<td>$g_{K \Xi, B_0}^0$</td>
<td>-</td>
<td>-0.08</td>
</tr>
<tr>
<td>$m_{B}^0/\text{MeV}$</td>
<td>-</td>
<td>1714</td>
</tr>
</tbody>
</table>

$\chi^2/ (120 \text{ data})$ | $U\chi PT$ | 166 | 177 |

Pole 1 (MeV) | 1379 $- i 71$ | 1333 $- i 85$ | 1338 $- i 89$ |
Pole 2 (MeV) | 1412 $- i 20$ | 1428 $- i 23$ | 1430 $- i 22$ |
Finite Volume Λ Spectrum for \( L = 3 \) fm
Finite Volume Λ Spectrum for $L = 3$ fm
(a) State 1

(b) State 2

(c) State 3
Finite Volume Λ Spectrum for $L = 3 \text{ fm}$
Conclusions - $\Lambda(1405)$ Resonance

- Lattice QCD calculations have revealed there are
  - No low-lying three-quark dominated states in the $\Lambda(1405)$ Resonance mass region.
  - Three-quark dominated states are associated with the octet states.
Conclusions - $\Lambda(1405)$ Resonance

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- $\Lambda(1405)$ structure is dominated by a $\bar{K}N$ molecule. Signified by:
  - The vanishing of the strange quark contribution to the magnetic moment of the $\Lambda(1405)$, and
  - The dominance of the $\bar{K}N$ component in finite-volume EFT.
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  - No low-lying three-quark dominated states in the \( \Lambda(1405) \) Resonance mass region.
  - Three-quark dominated states are associated with the octet states.
- \( \Lambda(1405) \) structure is dominated by a \( \bar{K}N \) molecule. Signified by:
  - The vanishing of the strange quark contribution to the magnetic moment of the \( \Lambda(1405) \), and
  - The dominance of the \( \bar{K}N \) component in finite-volume EFT.
- The three-quark flavour-singlet \( \Lambda(1405) \) anticipated by the quark model exists only at quark masses approaching the strange quark mass.
Low-lying negative-parity $N^*$ Spectrum
Constrain model parameters to experimental data

- Consider $\pi N$ and $\eta N$ and bare state interactions.
- Fit to phase shift and inelasticity
- Fit yields a pole at $1531 \pm 29 - i 88 \pm 2$ MeV.
- Compare PDG estimate $1510 \pm 20 - i 85 \pm 40$ MeV.
Hamiltonian Model $N^*$ Spectrum: 2 fm
(a) State 1

(b) State 2

(c) State 3
Hamiltonian Model $N^*$ Spectrum: 2 fm
Hamiltonian Model $N^*$ Spectrum: 3 fm
(a) State 1  (b) State 2  (c) State 4
Hamiltonian Model $N^*$ Spectrum: 3 fm
Conclusions - Odd-Parity Nucleon Resonances

- Lattice QCD calculations have revealed there are
  - Two low-lying states in the $N(1535)$ and $N(1650)$ resonance mass regions with large bare three-quark basis components.
  - These components compose $\sim 50\%$ of the states at the physical quark masses.
Conclusions - Odd-Parity Nucleon Resonances

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- $\eta N$ contributions dominate the meson-baryon dressings of these states.
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- $\eta N$ contributions dominate the meson-baryon dressings of these states.
- Strong single-particle components make these states ideal for lattice studies of the form factors and transition moments.
- The lattice scattering state energy calculated by Lang and Verduci is described accurately by Hamiltonian effective field theory constrained to experiment.
  - Two-to-two particle meson-baryon interactions are essential to describing the lattice results.
What about the Roper? Lattice results at $L \approx 3 \text{ fm}$

Filled Symbols: CSSM
Open Symbols: Cyprus Collaboration
What about the Roper? Lattice results at $L \approx 3$ fm

See Adrian Kiritidis’ talk, Tuesday R6 2:40

Filled Symbols: CSSM
Open Symbols: Cyprus Collaboration
Athens Model Independent Analysis Scheme (AMIAS)

- “Novel analysis method for excited states in lattice QCD: The nucleon case,”
  C. Alexandrou, T. Leontiou, C. N. Papanicolas and E. Stiliaris,
  Phys. Rev. D 91 (2015) 1, 014506
  arXiv:1411.6765 [hep-lat].
Bare Roper Case: $m_0 = 2.03$ GeV

- Consider $\pi N$, $\pi\Delta$ and $\sigma N$ channels, dressing a bare state.
- Fit to phase shift and inelasticity
- Fit yields a pole at $1380 - i 87$ MeV.
- Compare PDG estimate $1365 \pm 15 - i 95 \pm 15$ MeV.
Bare Roper: Hamiltonian Model $N'$ Spectrum
(a) State 2  
(b) State 3  
(c) State 4
Bare Roper: Hamiltonian Model $N'$ Spectrum
Bare Nucleon Case: $m_0 = 1.17$ GeV

- Consider $\pi N$, $\pi\Delta$ and $\sigma N$ channels, dressing a bare state.
- Fit to phase shift and inelasticity.
- Fit yields a pole at $1357 - i 36$ MeV.
- Compare PDG estimate $1365 \pm 15 - i 95 \pm 15$ MeV.
Bare Nucleon: Hamiltonian Model $N'$ Spectrum
(a) State 1  
(b) State 6  
(c) State 7
Bare Nucleon: Hamiltonian Model $N'$ Spectrum
Conclusions - Roper Resonance

- Lattice QCD calculations have revealed there are
  - No low-lying three-quark dominated states in the Roper Resonance mass region.
  - No localised meson-baryon states at light quark masses.
Conclusions - Roper Resonance

• Lattice QCD calculations have revealed there are
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  ○ No localised meson-baryon states at light quark masses.

• Roper of the Constituent Quark Model has been seen on the lattice.
  ○ Node structure and density is similar to model expectations.
Conclusions - Roper Resonance

- Lattice QCD calculations have revealed there are
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- Roper of the Constituent Quark Model has been seen on the lattice.
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- Hamiltonian Effective field theory explains the states observed in Lattice QCD as
  - Having a small three-quark component associated with a light bare-nucleon basis state.
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- Like the $\Lambda(1405)$, the Roper resonance is dominated by meson-baryon degrees of freedom.
Conclusions - Roper Resonance

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  - Node structure and density is similar to model expectations.
- Hamiltonian Effective field theory explains the states observed in Lattice QCD as
  - Having a small three-quark component associated with a light bare-nucleon basis state.
- Like the $\Lambda(1405)$, the Roper resonance is dominated by meson-baryon degrees of freedom.
- Conclude that the Roper Resonance is dynamically generated
  - dominated by the direct two-to-two particle meson-baryon interactions.
Artistic view of $\Lambda(1405)$ Structure

or the neutral Roper upon $s \rightarrow d$. 
Supplementary Information

The following slides provide additional information which may be of interest.
Variational Analysis

- Consider a basis of interpolating fields $\chi_i$
Variational Analysis

- Consider a basis of interpolating fields $\chi_i$
- Construct the correlation matrix

\[
G_{ij}(p; t) = \sum_x e^{-i p \cdot x} \text{tr} \left( \Gamma \left\langle \Omega | \chi_i(x) \chi_j(0) \right| \Omega \right) \\
= \sum_\alpha A_i^\alpha A_j^{\dagger \alpha} \exp (-E_\alpha(p) t) .
\]
Variational Analysis

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- Construct the correlation matrix

$$G_{ij}(p; t) = \sum_x e^{-i p \cdot x} \text{tr} \left( \Gamma \langle \Omega | \chi_i(x) \chi_j(0) | \Omega \rangle \right)$$

$$= \sum_\alpha A_i^\alpha A_j^{\dagger \alpha} \exp \left( -E_\alpha(p) t \right).$$

- Seek linear combinations of the interpolators $\{ \chi_i \}$ that isolate individual energy eigenstates, $\alpha$, at momentum $p$:

$$\phi^\alpha = v_i^{\alpha}(p) \chi_i, \quad \bar{\phi}^\alpha = u_i^{\alpha}(p) \bar{\chi}_i.$$
Variational Analysis

• When successful, only state $\alpha$ participates in the correlation function, and one can write recurrence relations

$$G(p; t_0 + \delta t) u^\alpha(p) = e^{-E_\alpha(p) \delta t} G(p; t_0) u^\alpha(p)$$

$$v^{\alpha^T}(p) G(p; t_0 + \delta t) = e^{-E_\alpha(p) \delta t} v^{\alpha^T}(p) G(p; t_0)$$

a Generalised Eigenvalue Problem (GEVP).
Variational Analysis

- When successful, only state $\alpha$ participates in the correlation function, and one can write recurrence relations

\[
G(p; t_0 + \delta t) \ u^\alpha(p) = e^{-E_\alpha(p) \delta t} \ G(p; t_0) \ u^\alpha(p)
\]

\[
\mathbf{v}^\alpha^T(p) \ G(p; t_0 + \delta t) = e^{-E_\alpha(p) \delta t} \ \mathbf{v}^\alpha^T(p) \ G(p; t_0)
\]

a Generalised Eigenvalue Problem (GEVP).

- Solve for the left, $\mathbf{v}^\alpha(p)$, and right, $\mathbf{u}^\alpha(p)$, generalised eigenvectors of $G(p; t_0 + \delta t)$ and $G(p; t_0)$.
Using these optimal eigenvectors, create eigenstate-projected correlation functions

\[ G_\alpha(p; t) = \sum_x e^{-ip \cdot x} \langle \Omega | \phi_\alpha(x) \bar{\varphi}_\alpha(0) | \Omega \rangle , \]

\[ = \sum_x e^{-ip \cdot x} \langle \Omega | v_\alpha^T(p) \chi_i(x) \bar{\chi}_j(0) u_j^\alpha(p) | \Omega \rangle , \]

\[ = v_\alpha^T(p) G(p; t) u_\alpha^\alpha(p) . \]

\[ G_\alpha(p; t) = A_\alpha \exp(-E_\alpha(p)t) . \]
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\[ = v^\alpha_T(p) G(p; t) u^\alpha(p). \]

\[ G^\alpha(p; t) = A_\alpha \exp(-E_\alpha(p) t). \]

Here \( t \) is different from \( t_0 \) and \( \delta t \) and can become large.
Smeared Source to Point Sink Correlation Functions
States Tracked via Orthogonal Eigenvectors
Positive Parity Nucleon Spectrum: CSSM
Further Information

• “Roper Resonance in 2+1 Flavor QCD,”
  M. S. Mahbub, et al. [CSSM],
  arXiv:1011.5724 [hep-lat],

• “Low-lying Odd-parity States of the Nucleon in Lattice QCD,”
  M. Selim Mahbub, et al. [CSSM],
  Phys. Rev. D Rapid Comm. 87 (2013) 011501,
  arXiv:1209.0240 [hep-lat]

• “Structure and Flow of the Nucleon Eigenstates in Lattice QCD,”
  M. S. Mahbub, et al. [CSSM],
  Phys. Rev. D 87 (2013) 9, 094506
  arXiv:1302.2987 [hep-lat].
Sequential Empirical Bayesian (SEB) Analysis: χQCD Collaboration

\[ \chi \text{QCD Collaboration} \]

\[ m^2 (\text{GeV}^2) \]

\[ m_N (\text{GeV}) \]

\[ J^P = \frac{1}{2}^+ \]

\[ \text{Experiment} \quad \text{BGR} \quad \text{CSSM} \quad \text{JLAB} \quad \text{Twisted Mass} \quad \text{Clover} \quad \chi \text{QCD} \]
χQCD & HSC Systematic Comparison - Same Correlators Examined

![Graph showing data points and lines representing different correlators and models.]

Note: $28 \times 28 = 784$ correlators versus 1.


Twisted Mass (this work)
Clover (this work)
CSSM
JLAB
BGR
Experiment

$J^P = \frac{1}{2}^+$

$m^2_N$ [GeV]

$m_N$ [GeV]
Positive Parity Spectrum: Cyprus (Twisted Mass) Collaboration: Jan. '14

The diagram shows a scatter plot with two variables: $m^2_{\pi}$ [GeV] on the x-axis and $m$ [GeV] on the y-axis. The plot includes data points from different collaborations:

- Twisted Mass (this work)
- Clover (this work)
- JLAB
- BGR
- Experiment

The plot also includes a note for $J^P = \frac{1}{2}^+$.
$d$-quark probability density in ground state proton: $m_\pi = 156$ MeV (CSSM)
$d$-quark probability density in first excited proton: $m_π = 156$ MeV (CSSM)
Positive Parity Nucleon Spectrum: only small smearing: Cyprus
Positive Parity Nucleon Spectrum: $r_{RMS}$ smearing of 8.6 lu: Cyprus
**Athens Model Independent Analysis Scheme (AMIAS)**

- “Novel analysis method for excited states in lattice QCD: The nucleon case,”
  C. Alexandrou, T. Leontiou, C. N. Papanicolas and E. Stiliaris,
  Phys. Rev. D **91** (2015) 1, 014506
  arXiv:1411.6765 [hep-lat].
Athens Model Independent Analysis Scheme (AMIAS)

- Does not rely on plateau identification of effective masses
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- Exploits small time separations where the excited states contribute and statistical errors are small.
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- Exploits small time separations where the excited states contribute and statistical errors are small.
- The Correlation matrix has the spectral decomposition

\[ G_{ij}(t) = \sum_{\alpha=0}^{N_{\text{states}}} A_i^\alpha A_j^{\dagger \alpha} e^{-E_\alpha t}. \quad i, j = 1, \ldots, N_{\text{interpolators}}. \]
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  - A parallel tempering algorithm is used to avoid local minima traps.
- Parameters are determined by fitting a Gaussian to their probability distributions.
- Increase \( N_{\text{states}} \) until there is no sensitivity to additional exponentials.
Determining $N_{\text{states}} \equiv n_{\text{max}}$ (Cyprus)

$n_{\text{max}} = 2$

$n_{\text{max}} = 3$

$n_{\text{max}} = 4$

$n_{\text{max}} = 5$

E (GeV)

Probability

$E_0$, $E$, $E_2$, $E_3$, $E_4$
Analysis of Correlation Matrix is Essential

\[
\begin{align*}
E_0 & \quad E_1 \\
C_{11}^{(1,1)} & \quad C_{11}^{(i,j)} \quad i,j=1,2,3 \\
E_0 & \quad E_1 \quad E_2 \\
C_{11}^{(5,5)} & \quad C_{11}^{(i,j)} \quad i,j=1,2,4 \\
E_0 & \quad E_1 \\
C_{11}^{(i,j)} \quad i,j=1,\ldots,5
\end{align*}
\]
Dispersion Relation Test for the $\Lambda(1405)$

\[
E \left[ \text{GeV}/c^2 \right] = \sqrt{m^2 + p^2}
\]

where $m$ and $p$ are the mass and momentum, respectively.
$G_E$ for the $\Lambda(1405)$

When compared to the ground state, the results for $G_E$ are consistent with the development of a non-trivial $\bar{K}N$ component at light quark masses.
$G_E$ for the $\Lambda(1405)$

When compared to the ground state, the results for $G_E$ are consistent with the development of a non-trivial $\overline{K}N$ component at light quark masses.

- Noting that the centre of mass of the $\overline{K}(s, \bar{\ell})\ N(\ell, u, d)$ is nearer the heavier $N$,
  - The anti–light-quark contribution, $\bar{\ell}$, is distributed further out by the $\overline{K}$ and leaves an enhanced light-quark form factor.
$\mathcal{G}_E$ for the $\Lambda(1405)$
$G_E$ for the $\Lambda(1405)$

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- Noting that the centre of mass of the $\overline{K}(s, \ell)\ N(\ell, u, d)$ is nearer the heavier $N$,
  - The anti–light-quark contribution, $\overline{\ell}$, is distributed further out by the $\overline{K}$ and leaves an enhanced light-quark form factor.
  - The strange quark may be distributed further out by the $\overline{K}$ and thus have a smaller form factor.
$G_E$ for the $\Lambda(1405)$

![Graph showing $G_E$ vs $m^2_\pi$ for different values of $\Lambda$ and $\Lambda(1405)$](image)
Excited State Form Factors

The eigenstate-projected three-point correlation function is

\[
G_{\alpha}(p', p; t_2, t_1) = \sum_{x_1, x_2} e^{-i p' \cdot x_2} e^{i(p' - p) \cdot x_1} \times
\]

\[
\langle \Omega | v_{i}(p') \chi_{i}(x_2) j^{\mu}(x_1) \bar{\chi}_{j}(0) u_{i}(p) | \Omega \rangle
\]

\[
= v_{\alpha}^{T}(p') G_{ij}^{\mu}(p', p; t_2, t_1) u_{\alpha}(p)
\]

where

\[
G_{ij}^{\mu}(p', p; t_2, t_1) = \sum_{x_1, x_2} e^{-i p' \cdot x_2} e^{i(p' - p) \cdot x_1} \langle \Omega | \chi_{i}(x_2) j^{\mu}(x_1) \bar{\chi}_{j}(0) | \Omega \rangle
\]

is the matrix constructed from the three-point correlation functions of the original operators \{ \chi_i \}. 
To eliminate the time dependence of the three-point correlation function, we construct the ratio

\[ R_\alpha^\mu(p', p; t_2, t_1) = \left( \frac{G_\alpha^\mu(p', p; t_2, t_1) G_\alpha^\mu(p, p'; t_2, t_1)}{G_\alpha(p'; t_2) G_\alpha(p; t_2)} \right)^{1/2} \]
To eliminate the time dependence of the three-point correlation function, we construct the ratio

$$R^\mu_\alpha(p', p; t_2, t_1) = \left( \frac{G^\mu_\alpha(p', p; t_2, t_1) G^\mu_\alpha(p, p'; t_2, t_1)}{G_\alpha(p'; t_2) G_\alpha(p; t_2)} \right)^{1/2}$$

To further simplify things, we define the reduced ratio

$$\overline{R}^\mu_\alpha = \left( \frac{2E_\alpha(p)}{E_\alpha(p) + m_\alpha} \right)^{1/2} \left( \frac{2E_\alpha(p')}{E_\alpha(p') + m_\alpha} \right)^{1/2} R^\mu_\alpha$$
Current Matrix Element for Spin-1/2 Baryons

The current matrix element for spin-1/2 baryons has the form

\[
\langle p', s' | j^\mu | p, s \rangle = \left( \frac{m_\alpha^2}{E_\alpha(p) E_\alpha(p')} \right)^{1/2} \times \\
\times \bar{u}(p') \left( F_1(q^2) \gamma^\mu + i F_2(q^2) \sigma^{\mu \nu} \frac{q^\nu}{2m_\alpha} \right) u(p)
\]
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\]

- The Dirac and Pauli form factors are related to the Sachs form factors through

\[
G_E(q^2) = F_1(q^2) - \frac{q^2}{(2m_\alpha)^2} F_2(q^2)
\]

\[
G_M(q^2) = F_1(q^2) + F_2(q^2)
\]
Sachs Form Factors for Spin-1/2 Baryons

- A suitable choice of momentum \((q = (q, 0, 0))\) and the (implicit) Dirac matrices allows us to directly access the Sachs form factors:
  - for \(G_E\): using \(\Gamma_4^\pm\) for both two- and three-point,
    \[
    G_E^\alpha(q^2) = \bar{R}_\alpha^4(q, 0; t_2, t_1)
    \]
  - for \(G_M\): using \(\Gamma_4^\pm\) for two-point and \(\Gamma_j^\pm\) for three-point,
    \[
    |\varepsilon_{ijk} q^i| G_M^\alpha(q^2) = (E_\alpha(q) + m_\alpha) \bar{R}_\alpha^k(q, 0; t_2, t_1)
    \]
  - where for positive parity states,
    \[
    \Gamma_j^+ = \frac{1}{2} \begin{bmatrix} \sigma_j & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_4^+ = \frac{1}{2} \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}
    \]
    and for negative parity states,
    \[
    \Gamma_j^- = -\gamma_5 \Gamma_j^+ \gamma_5 = -\frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & \sigma_j \end{bmatrix}, \quad \Gamma_4^- = -\gamma_5 \Gamma_4^+ \gamma_5 = -\frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{bmatrix}
    \]
Finite Volume Dependence of the Λ Spectrum