# Inelastic scattering parameters from finite volume eigenstates

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# Outline

Resolving scattering in lattice QCD is not straightforward

Lattice QCD can determine (finite volume) energy eigenstates

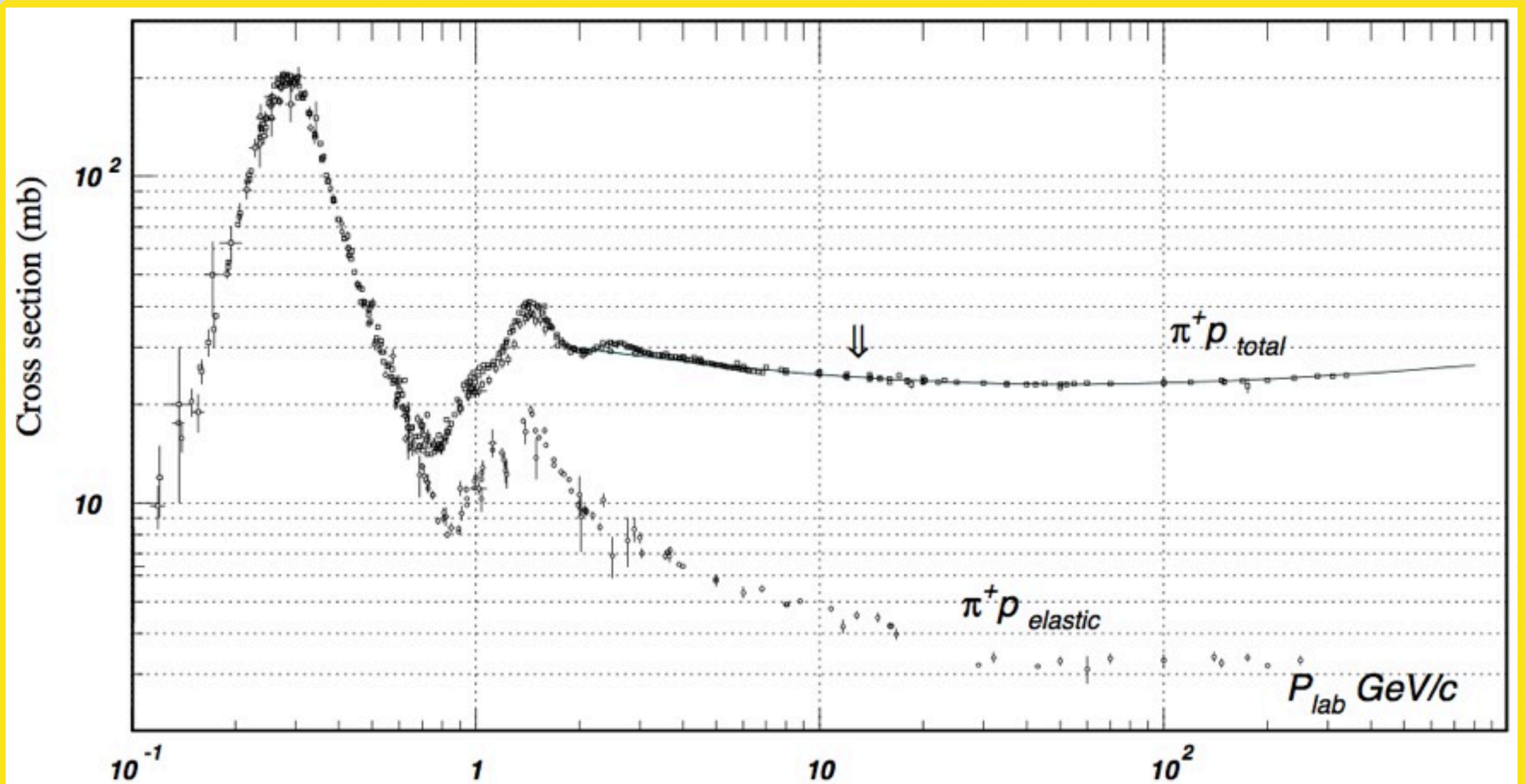
**Problem**:

Energy eigenstates → scattering parameters?

Lüscher method

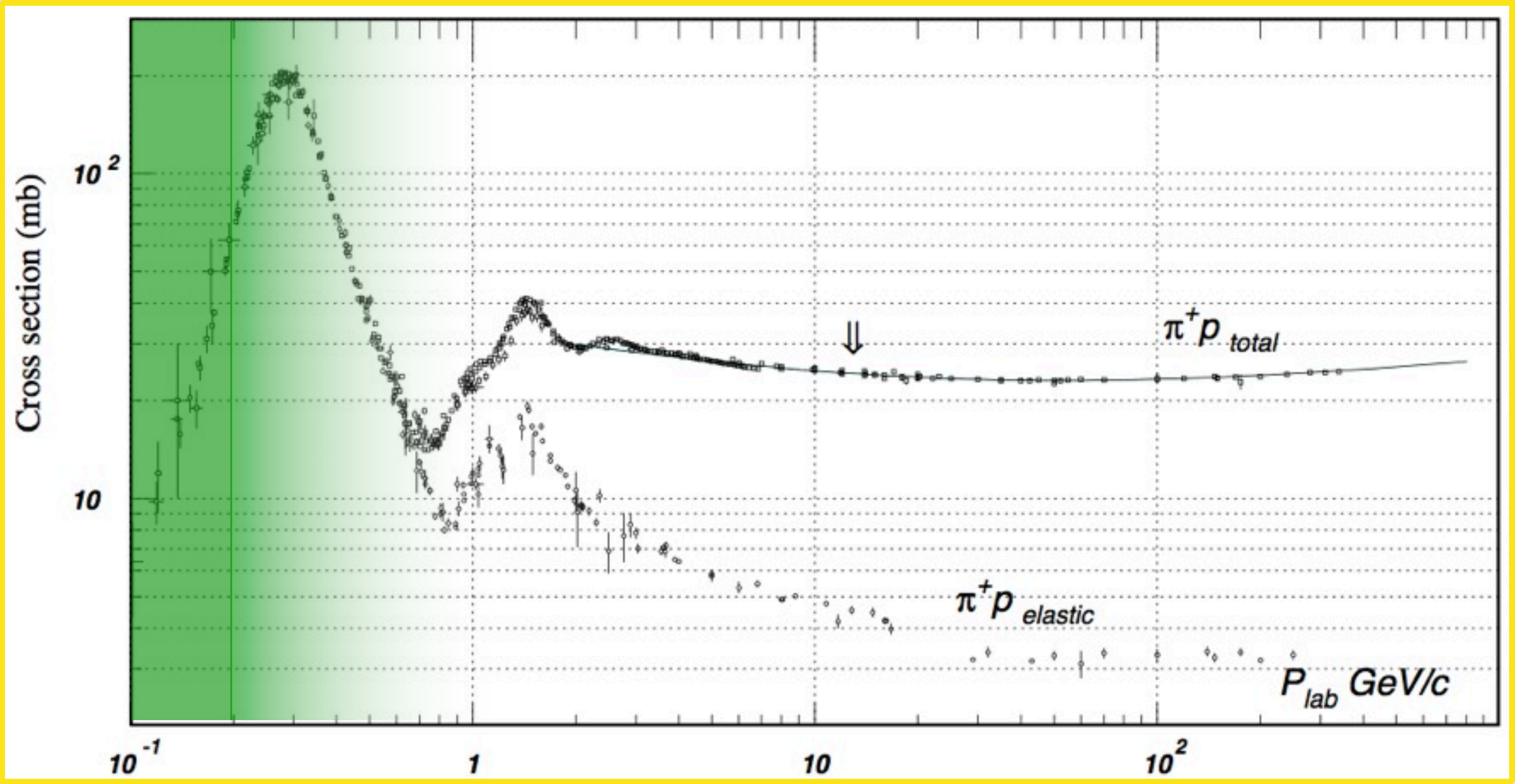
Hamiltonian approach

Multi-channel systems $(\pi\pi \leftrightarrow K\overline{K})$




# QCD has a rich spectrum

EFT

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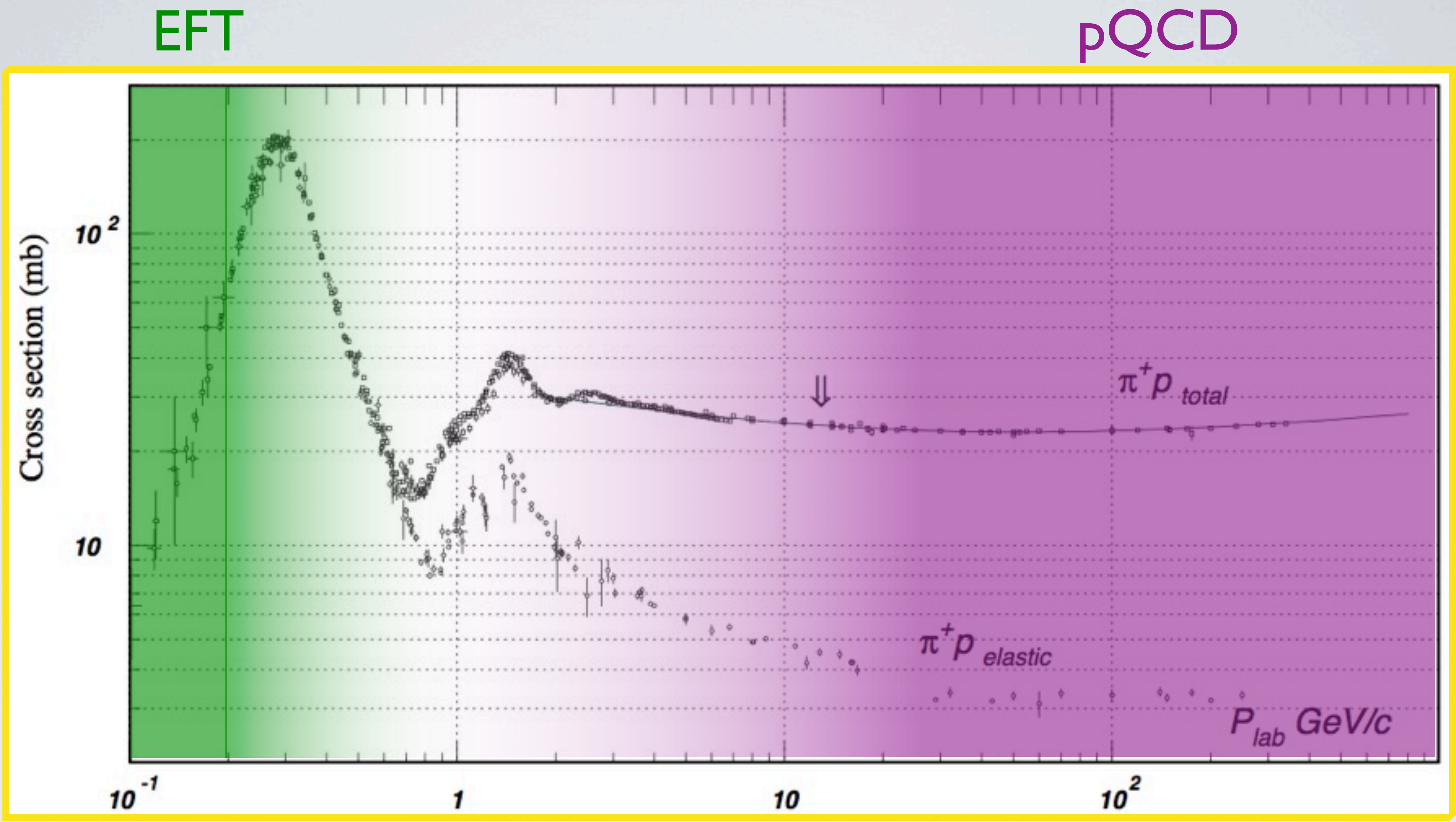


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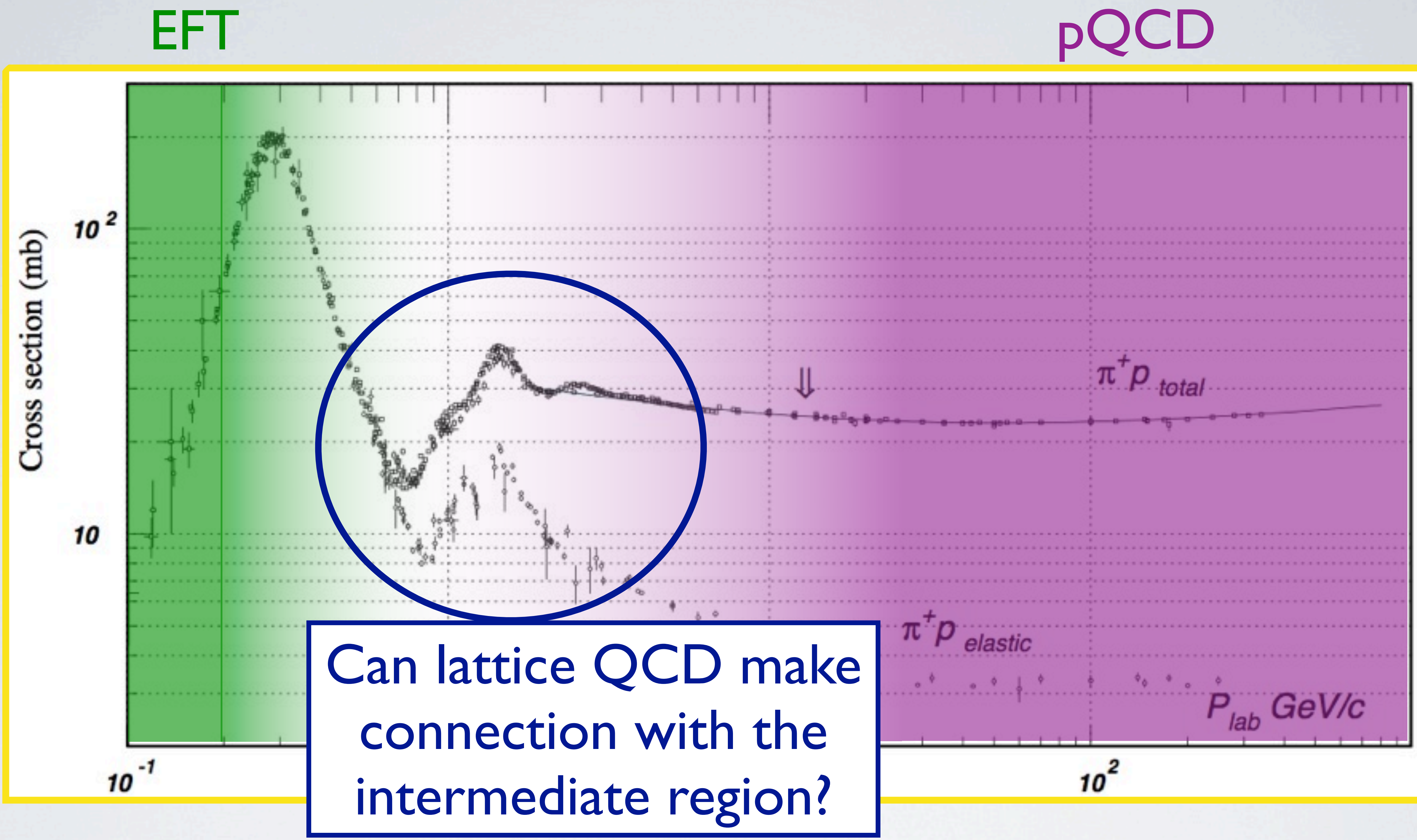
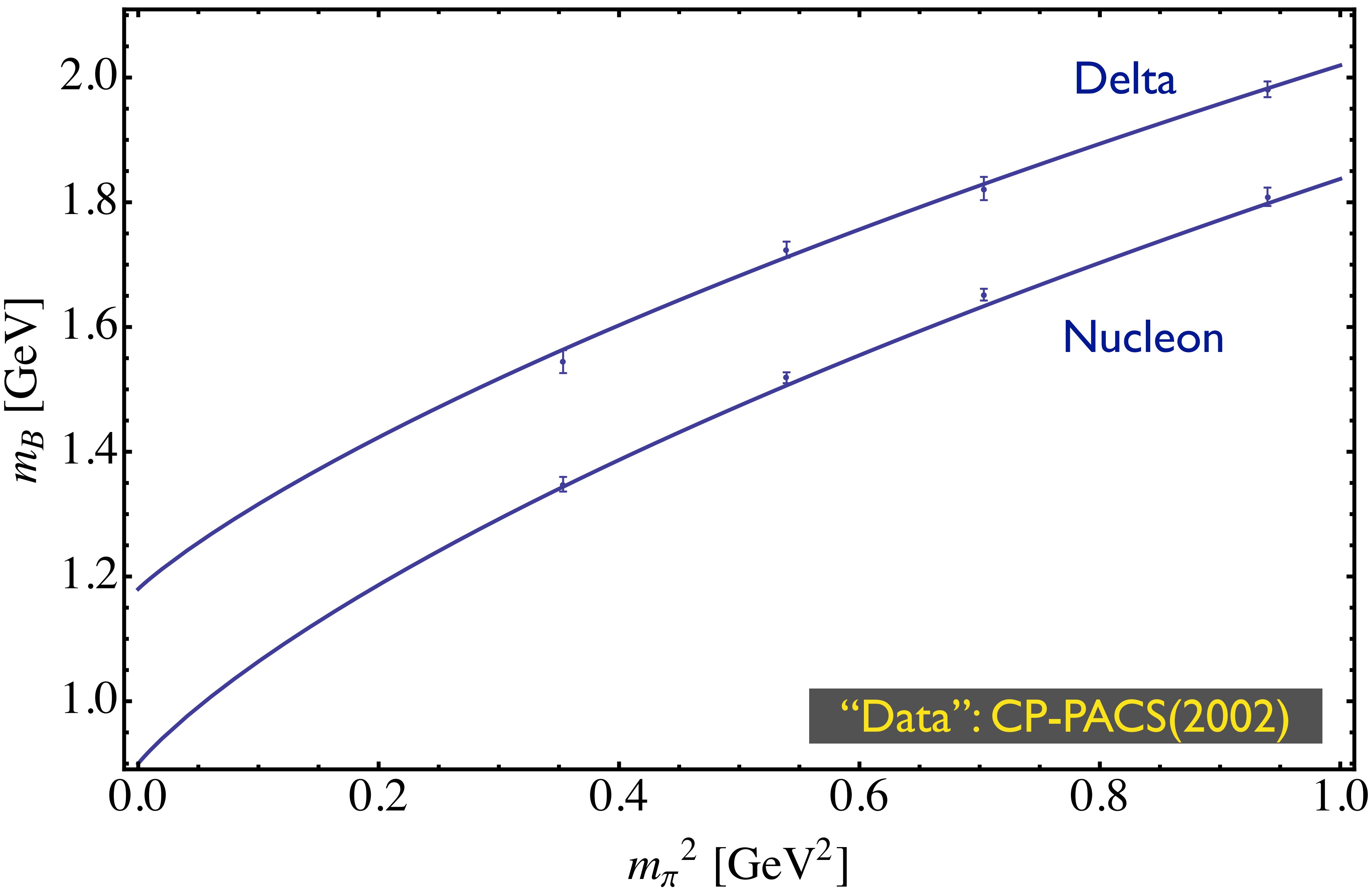



# QCD has a rich spectrum

# Spectrum in lattice QCD

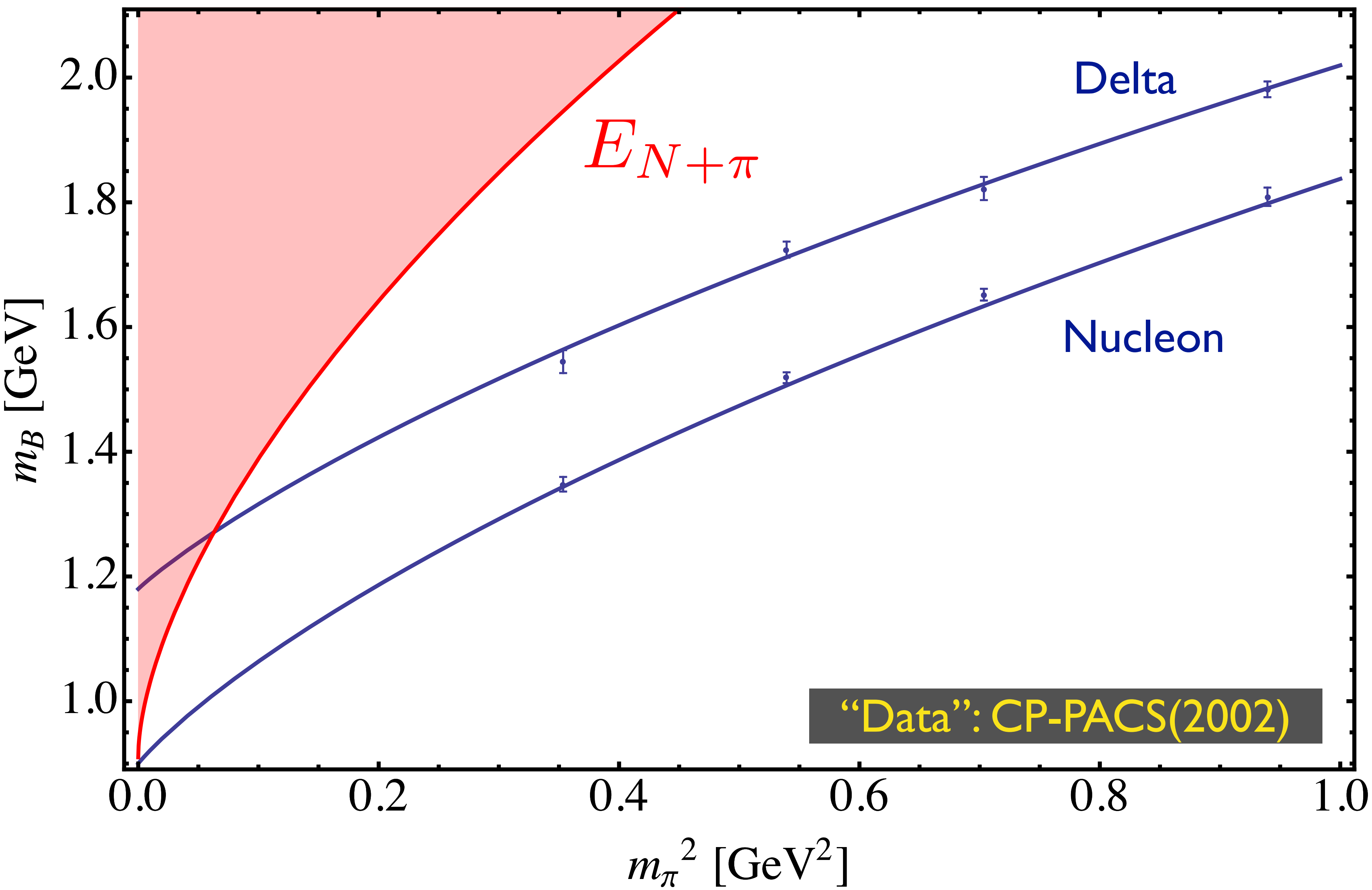



# In the good old days

Lattice QCD was forced to use large quark masses

Low-lying spectrum comprised of stable particles

[Curves are just a sketch: *not* a fit]

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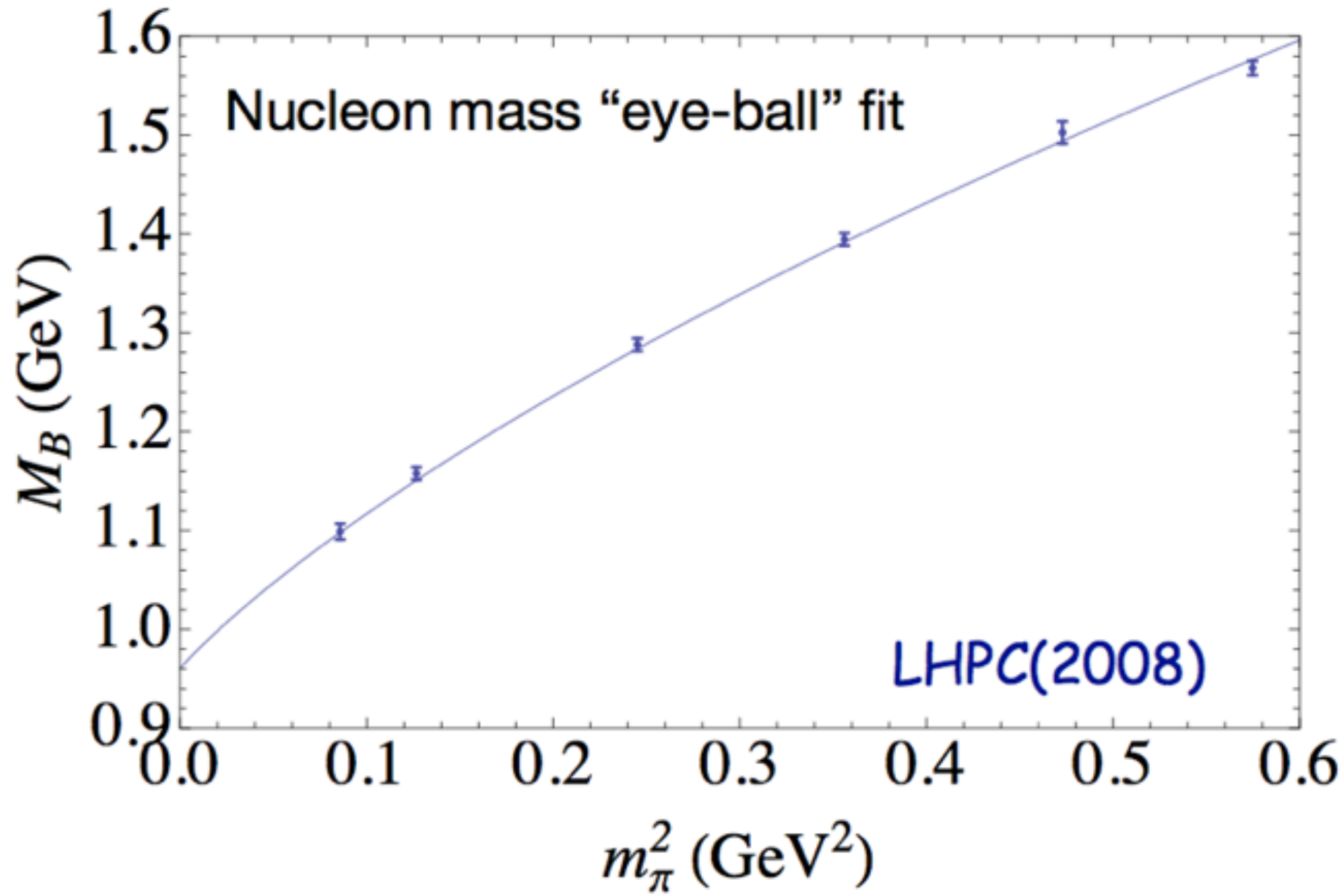
# Baryon mass: "Sketching a curve"

$$M_B = M_B^0 \left( \frac{1}{1 + \frac{3}{2}\frac{m_\pi}{M_B^0}} + \frac{3}{2}\frac{m_\pi}{M_B^0} \right)$$



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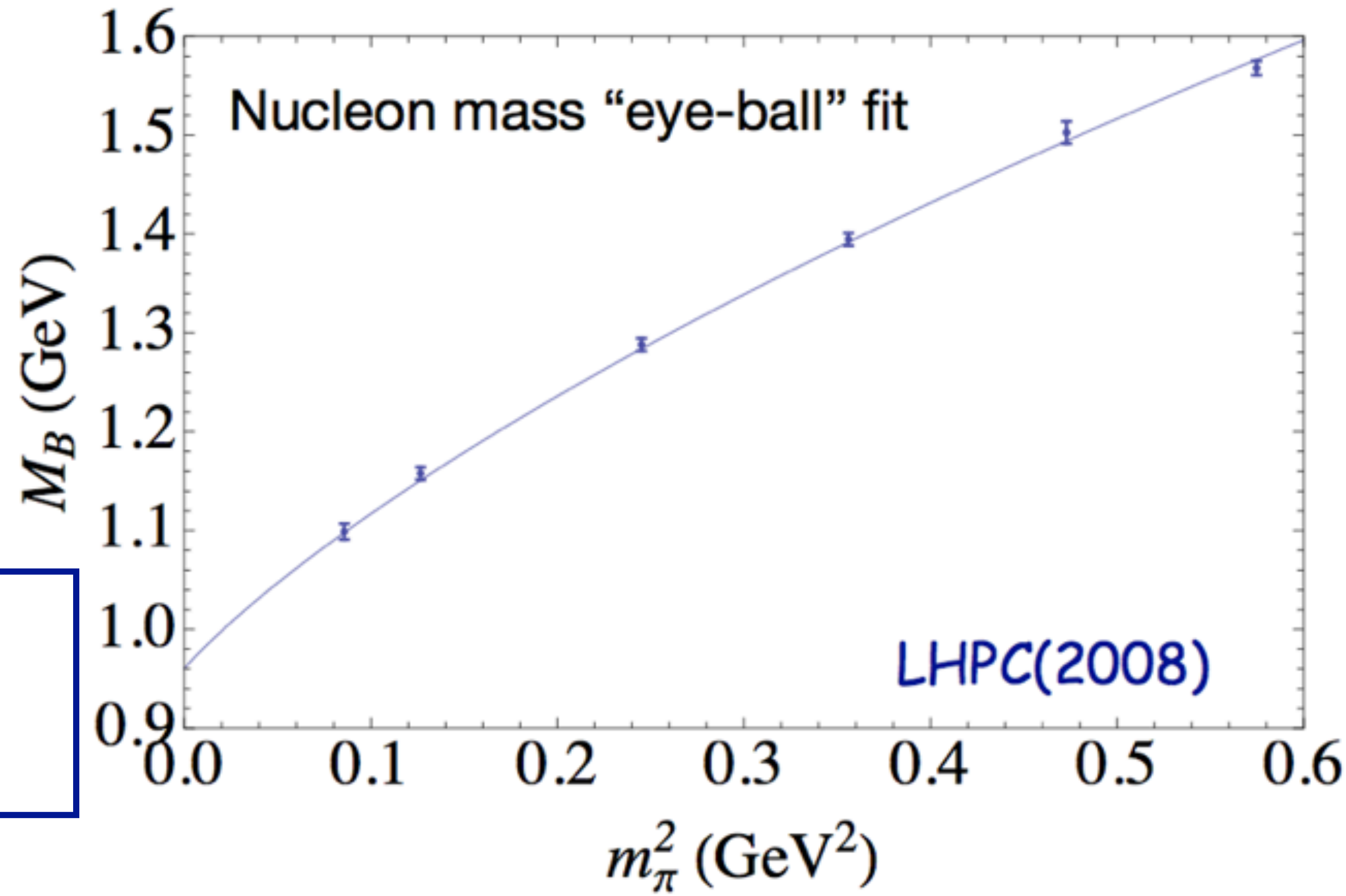
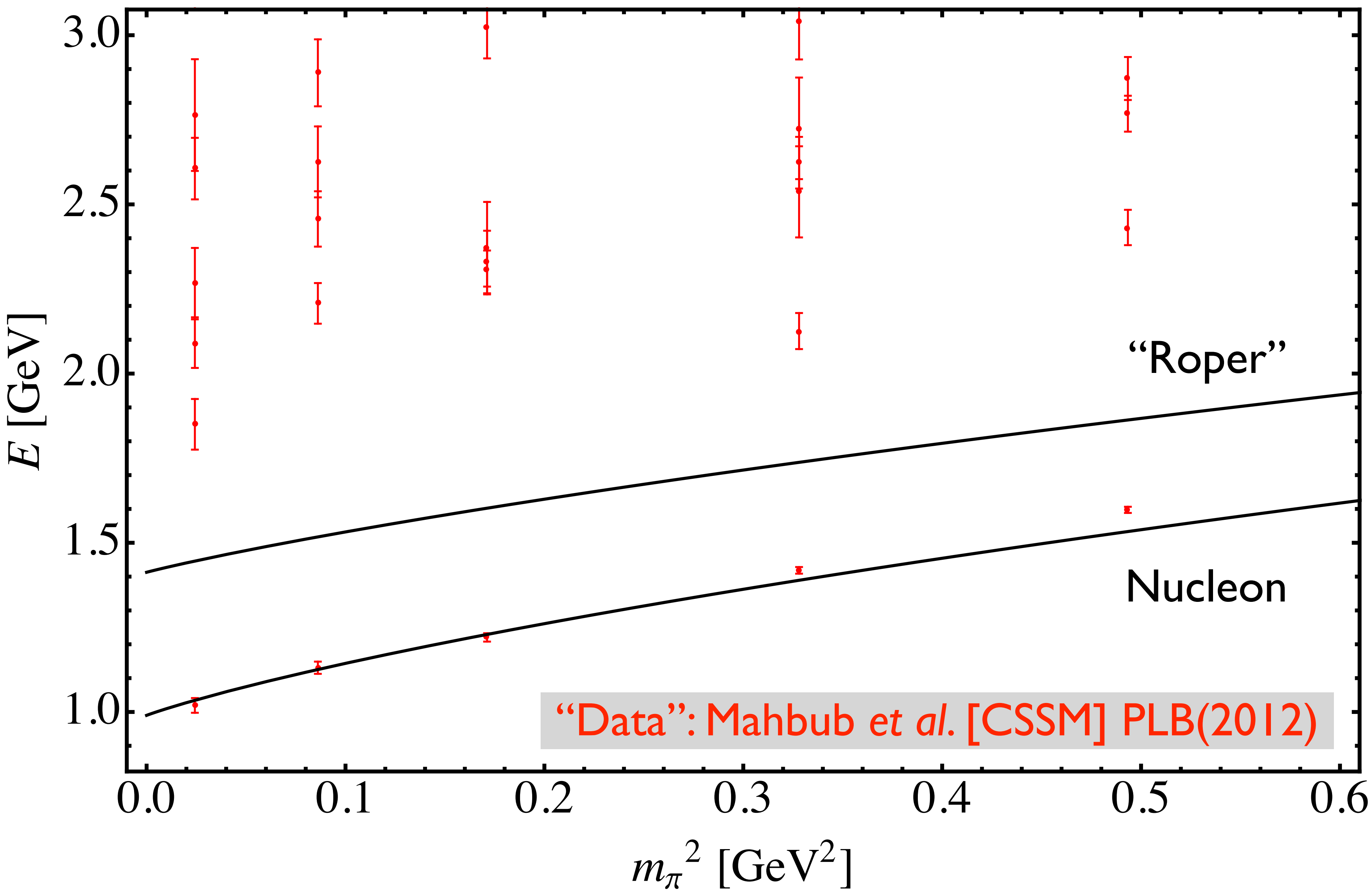
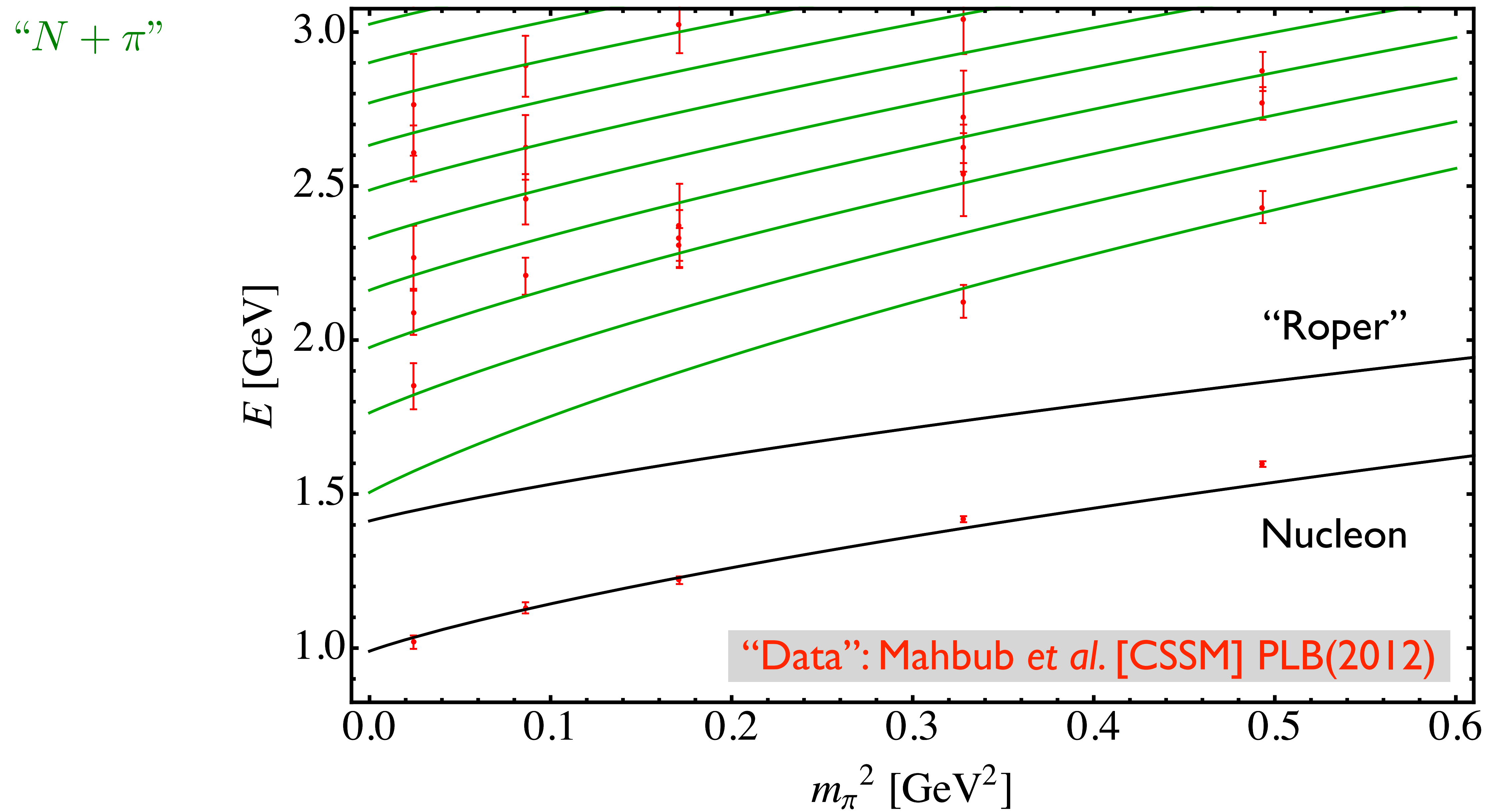



One parameter = magnitude, slope & curvature!




# State-of-the-art LQCD spectrum

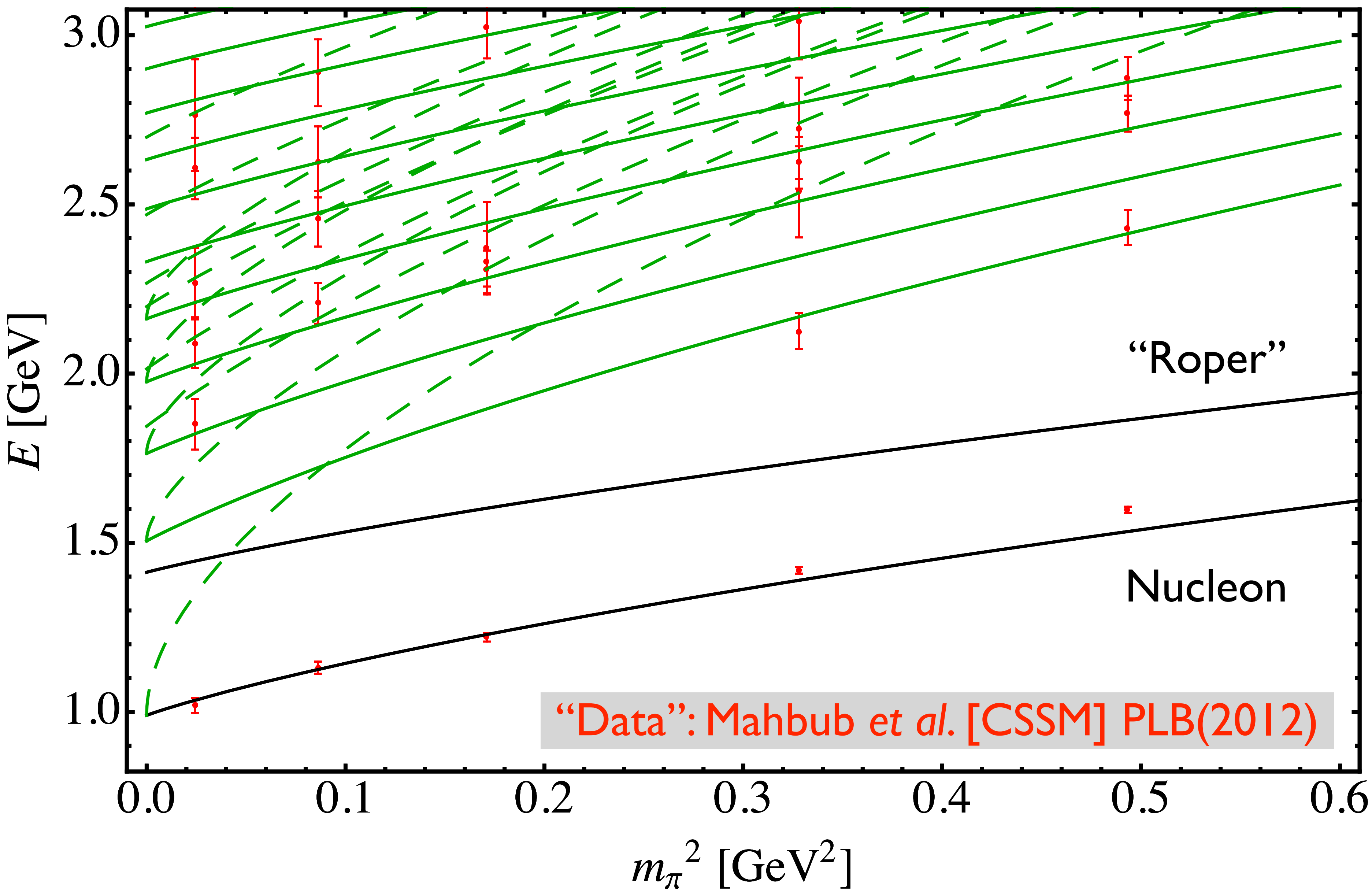
*1/2*$^+$: "Free-particle" energies on a 2.9 fm box
[Interactions will shift energy levels]




# State-of-the-art LQCD spectrum

*1/2*$^+$: "Free-particle" energies on a 2.9 fm box
[Interactions will shift energy levels]

“$N + \pi$”
“$N + \pi\pi$”




# State-of-the-art LQCD spectrum

*1/2*⁺: “Free-particle” energies on a 2.9 fm box
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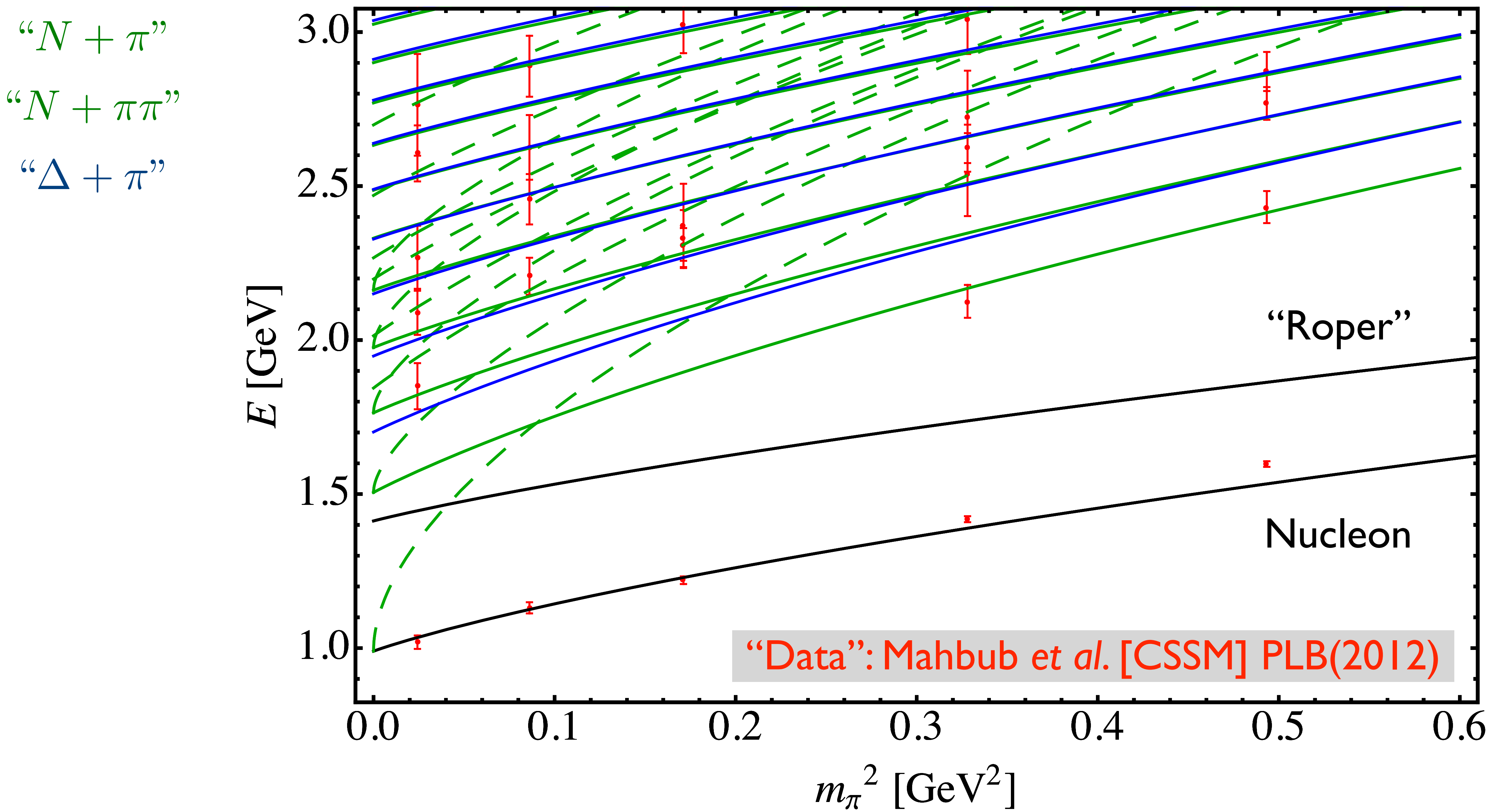


# State-of-the-art LQCD spectrum

*1/2*+: "Free-particle" energies on a 2.9 fm box
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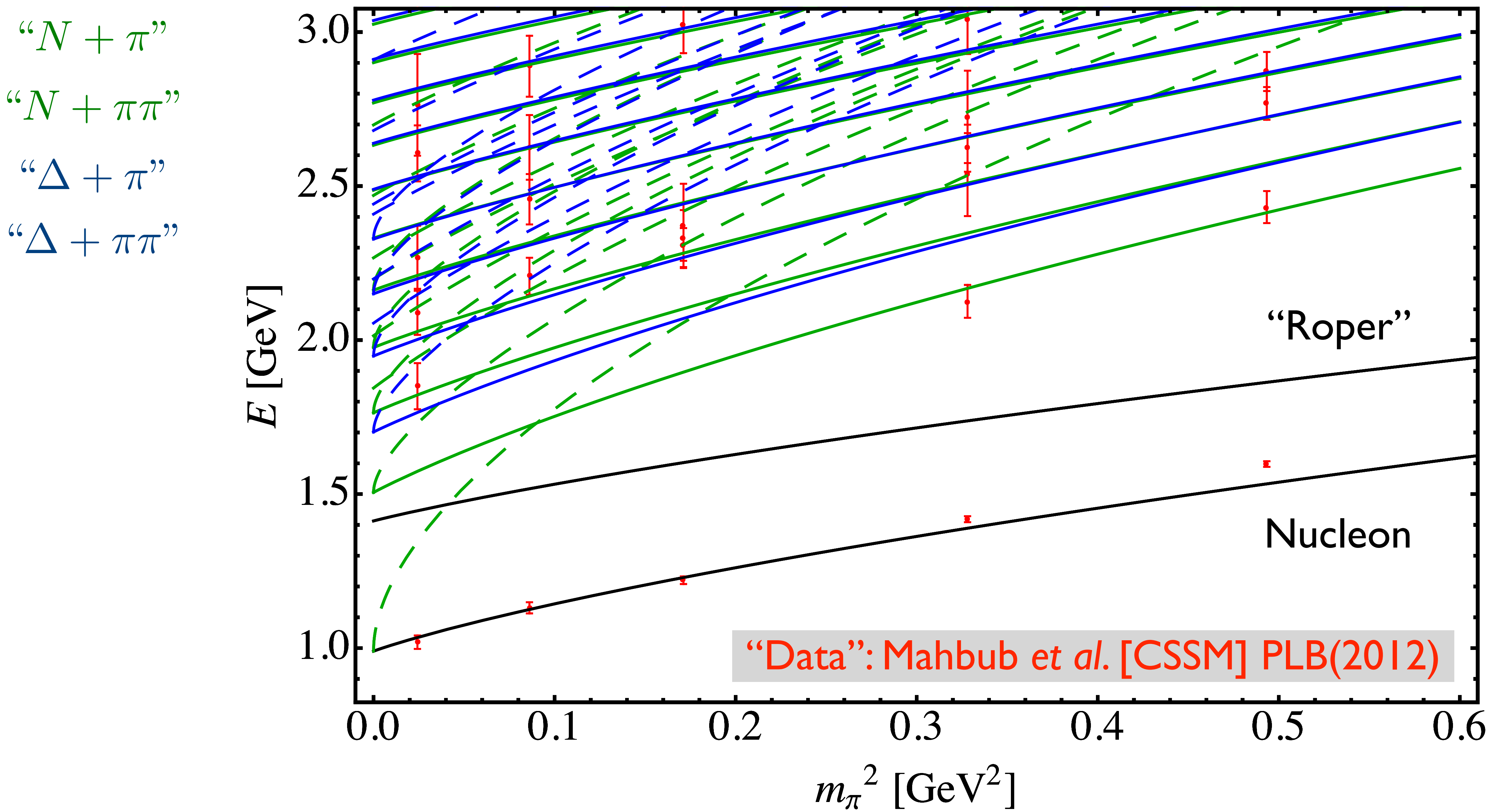



# State-of-the-art LQCD spectrum

*1/2*+: "Free-particle" energies on a 2.9 fm box
[Interactions will shift energy levels]

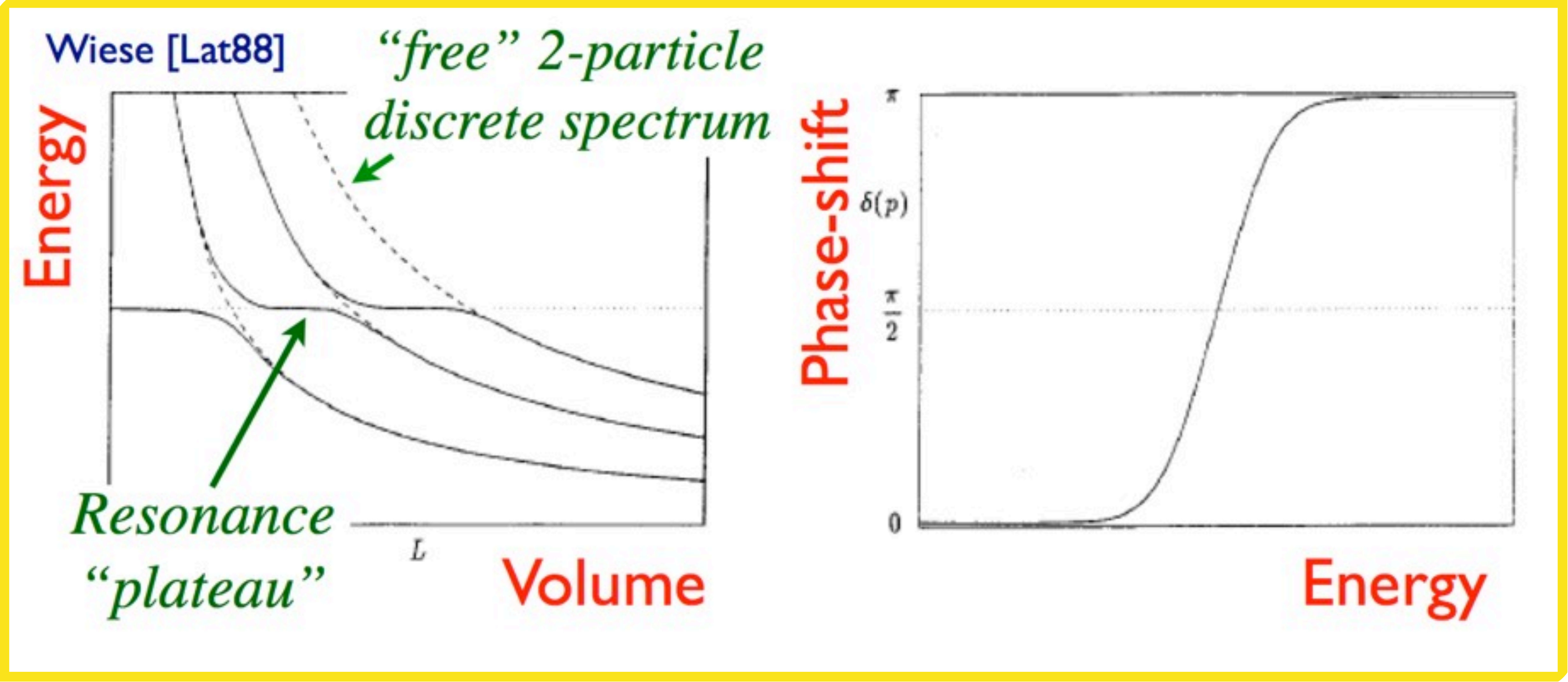
Resonance spectrum is no longer separated from multiparticle thresholds

[where widths are greater than level spacing]

# Resonance parameters?

Lüscher: 2-body elastic "scattering", one can map finite-volume energies directly to phase shifts

Map out volume dependence of energy levels



# Scattering parameters?

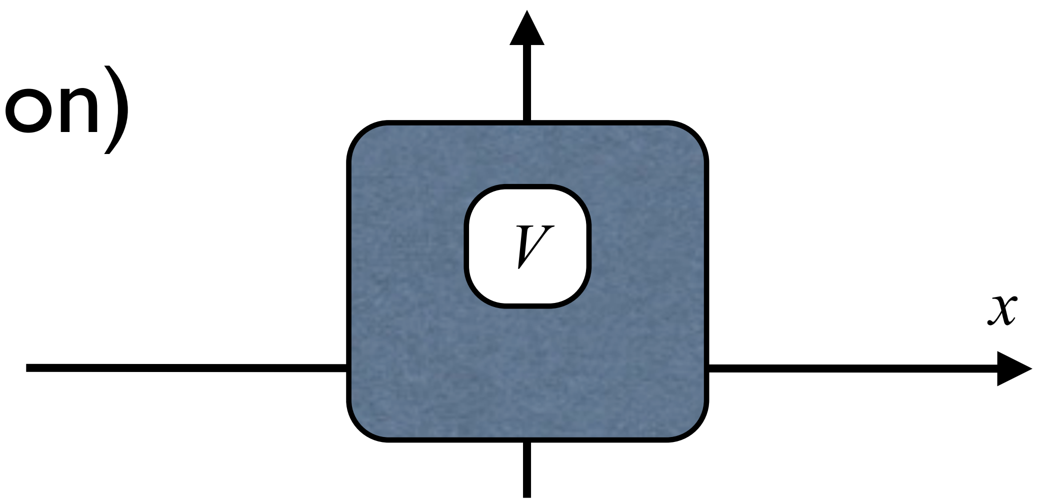
Finite volume spectra in lattice QCD encode a quantum mechanical admixture of scattering states and resonance-like structures

How can we use finite-volume spectra to identify scattering parameters (eg. *S*-matrix)?

# Lüscher: Qualitative (1-D QM)

1-D Scattering of 2 identical bosons; finite-range potential

General solution (non-interacting region)




Right $\psi^{(r)}(x) = Ae^{-ikx} + Be^{ikx}$

[Left by Bose symmetry]

Steady-state: conservation of probability $\Rightarrow |B| = |A|$

Define $\dfrac{B}{A} = e^{i2\delta} \quad \Rightarrow \psi^{(r)}(x) = A\left(e^{-ikx} + e^{i2\delta(k)}e^{ikx}\right)$

For any potential, scattering process is entirely determined by the phase shift $\delta(k)$

# Lüscher: Qualitative (1-D QM)

## Periodic boundary conditions

"Box" length $L$ $\quad \psi(x+L) = \psi(x)$

## Boundary $\quad x = \pm L/2$

Continuous: Bose symmetry $\quad \Rightarrow \; \psi^{(r)}(L/2) = \psi^{(l)}(-L/2)$

Smooth $\quad -ik\left(Ae^{ikL/2} - Be^{-iKL/2}\right) = ik\left(Ae^{ikL/2} - Be^{-iKL/2}\right),$

$$\Rightarrow \quad \frac{B}{A} = e^{-ikL}\,.$$

## Potential defines phase shift

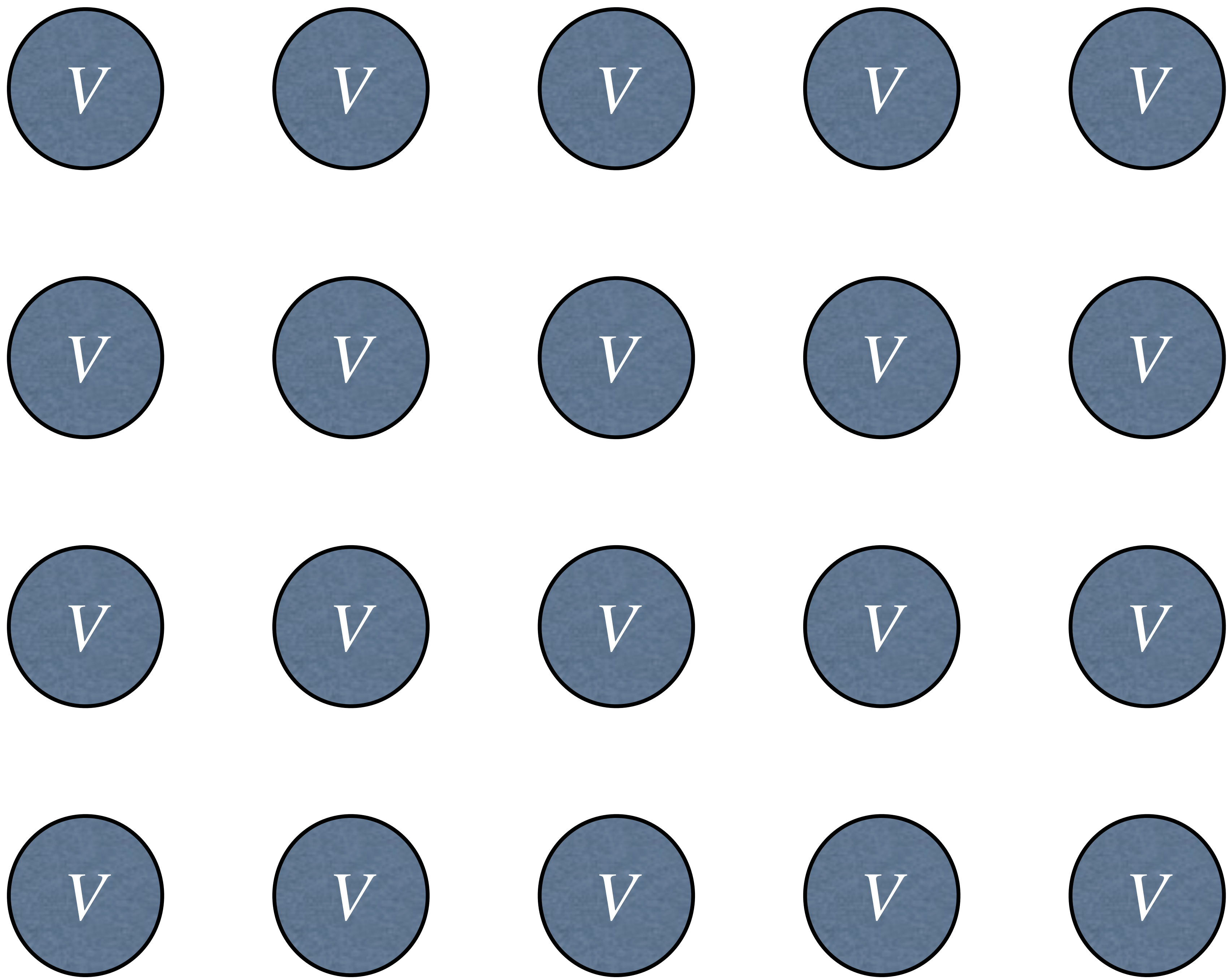
Eigenvalue equation

$$e^{i2\delta(k)} - e^{-ikL} = 0$$

Scattering phase $\qquad$ Lattice geometry

# Lüscher: Extension to 3-D

For 3-D, we require general solution for free particles outside spherical potential region with cubic periodicity



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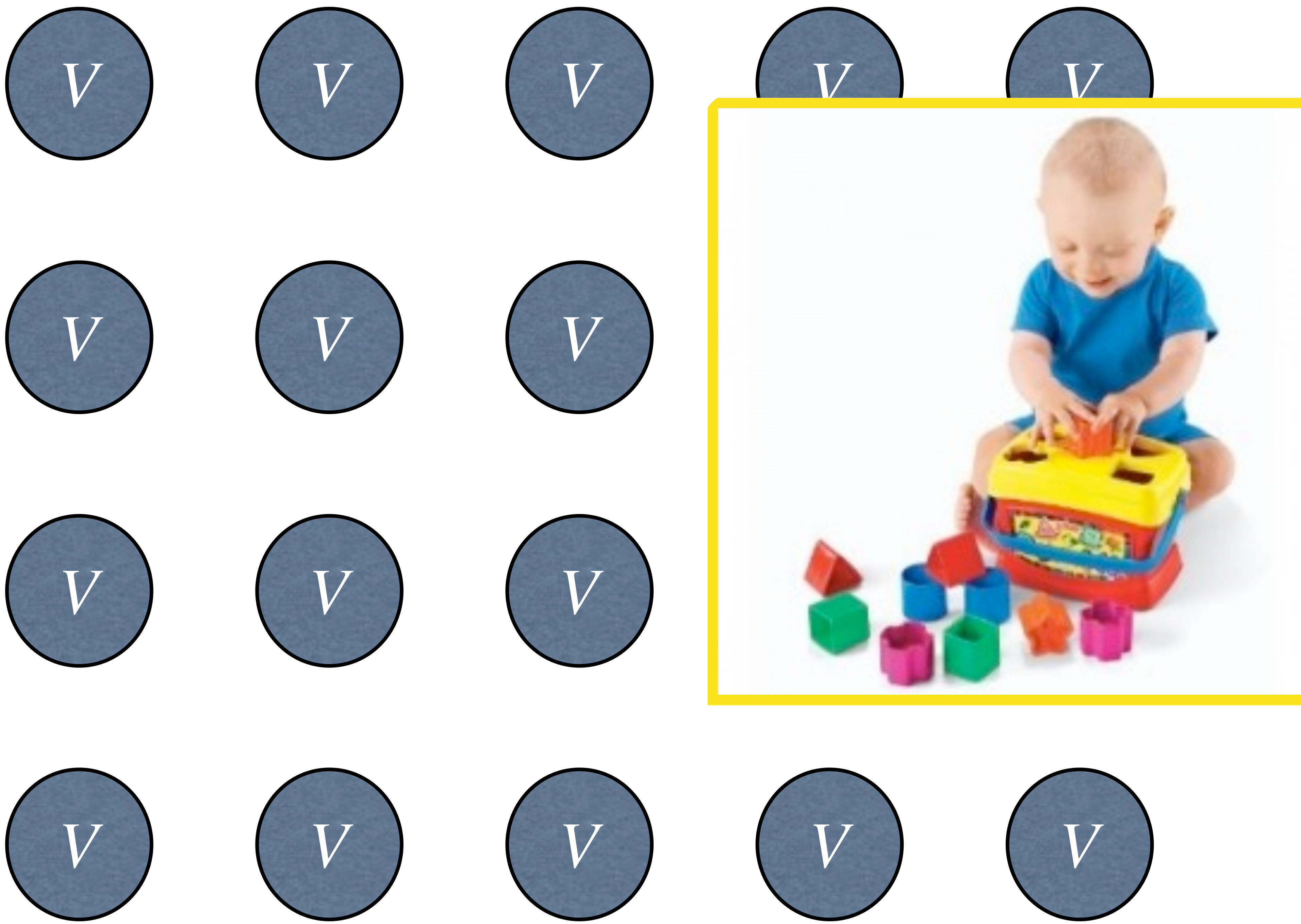


# Lüscher (3-D)

Eigenvalue equation

$$\det\left\{U_{l'm';lm} - e^{i2\delta_{l'}}\delta_{ll'}\delta_{mm'}\right\} = 0$$

Lattice geometry    Scattering phases

# Hamiltonian alternative

Construct a basis of non-interacting multi-particle and (bare) resonance states satisfying the peridocity of the lattice

$$\{\Delta, (N\pi)_1, (N\pi)_2, (N\pi)_3, \ldots\}$$

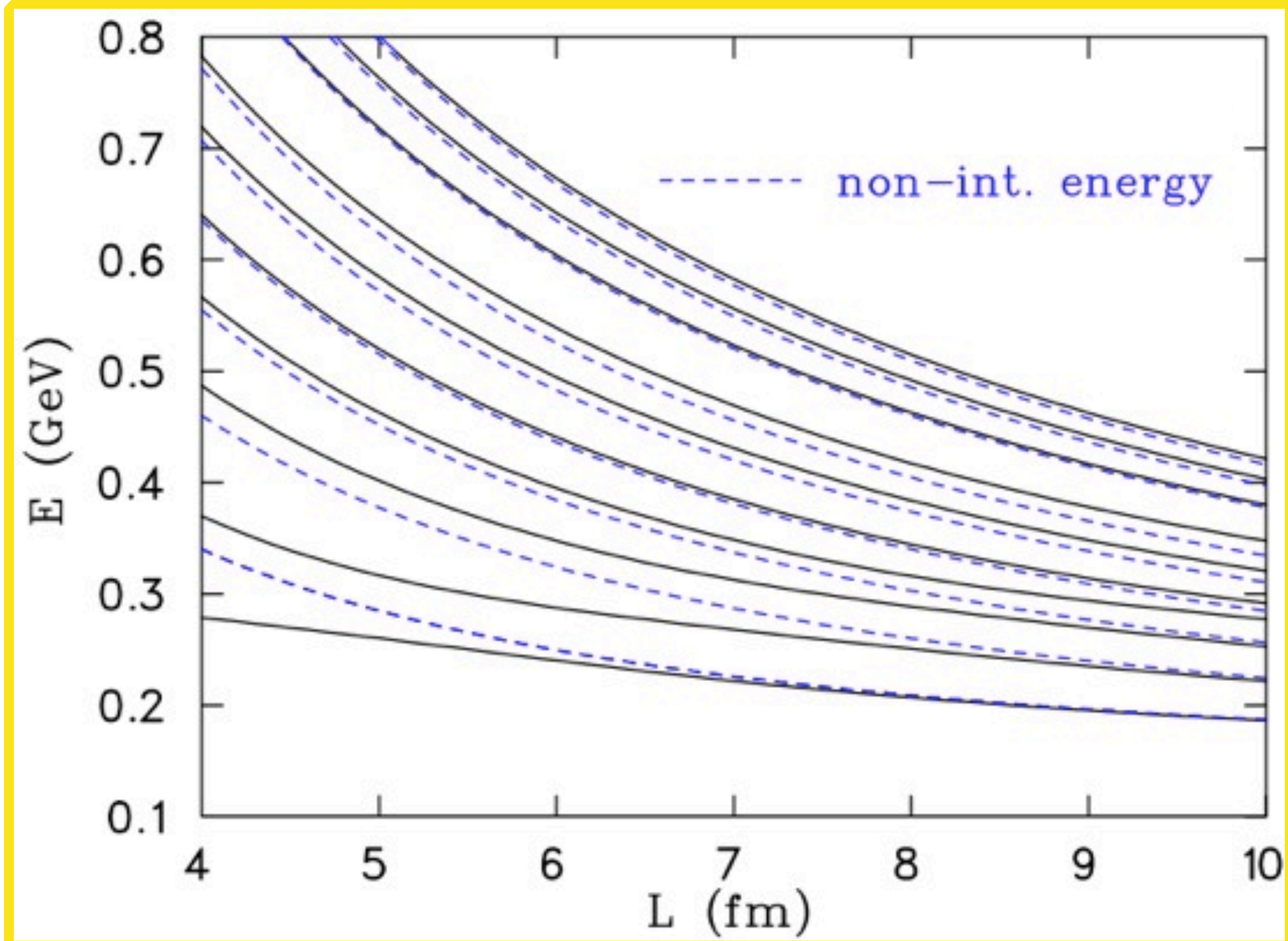
Bare Hamiltonian

$$H_0|\Delta\rangle = \Delta^0|\Delta\rangle; \qquad H_0|(N\pi)_j\rangle = \sqrt{m_\pi^2 + k_j^2}|(N\pi)_j\rangle \qquad k_j^2 = j\left(\frac{2\pi}{L}\right)^2$$

Interaction

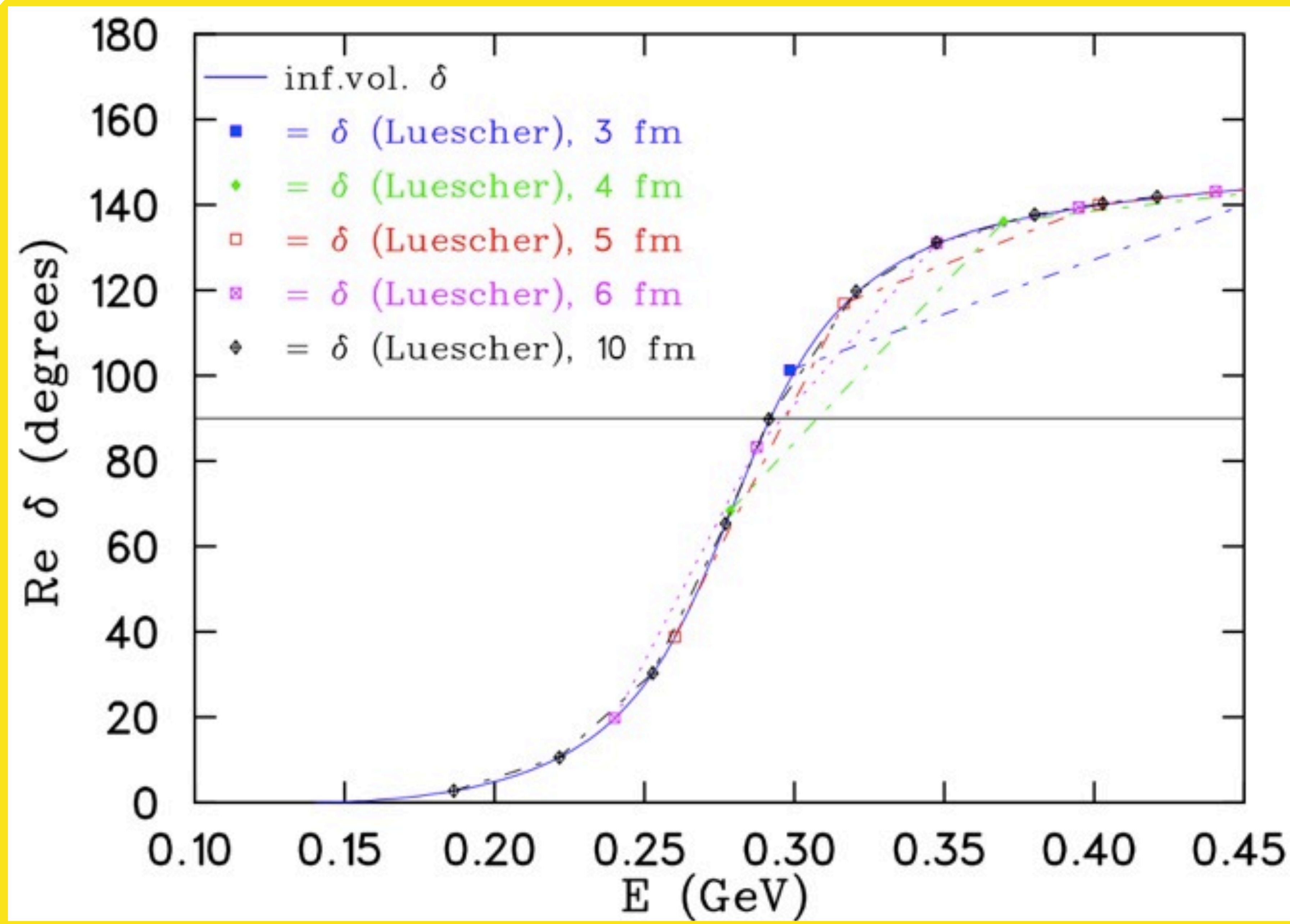
$$\langle\Delta|H_I|(N\pi)_j\rangle \sim g(k_j)$$

Lattice eigenstates determined by $\det(H - \lambda I) = 0$



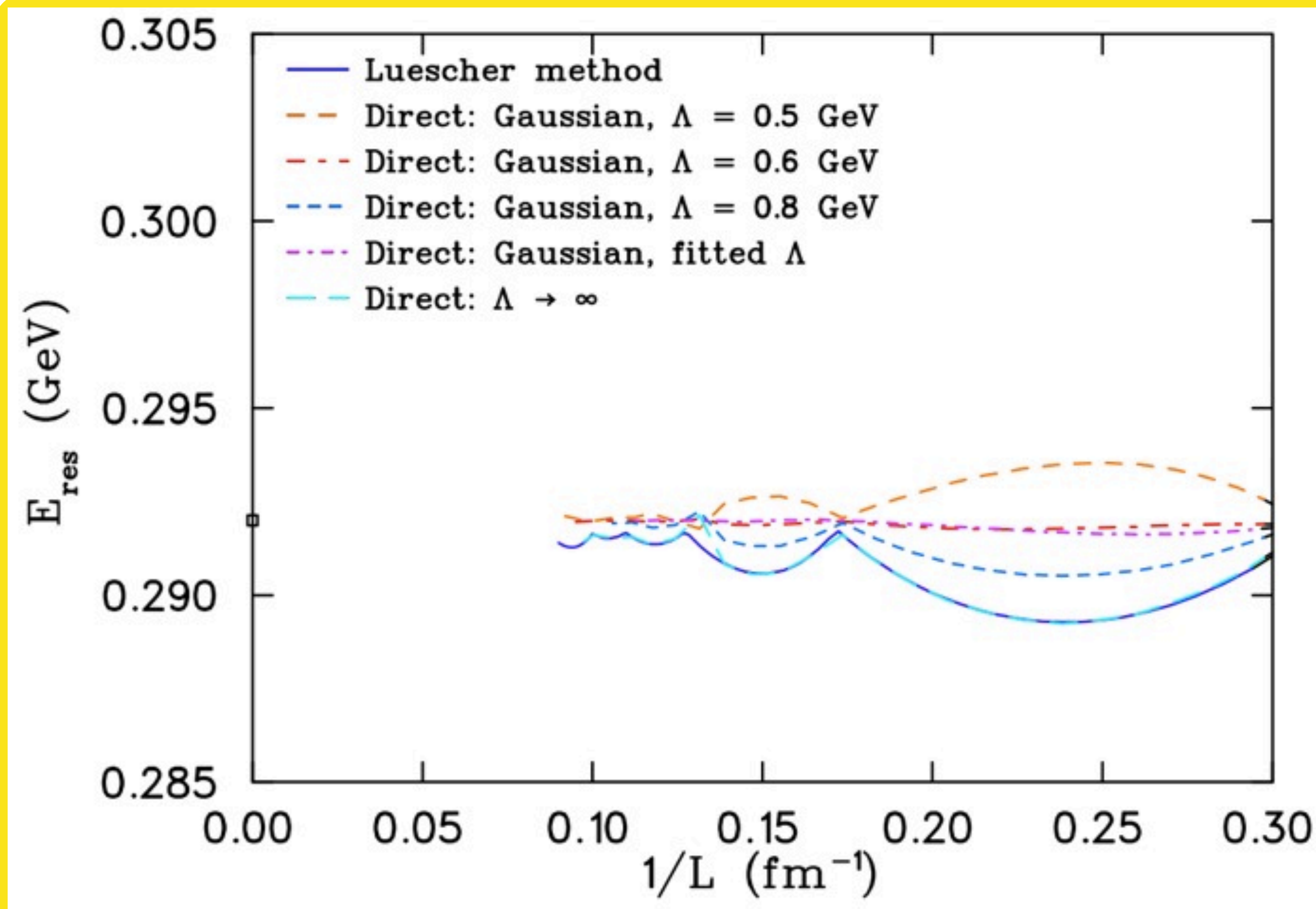

# Hamiltonian spectrum

Identical to Lüscher (to the eye)




# Resonance parameter estimation

On each volume determine phase shift from Lüscher equation
Use Breit-Wigner to interpolate points and extract mass and width
**OR** Directly fit Hamiltonian parameters to spectrum




# Resonance mass

Lüscher & Hamiltonian approach both give a reliable estimate of resonance mass at finite $L$

# Inelastic channels

# Extension to coupled channel

Extension of Lüscher by He, Feng & Liu JHEP(2005)

$$S^{(l)}(E) = \begin{pmatrix} \eta_l e^{2i\delta_1^l} & i\sqrt{1-\eta_l^2}e^{i(\delta_1^l+\delta_2^l)} \\ i\sqrt{1-\eta_l^2}e^{i(\delta_1^l+\delta_2^l)} & \eta_l e^{2i\delta_2^l} \end{pmatrix}$$

Scattering parameters as a function of E

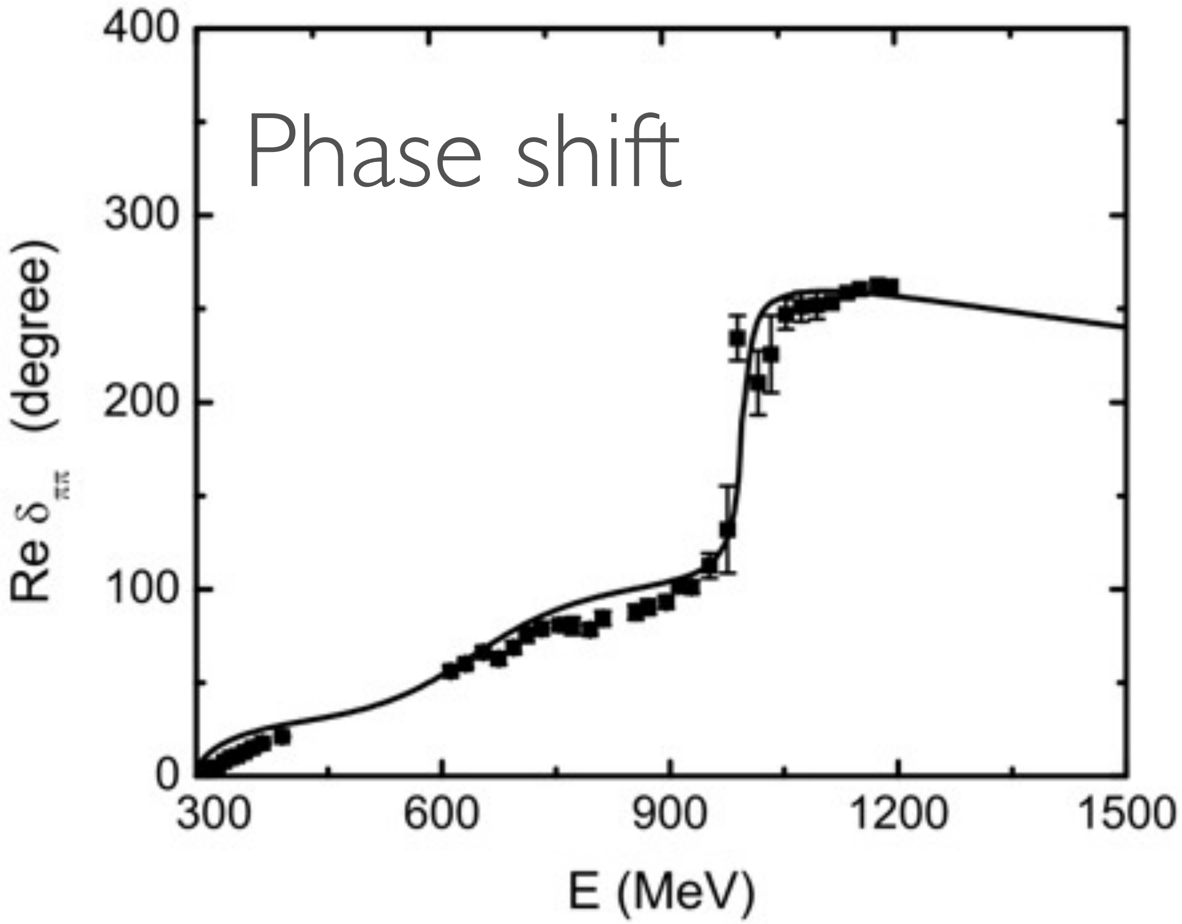
$$\delta_1,\ \delta_2,\ \eta$$

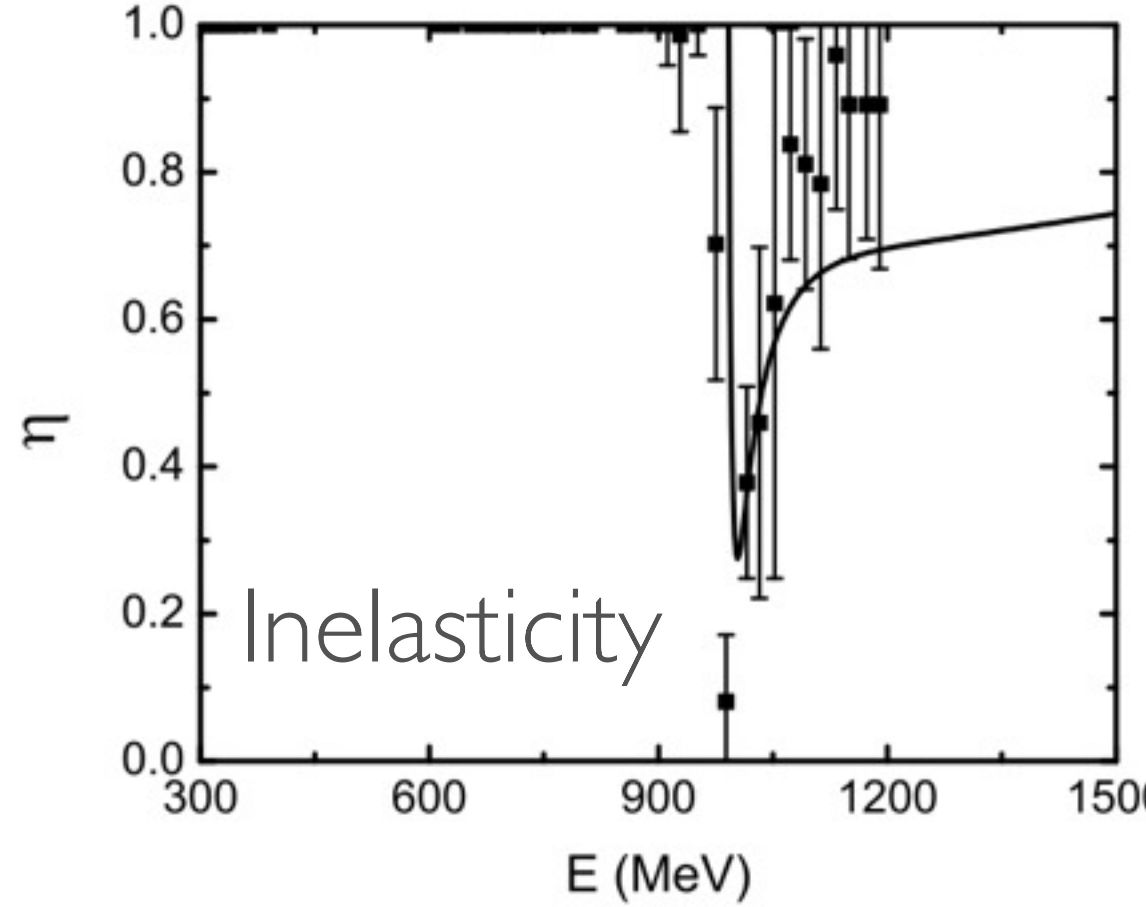
Finite-volume eigenvalue equation (s-wave)

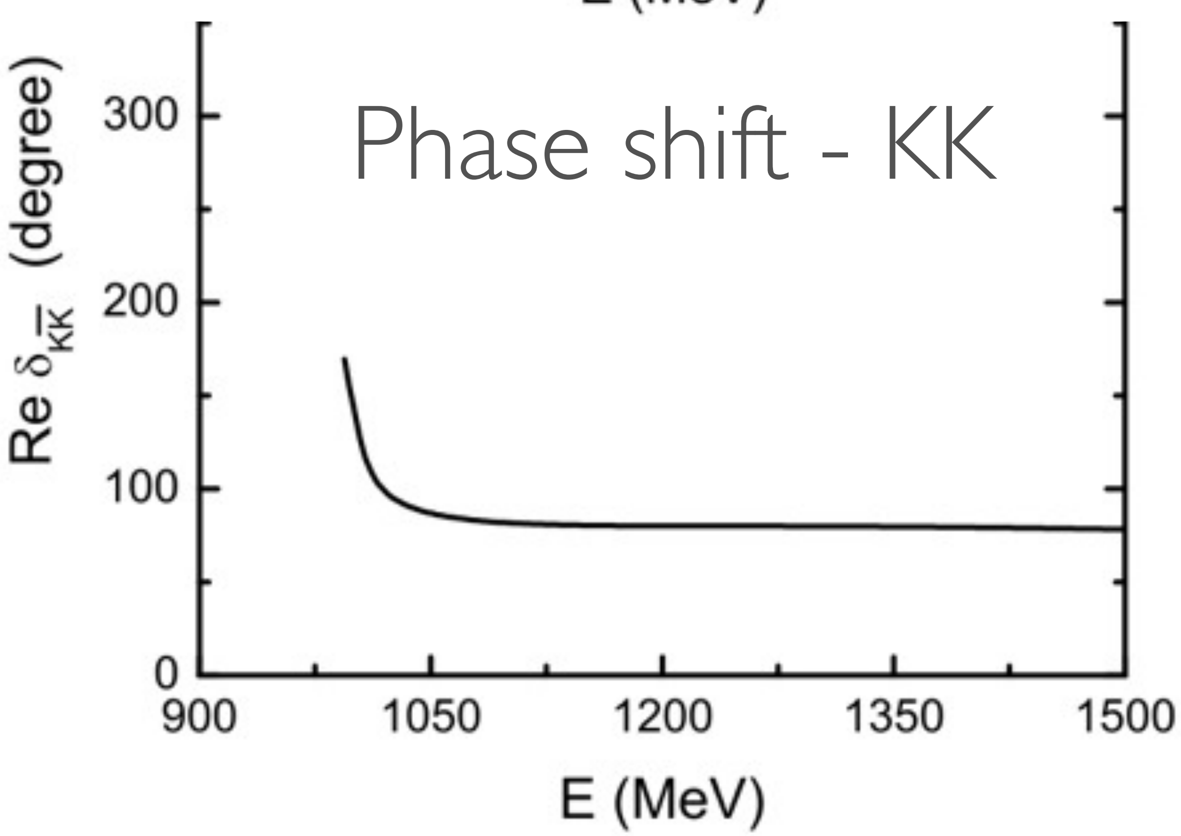
$$\cos(\Delta_1 + \Delta_2 - \delta_1^0 - \delta_2^0) = \eta\cos(\Delta_1 - \Delta_2 - \delta_1^0 + \delta_2^0)$$

$$\Delta_i = \mathcal{M}_{00;00}(q_i^2) = \frac{\mathcal{Z}_{00}(1;q_i^2)}{\pi^{3/2}q_i}$$

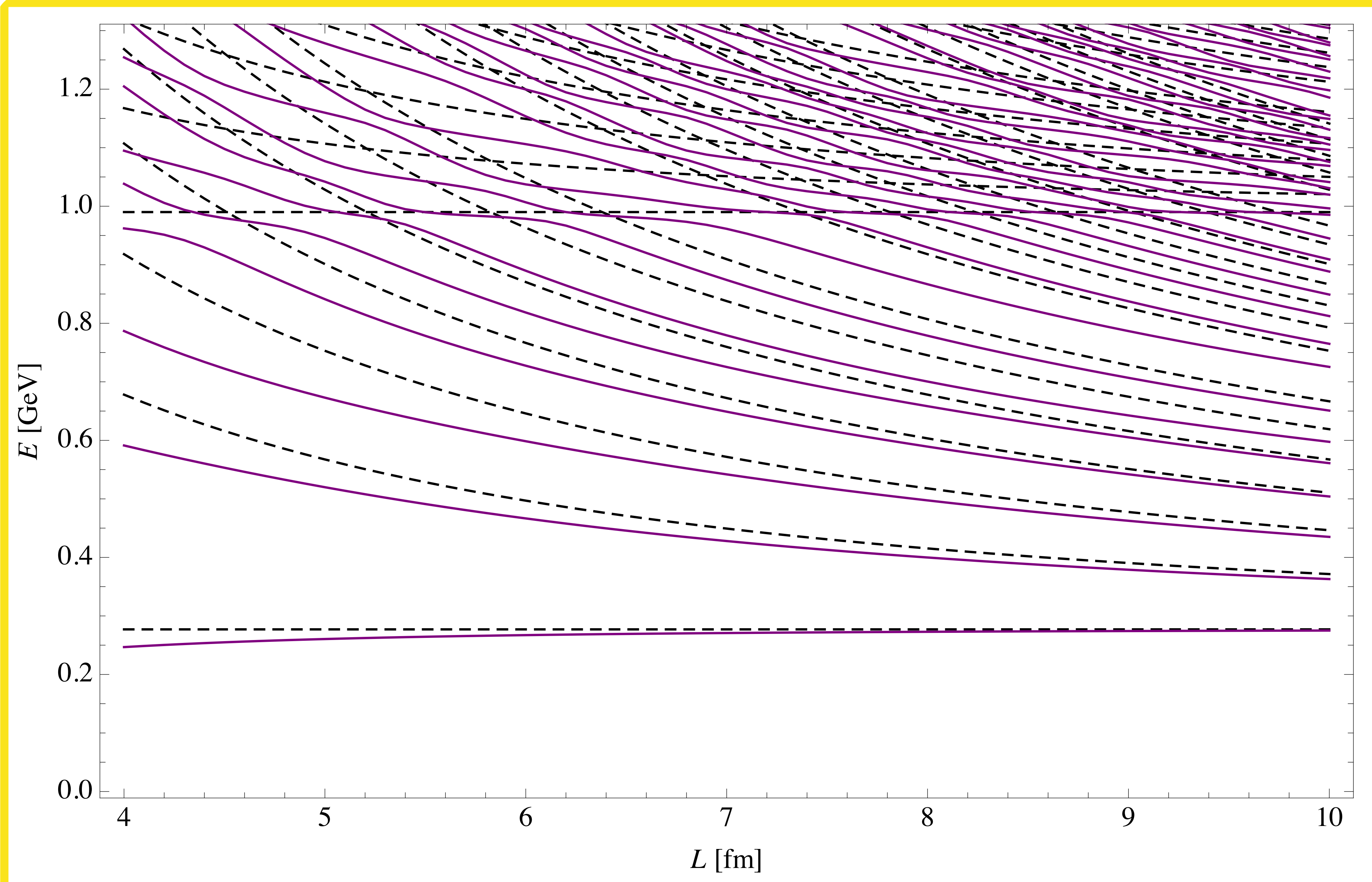
# J=0 Isoscalar $\pi\pi, \; K\bar{K}$



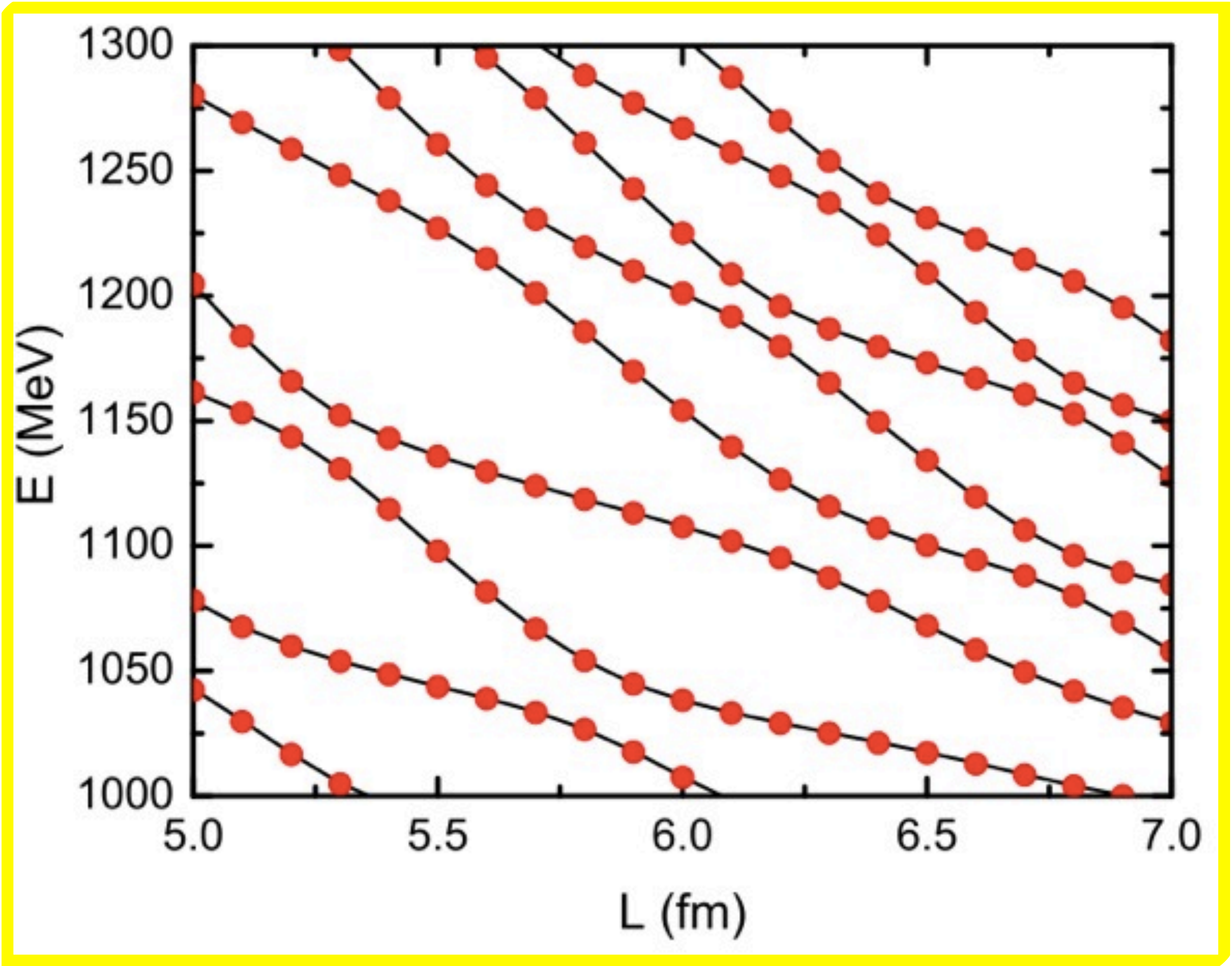

Wu, Lee, Thomas, RDY, arXiv:1402.4868




# Hamiltonian spectrum

$$\{\sigma, (\pi\pi)_0, (\pi\pi)_1, (\pi\pi)_2, \ldots, (K\bar{K})_0, (K\bar{K})_1, (K\bar{K})_2, \ldots\}$$

Dashed lines are non-interacting levels

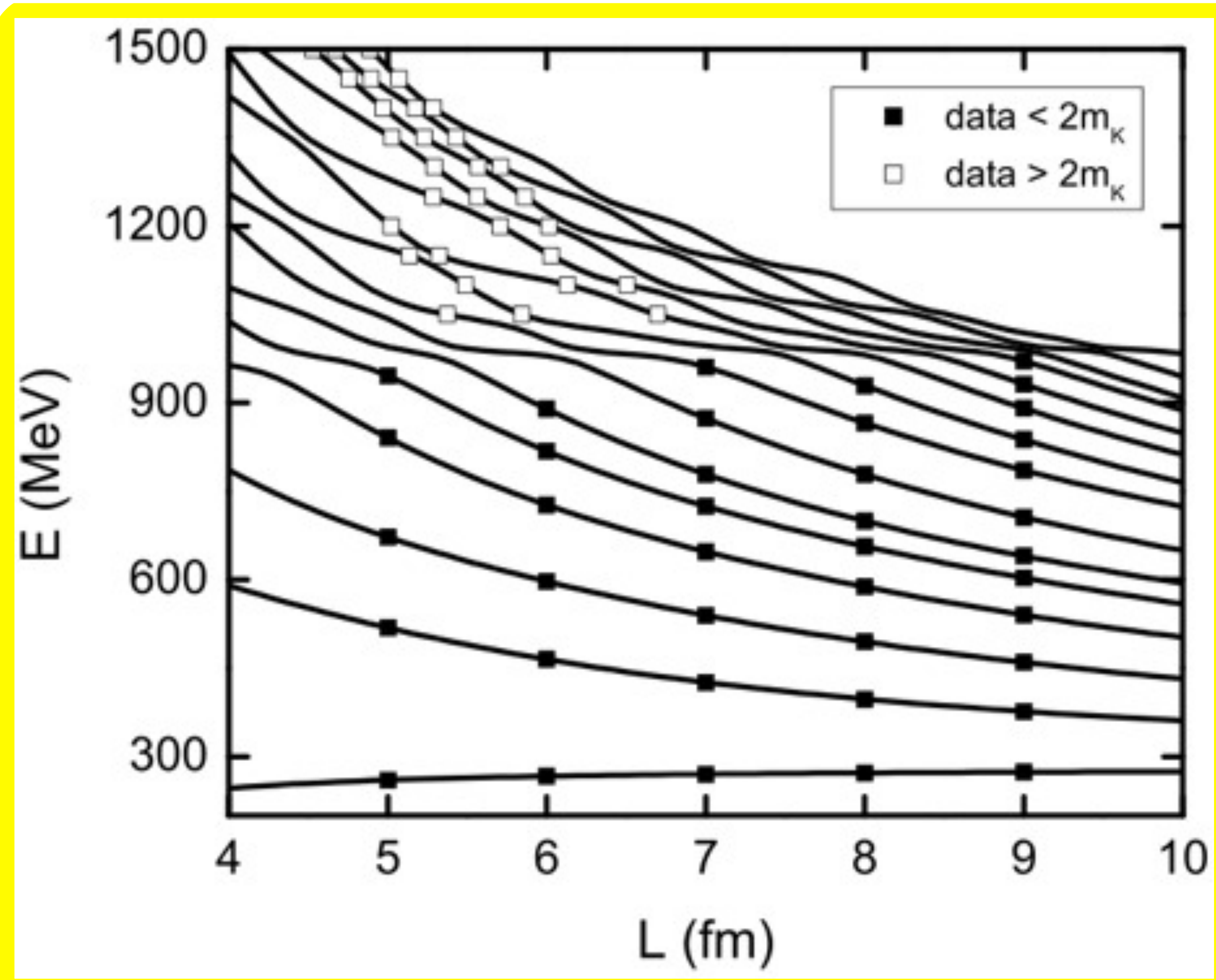


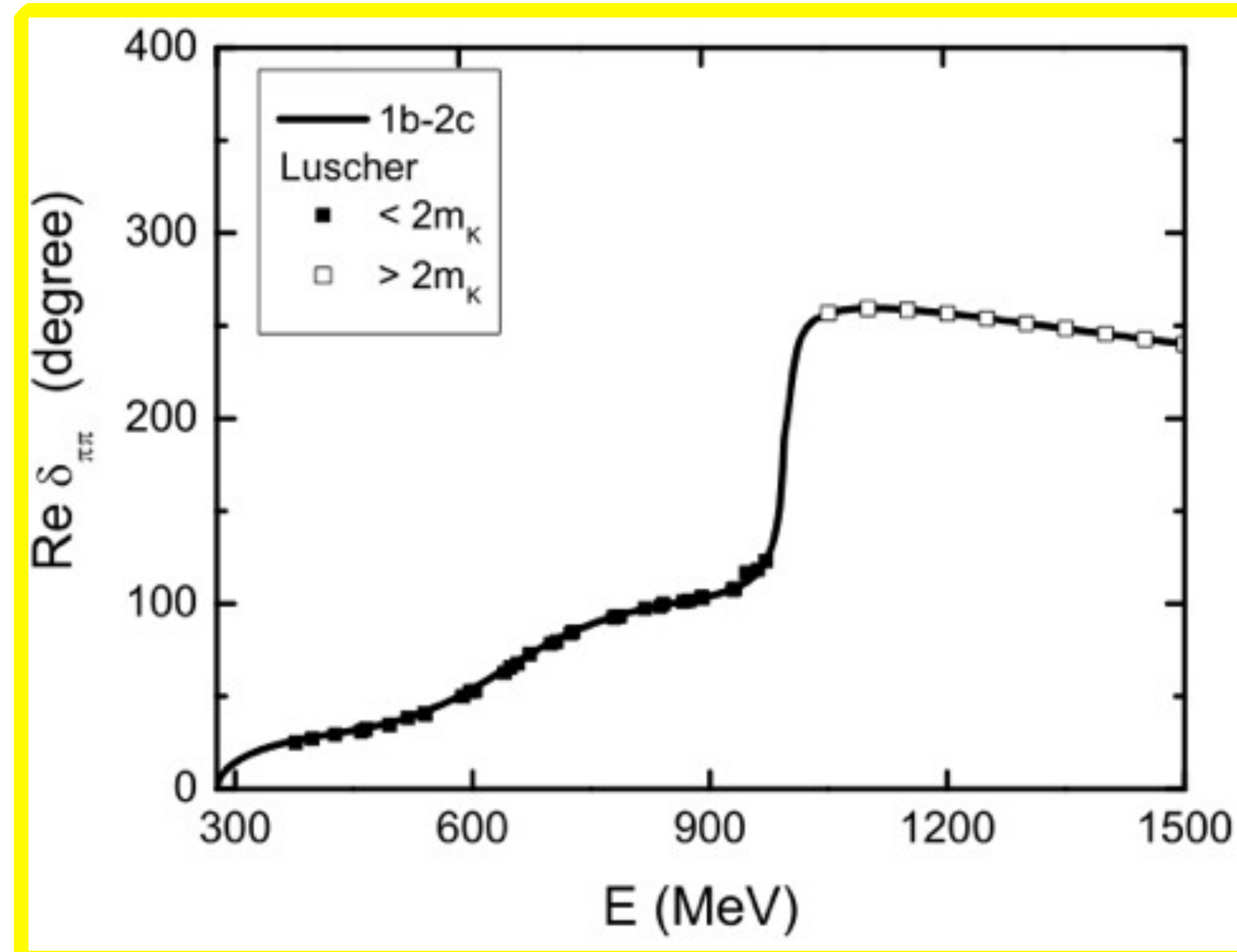

# FV Spectrum “Prediction”

Lüscher: red dots

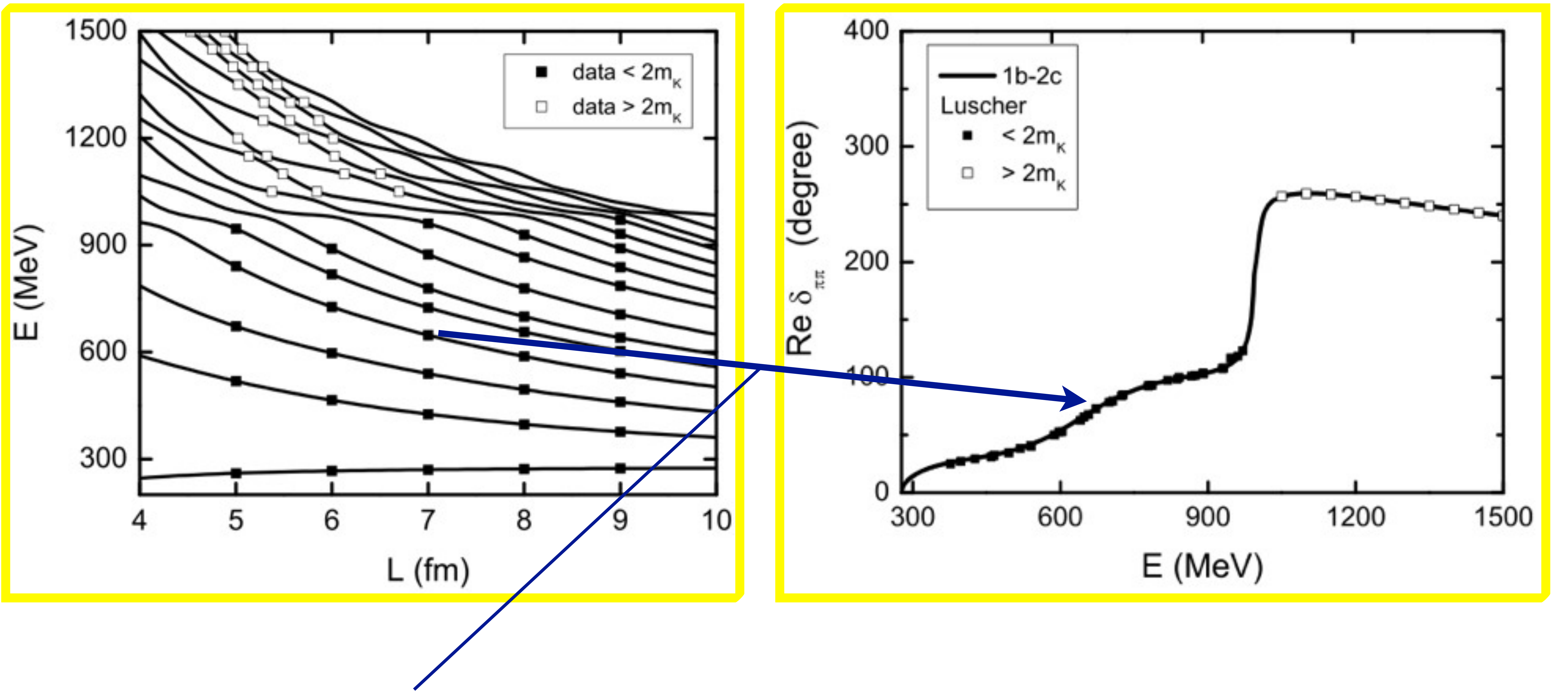
Hamiltonian: black curves

Applications for lattice QCD,
we want the inverse:
ie. *Spectrum* → *Phase shifts*



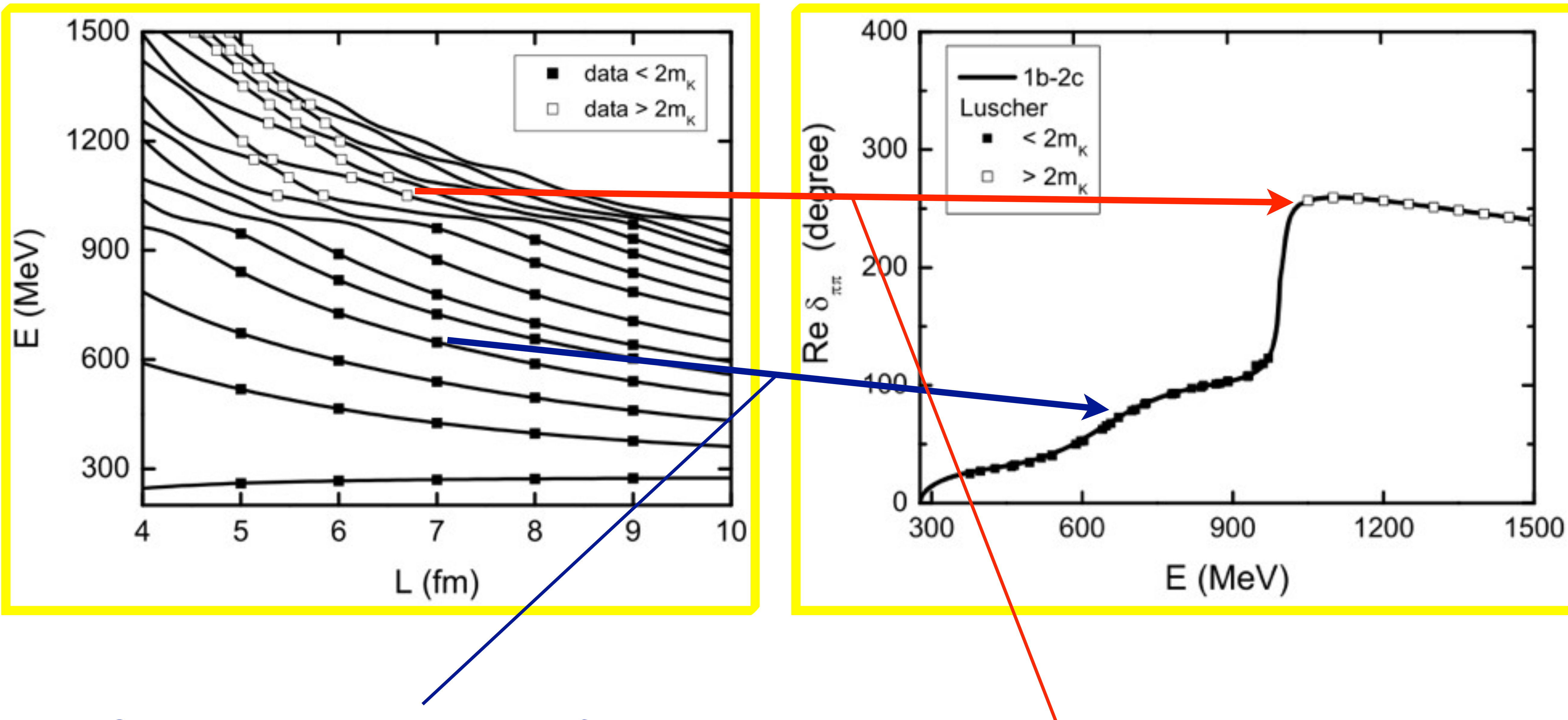




# Inversion with multi-channel Lüscher




Single channel: straightforward

# Inversion with multi-channel Lüscher



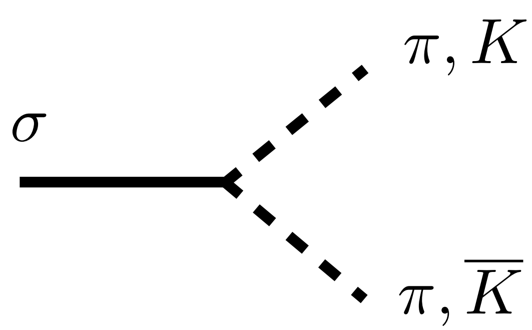
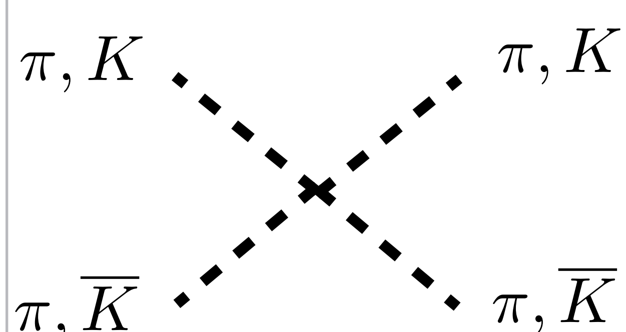

Single channel: straightforward

2 channels: requires degeneracy at independent L

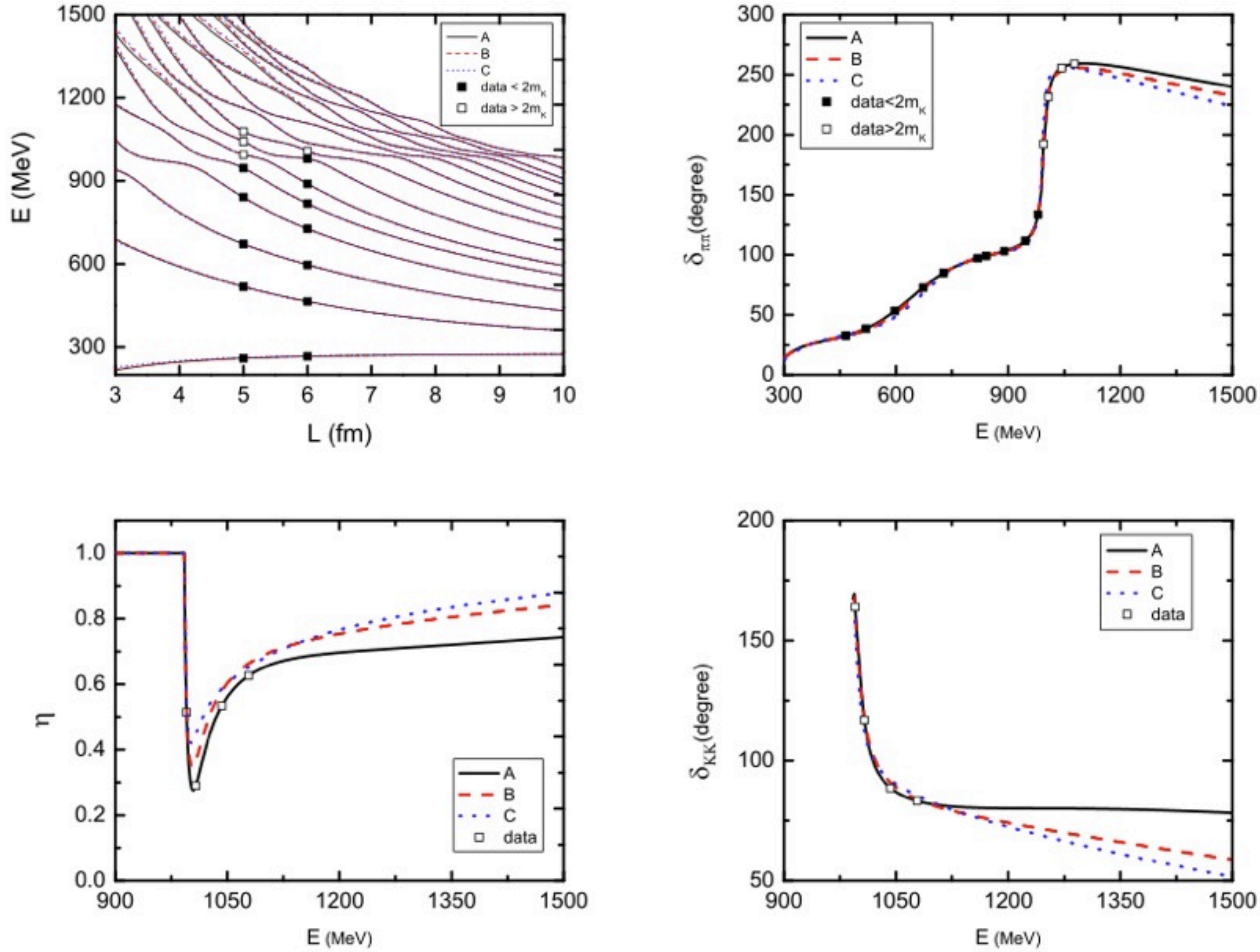
# Inversion with multi-channel Lüscher

# Inversion with Hamiltonian

Requires a choice of Hamiltonian

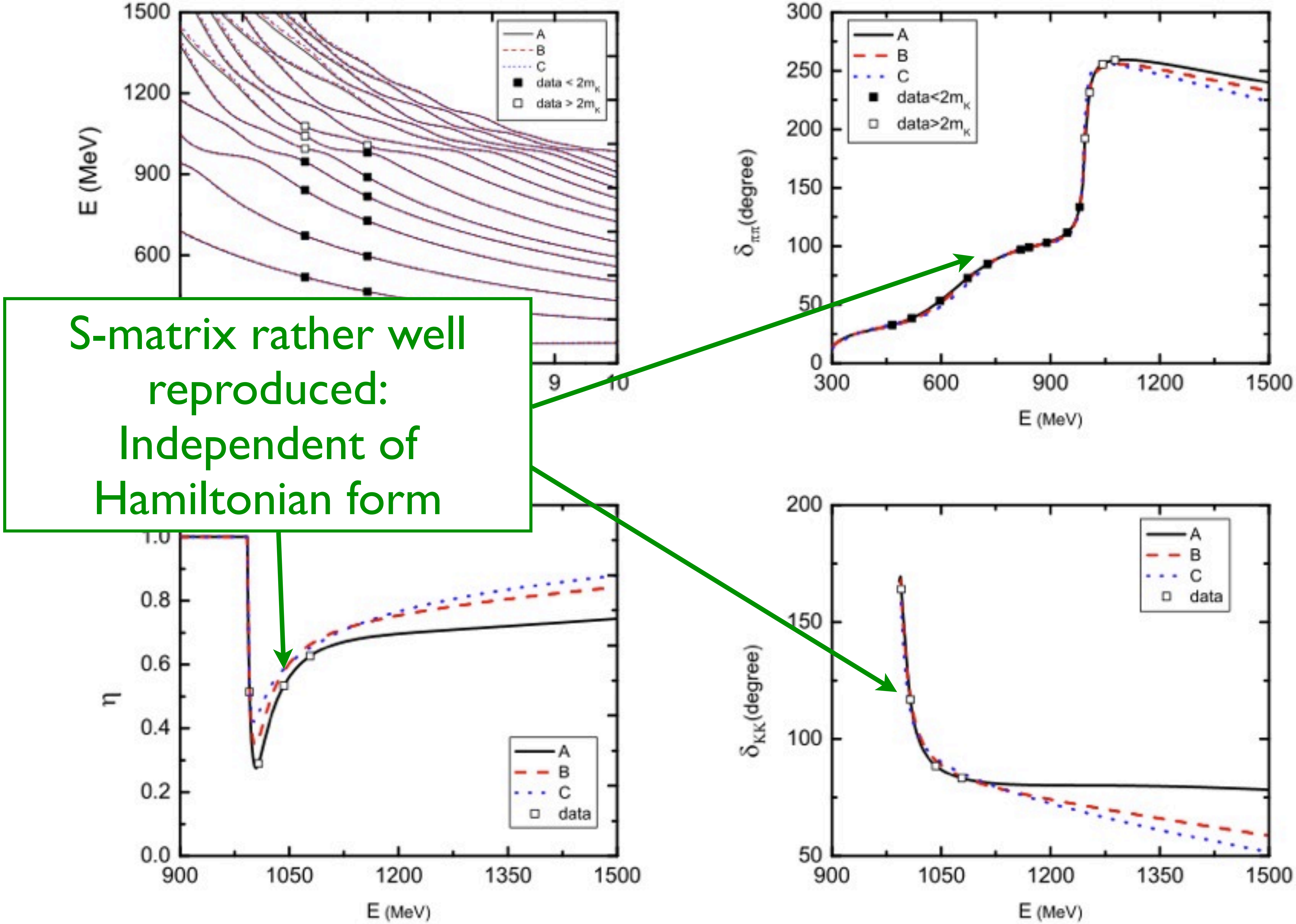
| | Model A | Model B | Model C |
|---|---|---|---|
|  | $\frac{g_{\sigma,\alpha}}{(1+(c_\alpha k)^2)}$ | $\frac{g_{\sigma,\alpha}}{(1+(c_\alpha k)^2)^2}$ | $g_{\sigma,\alpha} e^{-(c_\alpha k)^2}$ |
|  | $\frac{G_{\sigma,\alpha}}{(1+(d_\alpha k)^2)(1+(d_\beta k)^2)}$ | $\frac{G_{\sigma,\alpha}}{(1+(d_\alpha k)^2)^2(1+(d_\beta k)^2)^2}$ | $G_{\alpha,\beta} e^{-(d_\alpha k)^2} e^{-(d_\beta k)^2}$ |

$$\alpha, \beta = (\pi\pi), (K\overline{K})$$




# Inversion with Hamiltonian

Just 2 box sizes (8 eigenstates)

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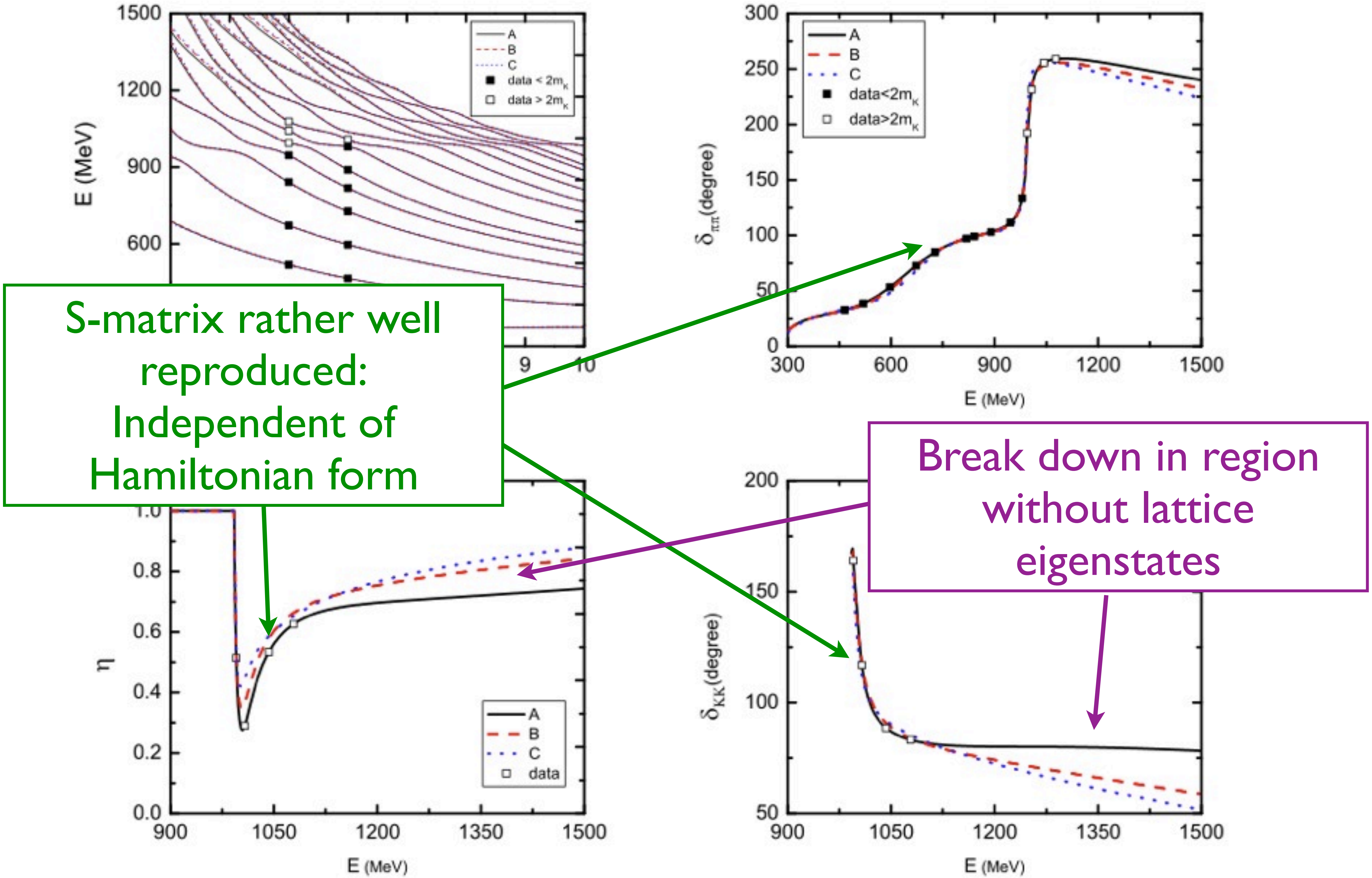


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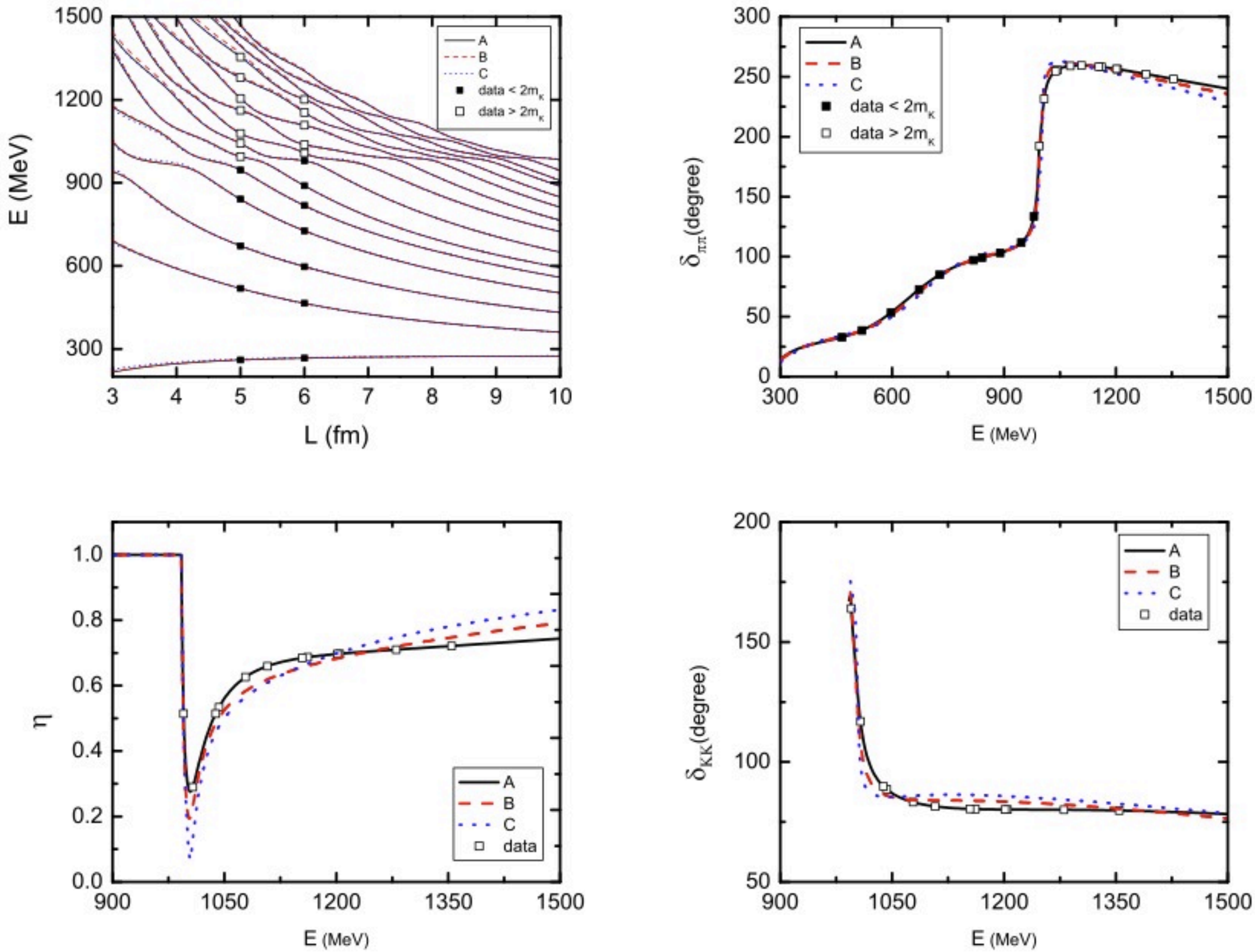
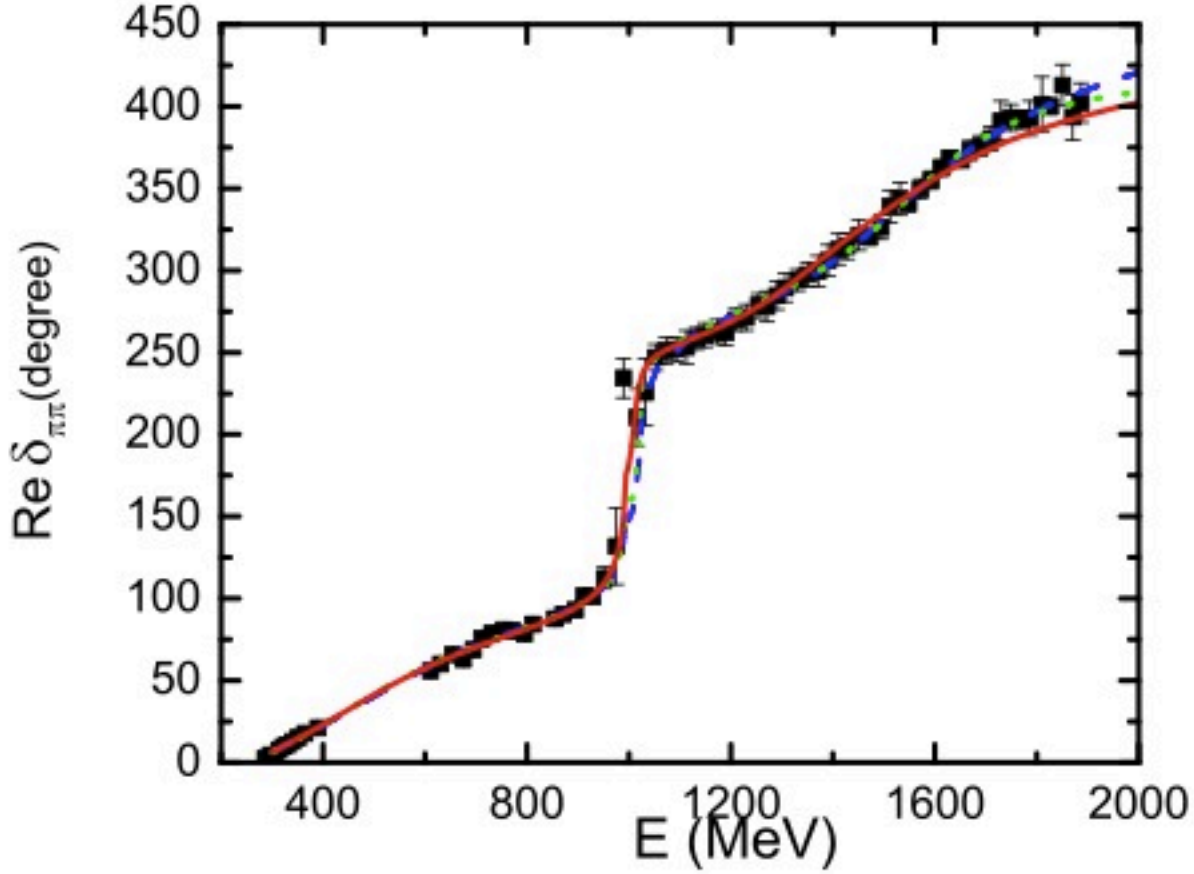
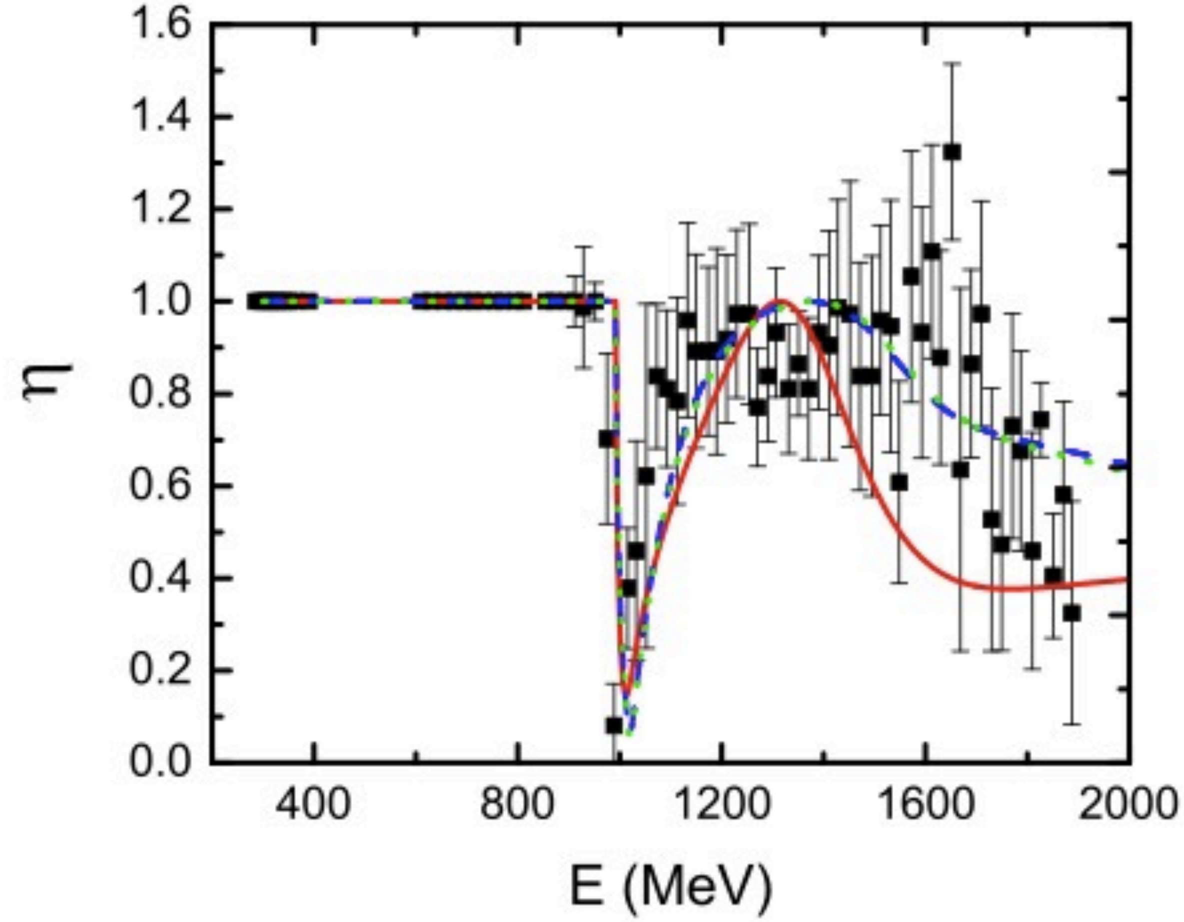
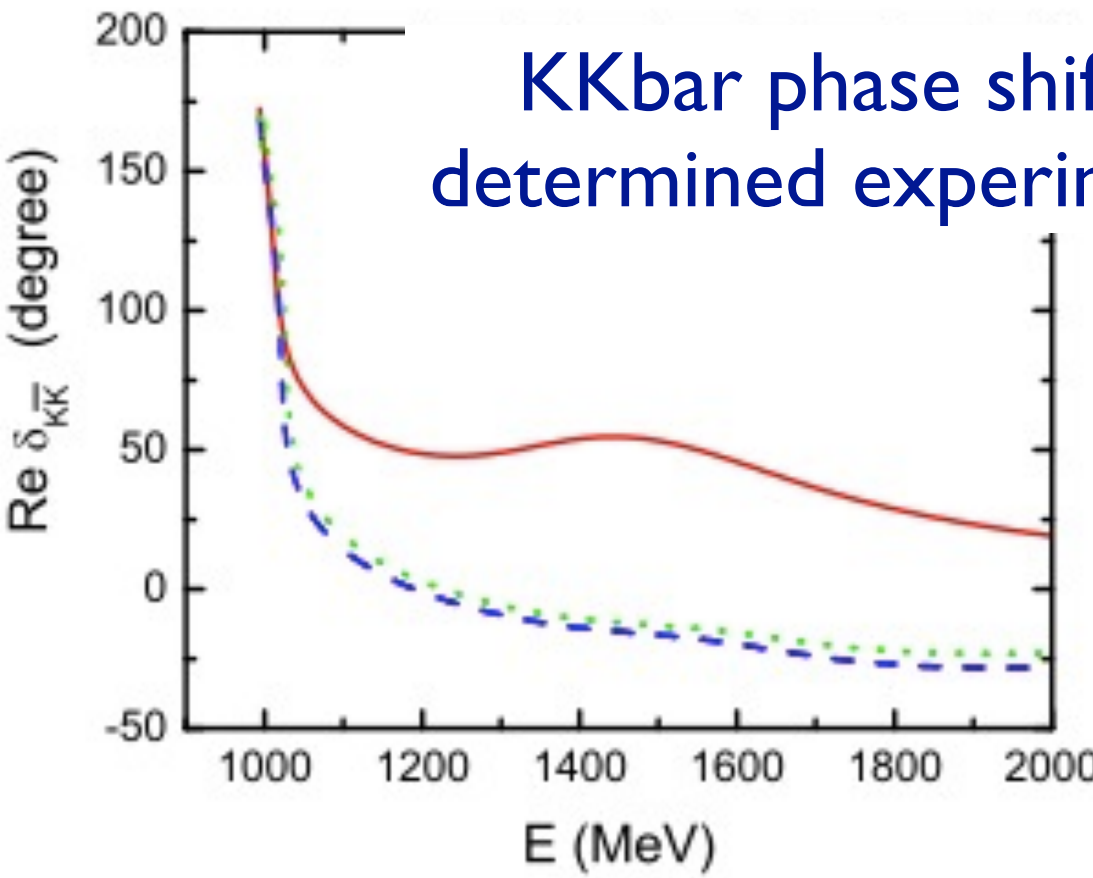



# Inversion with Hamiltonian

Just 2 box sizes (12 eigenstates)

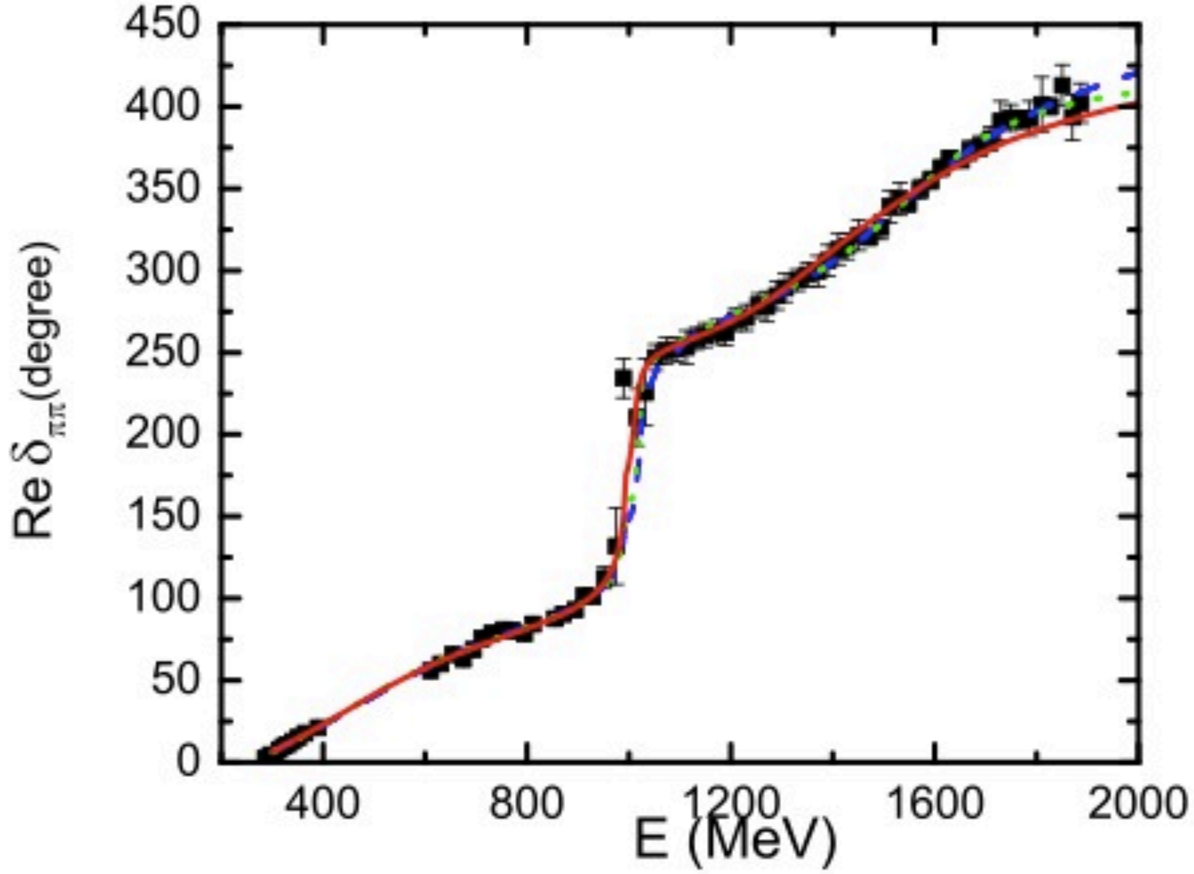


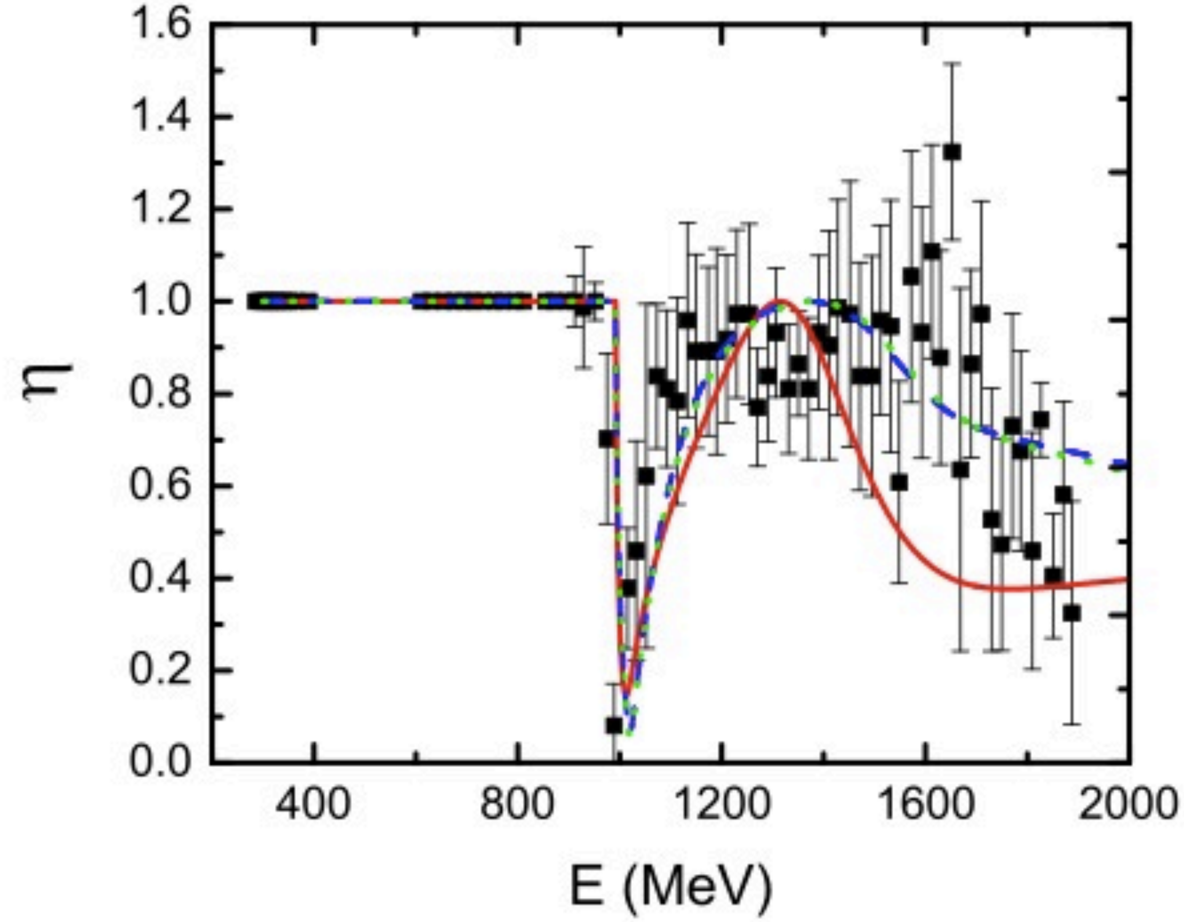



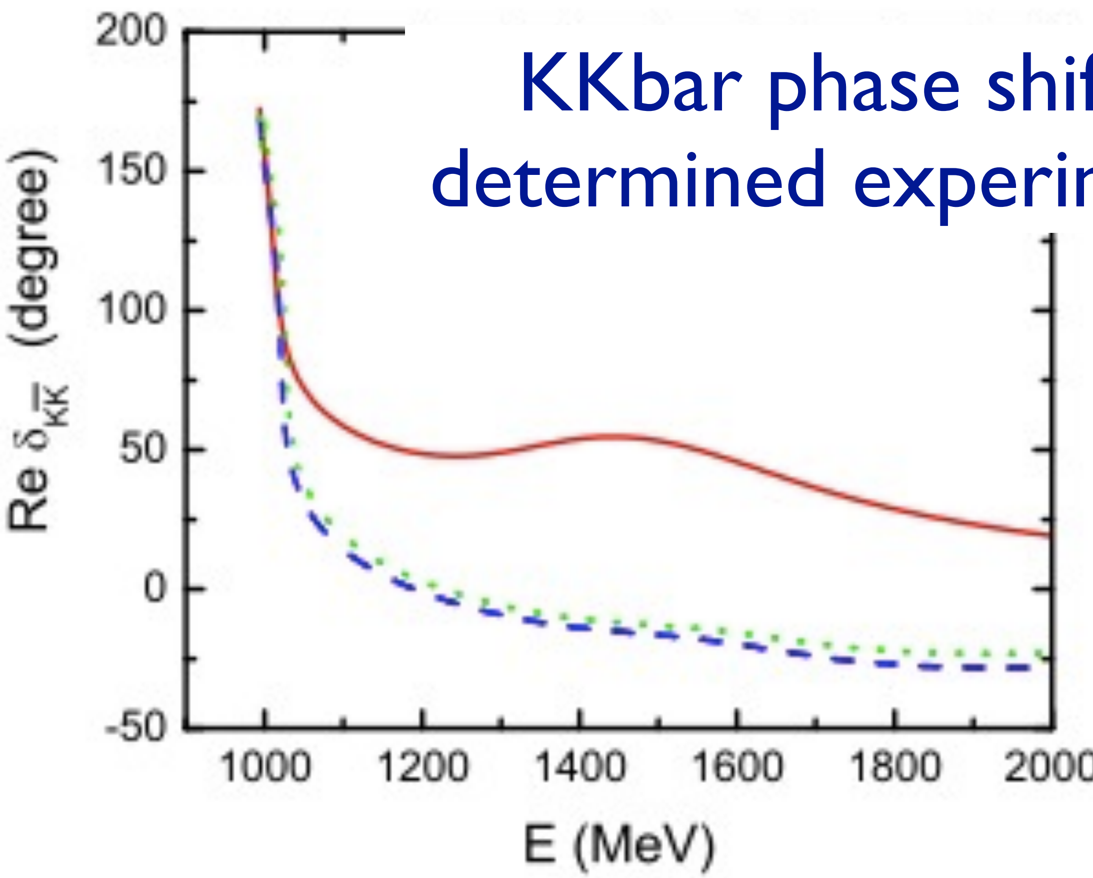




KKbar phase shift not determined experimentally

Application?




KKbar phase shift not determined experimentally

# Application?

Lattice eigenstates of ~50 MeV precision could discriminate model dependence

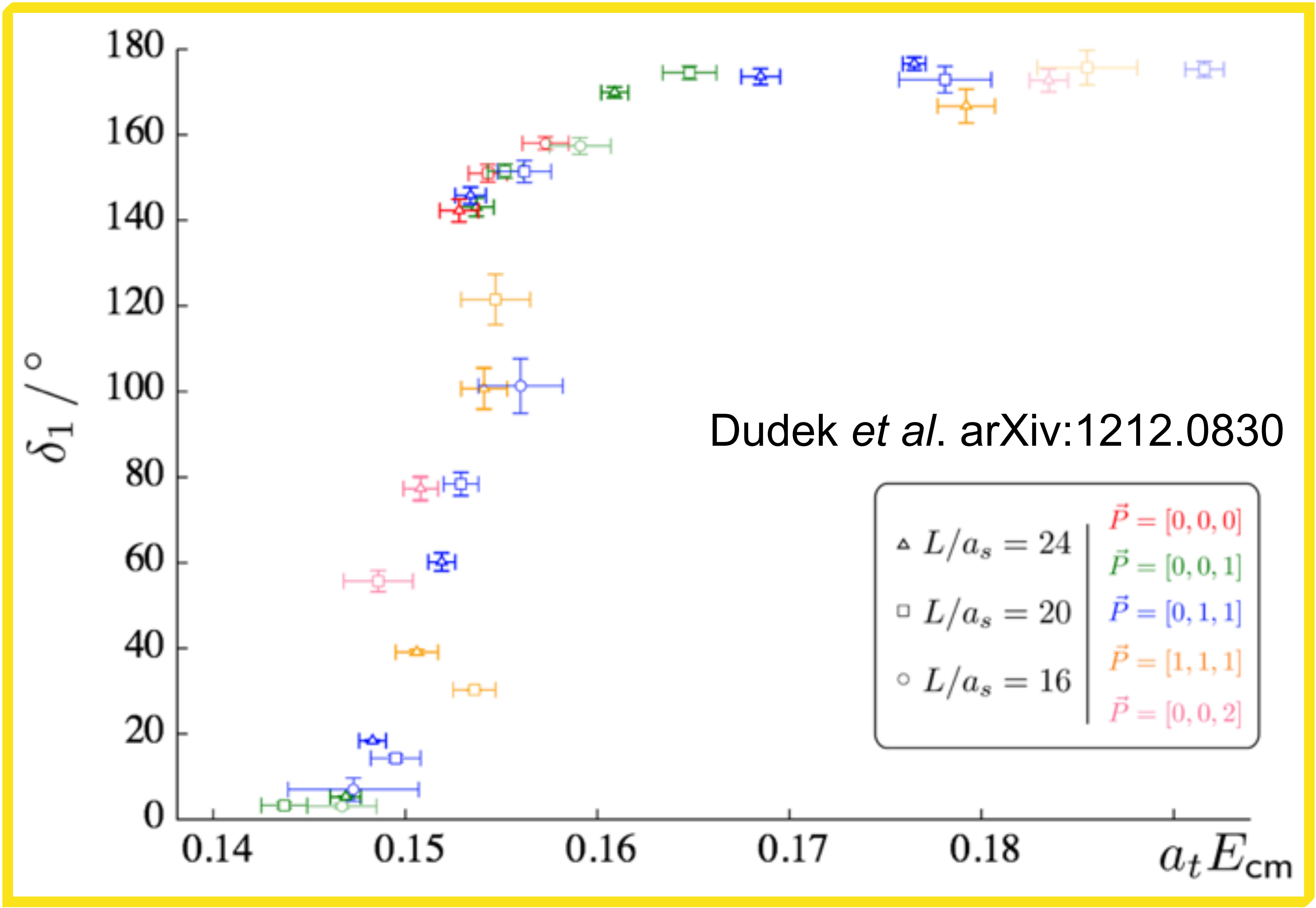
# Summary

For elastic 2-body scattering, we have a straightforward relation that maps finite-volume spectra to scattering parameters (Lüscher and Hamiltonian equivalent)

For coupled-channel systems, Hamiltonian approach appears to give a much more accessible determination of S-matrix

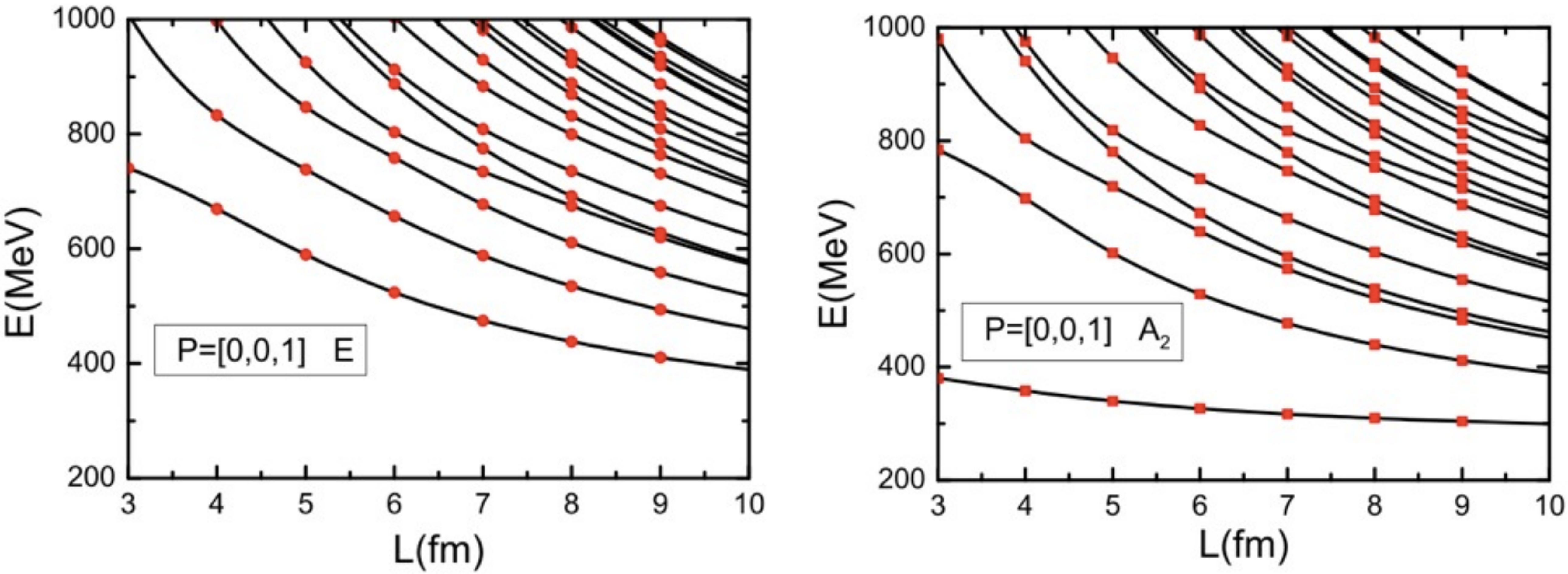
Hamiltonian formulation offers a natural method to extend to more complicated systems

eg. many-body systems; transition matrix elements




# Boosted momentum frame

Using boosted frames, can extract the scattering phases at many more eigenenergies (momenta)

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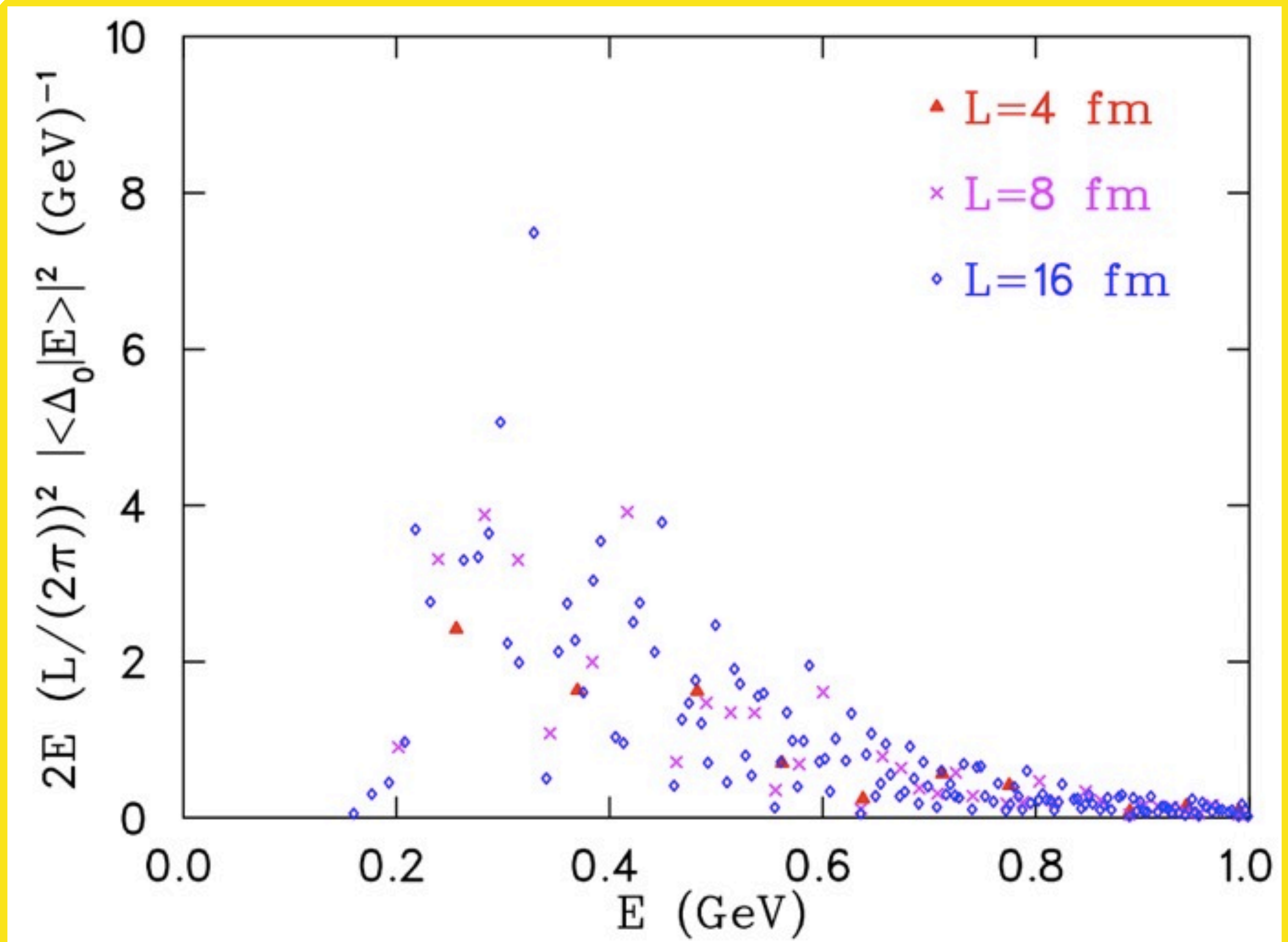


# Eigenvectors

Strength of coupling of bare Delta to energy eigenstates.
Clearly no distinction between "local" and scattering state (within width of resonance)