Electromagnetic Mass Splitting of Proton and Neutron

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Apr. 11, 2014

Physical interest

- The physical proton-neutron mass splitting has been measured extremely precisely,

$$M_p - M_n = -1.2933322(4)\ \mathrm{MeV}\ .$$

- Separation of the electromagnetic and $u - d$ quark mass difference contribution to the p-n mass difference is of enormous interest

$$M_p - M_n = (M_p - M_n)|_{QED} + (M_p - M_n)|_{QCD}$$

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QCD Contribution

- P.E. Shanahan *et al.*, Phys. Lett. B 718(2013)1148
  $SU(3)$ χPT $\oplus$ Lattice data on octet baryon masses

	$(M_p - M_n)\|_{QCD}$ [MeV]
PACS-CS	-2.9 ± 0.4
QCDSF-UKQCD	-2.4 ± 0.3

Table: Proton and neutron mass difference due to QCD interactions, corresponding to $R = \frac{m_u}{m_d} = 0.553 \pm 0.043$.

The electromagnetic self-energy of the nucleon,

$$\delta M^{\gamma} = \frac{i}{2M}\frac{\alpha}{(2\pi)^3}\int_R d^4q \frac{T^{\mu}_{\mu}(p,q)}{q^2 + i\epsilon} \ ,$$

where $T_{\mu\nu}$ is the spin averaged forward Compton scattering tensor

$$T_{\mu\nu} = \frac{i}{2}\sum_{\sigma}\int d^4x e^{iq\cdot x}\langle p\sigma|T\{J_{\mu}(x)J_{\nu}(0)\}|p\sigma\rangle \ ,$$

$$T_{\mu\nu}(\nu, q^2) =$$

$$-(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2})T_1(\nu, q^2) + \frac{1}{M^2}(p_{\mu} - \frac{p\cdot q}{q^2}q_{\mu})(p_{\nu} - \frac{p\cdot q}{q^2}q_{\nu})T_2(\nu, q^2$$

QED Contribution

The electromagnetic self-energy of the nucleon,

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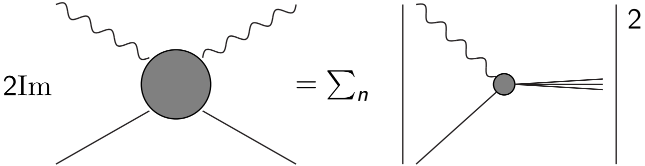
Optical Theorem



The imaginary part of a forward scattering amplitude arises from a sum of contributions from all possible intermediate states

W. Cottingham, Ann. Phys. (N.Y.) 25(1963)424

- Establish unsubtracted dispersion representations for $T_1(\nu, Q^2)$ and $T_2(\nu, Q^2)$ in respect to ν
- Separation between elastic and inelastic contributions

$$T_i = T_i^{el} + T_i^{inel}$$

where

$$T_i^{el} \sim G_{E,M} \quad T_i^{inel} \sim F_{1,2}(\nu, Q^2)$$

The dominate contribution comes from the elastic intermediate state exchange,

$$\delta M^{\gamma} \approx \delta M^{\gamma}_{el} = 0.76 \pm 0.30 \ \mathrm{MeV}$$

J. Gasser and H. Leutwyler, Phys. Rep. 87(1982)77

- Renormalization
  J.C. Collins, Nucl. Phys. B 149(1979)90

$$T_i(\nu, Q^2) \sim 1/Q^2 \ (Q^2 \to \infty)$$

- Possible Subtraction
  H. Harari, Phys. Rev. Lett. 17(1966)1303

  $T_1(\nu, Q^2)$: once subtraction
  $T_2(\nu, Q^2)$: no subtraction

Subtracted Dispersion Representation

Recently, Walker-Loud *et al.*(WLCM) reanalyzed EM mass splitting of proton and neutron with

- Subtracted dispersion representation

$$\delta M^{\gamma} = \delta M^{el} + \delta M^{inel} + \delta M^{el}_{sub} + \delta M^{inel}_{sub} + \delta \tilde{M}^{ct} \ .$$

- Modern knowledge on $G_{E,M}$ and $F_{1,2}(x, Q^2)$ from the latest accurate experiments.

A. Walker-Loud *et al.*, Phys. Rev. Lett. 108(2012)232301.

- Elastic contribution

$$\delta M^{el} = \frac{\alpha}{\pi}\int_0^{\Lambda_0} dQ \left\{ \frac{3\sqrt{\tau_{el}}G_M^2}{2(1+\tau_{el})} + \frac{[G_E^2 - 2\tau_{el}G_M^2]}{1+\tau_{el}} \times \left[1+\tau_{el})^{3/2} - \tau_{el}^{3/2} - \frac{3}{2}\sqrt{\tau_{el}} \right] \right\},$$

with $\tau_{el} = Q^2/4M^2$.

G_E, G_M: EM Form Factors using Kelly parametrization
Kelly, Phys. Rev. C 70(2004)068202

$$\delta M^{el}|_{p-n} = 1.39(02)\ \mathrm{MeV}$$

- Inelastic contribution

$$\delta M^{inel} = \frac{\alpha}{\pi}\int_0^{\Lambda_0} dQ \int_{\nu_{th}}^{\infty} \left\{ \frac{3F_1(\nu,Q^2)}{M}\left[\frac{\tau^{3/2}-\tau\sqrt{1+\tau}+\sqrt{\tau}/2}{\tau}\right] + \frac{F_2(\nu,Q^2)}{\nu}\left[(1+\tau)^{3/2}-\tau^{3/2}-\frac{3}{2}\sqrt{\tau}\right]\right\}$$

with $\tau = \nu^2/Q^2$.

F_1, F_2: Structure functions from Bosted and Christy
P.E. Bosted, M.E. Christy, Phys. Rev. C 77(2008)065206
M.E. Christy, P.E. Bosted, Phys. Rev. C 81(2010)055213

$$\delta M^{inel}|_{p-n} = 0.057(16)\ \mathrm{MeV}$$

- elastic subtraction term

$$\delta M^{el}_{sub} = -\frac{3\alpha}{16\pi M}\int_0^{\Lambda_0^2} dQ^2[2G_M^2(Q^2) - 2F_1^2(Q^2)] = -0.62(02)\ \mathrm{MeV}$$

- inelastic subtraction term

$$\delta M^{inel}_{sub} = -\frac{3\beta_M}{8\pi}\int_0^{\Lambda_0^2} dQ^2 Q^2 \left(\frac{m_0^2}{m_0^2 + Q^2}\right)^2$$

where $m_0^2 = 0.71\ \mathrm{GeV}^2$ and $\beta_{p-n} = (-1 \pm 1) \times 10^{-4}\ \mathrm{fm}^3$ taken from H.W. Griesshammer, *et al.*, Prog. Part. Nucl. Phys. 67(2012)841,

$$\delta M^{inel}_{sub}|_{p-n} = 0.47(47)\ \mathrm{MeV}$$

- counter term

$$\delta \tilde{M}^{ct}_{p-n} = 3\alpha \ln\left(\frac{\Lambda_0^2}{\Lambda_1^2}\right) \frac{e_u^2 m_u - e_d^2 m_d}{8\pi M\delta} \langle p|\delta(\bar{u}u - \bar{d}d)|p\rangle$$

where $\delta = (m_d - m_u)/2$.

In QCD, $m_{u,d} \sim \delta$, so this contribution is numerically second order in isospin breaking, $\mathcal{O}(\alpha\delta)$.

With $\Lambda_1^2 = 100\ \mathrm{GeV}^2$ and $\Lambda_0^2 = 2\ \mathrm{GeV}^2$,

$$|\delta \tilde{M}^{ct}_{p-n}| < 0.02\ \mathrm{MeV}.$$

Finally, the WLCM analysis led to a significantly larger numerical value for the electromagnetic contribution to the p-n mass difference but with a rather large error,

$$\delta M^{\gamma}|_{p-n} = 1.30(03)(47)\ \mathrm{MeV}\ ,$$

Lattice Simulations: QED+QCD

- T. Blum *et al.*, Phys. Rev. D 82(2010)094508

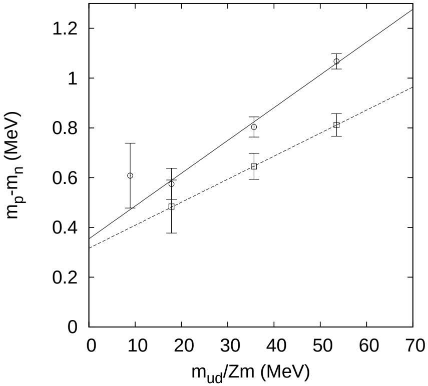


Figure: The p-n mass difference due to QED interactions. 16^3 (squares) and 24^3 (circles) lattice sizes. The lattice cutoff $a^{-1} \approx 1.78\ \mathrm{GeV}$.

LatticeSize	$(M_p - M_n)\|_{QED}$	$(M_p - M_n)\|_{QCD}$	$(M_p - M_n)$
16^3	0.33(11)	−2.265(70)	−1.93(12)
24^3	0.383(68)	−2.51(14)	−2.13(16)

Table: Proton and neutron mass difference due to QED and QCD interactions, in unit of MeV.

- Sz. Borsanyi *et al.* (BMW Collaboration), Phys. Rev. Lett. 111(2013)252001

$(M_p - M_n)\|_{QED}$	$(M_p - M_n)\|_{QCD}$	$(M_p - M_n)$
1.59(30)(35)	−2.28(25)(7)	−0.68(39)(36)

Table: Proton and neutron mass difference due to QED and QCD interactions, in units of MeV. These results are obtained by extrapolating to infinite volume limit.

Finite volume correction for EM contributions should be significant!

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Our work

- We make a combined analysis of WLCM formalism with lattice simulations.

$$WLCM(L, m_\pi) = LQCD(L, m_\pi)$$

- Provide constraint on δM_{sub}^{inel}
- Extract the isovector nucleon magnetic polarizability β_{p-n}
- Reduce the uncertainty of δM_{p-n}^{γ}

Finite volume version

$$\delta M^{el}(L) = \frac{2\pi\alpha}{L^3} \sum_{|Q|\neq 0} \frac{1}{Q^2} \left\{ \frac{3\sqrt{\tau_{el}}G_M^2}{2(1+\tau_{el})} + \frac{[G_E^2 - 2\tau_{el}G_M^2]}{1+\tau_{el}} \right.$$

$$\left. \times \left[1+\tau_{el})^{3/2} - \tau_{el}^{3/2} - \frac{3}{2}\sqrt{\tau_{el}} \right] \right\} ,$$

$$\delta M^{el}_{sub}(L) = -\frac{3\alpha\pi}{4M}\frac{1}{L^3}\sum_{Q\neq 0}\frac{1}{|Q|}[2G_M^2 - 2F_1^2]$$

The integral over continuous variable $\vec{Q}$ is replaed by a sum over the discrete values,

$$\vec{Q} = \frac{2\pi}{L}\vec{n} , \quad \vec{n} \in \mathbb{Z}^3 .$$

EM Form Factors

In order to evaluate Eqs. (1) as a function of quark, or equivalently pion mass, we use a parametrization of lattice data for the nucleon isovector and isoscalar form factors introduced in J.D. Ashley *et al.*, Eur. Phys. J. A 19(2004)9,

$$
\begin{aligned}
G_M^{v,s} &= \frac{\mu_{v,s}(m_\pi)}{(1+Q^2/(\Lambda_M^{v,s})^2)^2} \,, \\
G_E^{v,s} &= \frac{1}{(1+Q^2/(\Lambda_E^{v,s})^2)^2} \,.
\end{aligned}
$$

The electromagnetic form factors of proton and neutron can be reconstructed by

$$
G^p(Q^2, m_\pi) = \frac{1}{2}(G^s + G^v) \,, \quad G^n(Q^2, m_\pi) = \frac{1}{2}(G^s - G^v) \,.
$$

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Following D.B. Leinweber, Phys. Rev. D 60(1999)034014, one can use a Padé approximant to parametrize the magnetic moments

$$\mu_i(m_\pi) = \frac{\mu_0}{1 - \frac{\chi_i}{\mu_0} m_\pi + c m_\pi^2} \; ,$$

- $\chi_v = -8.82$ and $\chi_s = 0$ determined model independently from chiral perturbation theory
- μ_0 and c are determined by fitting lattice data M. Gockeler *et al.* (QCDSF Collaboration), Phys. Rev. D 71(2005)034508

The dipole masses of isovector magnetic and electric form factors are parametrized as

$$(\Lambda_M^v)^2 = \frac{12(1+A_1 m_\pi^2)}{A_0 + \frac{\chi_1}{m_\pi}\frac{2}{\pi}\arctan(\mu/m_\pi) + \frac{\chi_2}{2}\ln(\frac{m_\pi^2}{m_\pi^2+\mu^2})},$$

$$(\Lambda_E^v)^2 = \frac{12(1+B_1 m_\pi^2)}{B_0 + \frac{\chi_2}{2}\ln(\frac{m_\pi^2}{m_\pi^2+\mu^2})},$$

where

$$\chi_1 = \frac{g_A^2 m_N}{8\pi f_\pi^2 \kappa_v}, \qquad \chi_2 = -\frac{5g_A^2+1}{8\pi^2 f_\pi^2},$$

with $g_A = 1.27$ the axial coupling constant and $f_\pi = 93\mathrm{MeV}$ the pion decay constant. $m_N = 940\mathrm{MeV}$ is the nucleon mass and $\kappa_v = 4.2$ is the isovector anomalous magnetic moment of the nucleon (in the chiral limit).

The isoscalar dipole masses are observed to be roughly linear in m_π^2,

$$(\Lambda^s_{E,M})^2 = a_{E,M} + b_{E,M} m_\pi^2 \ .$$

The parameters are determined by fitting lattice data from QCDSF Collaboration M. Gockeler *et al.* (QCDSF Collaboration), Phys. Rev. D 71(2005)034508

$$\begin{aligned} A_0 &= 8.65\ , & A_1 = 0.28\ , \\ B_0 &= 11.71\ , & B_1 = 0.72\ , \\ a_E &= 1.09\ , & b_E = 0.85\ , \\ a_M &= 1.09\ , & b_M = 0.68\ , \end{aligned}$$

and the arbitrary scale parameter $\mu = 0.14\ \mathrm{GeV}$.

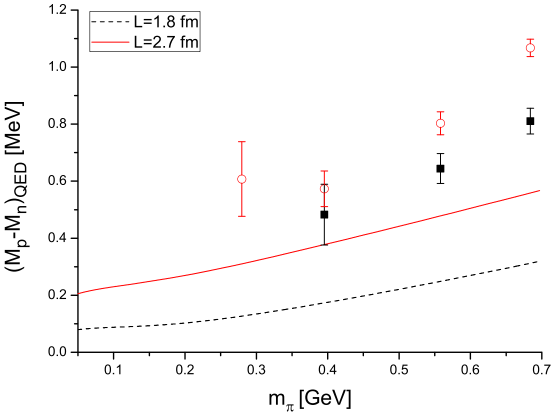

Figure: Total elastic contribution to nucleon mass splitting at finite volume. Lattice data are taken from Blum, PRD 82(2010)094508, 16^3 (squares) and 24^3 (circles) lattice sizes.

The discrepancy between the curves and lattice data may be compensated by including the inelastic contribution.

- $\delta M^{inel} = 0.057$ MeV (WLCM)
-

$$\delta M^{inel}_{sub}(L) = -\frac{3\pi\beta_{p-n}}{2}\frac{1}{L^3}\sum_{Q\neq 0}|Q|\left(\frac{(\Lambda^v_M)^2}{(\Lambda^v_M)^2+Q^2}\right)^3 , \quad (1a)$$

$$\delta M^{inel}_{sub}(L) = -\frac{3\pi\beta_{p-n}}{2}\frac{1}{L^3}\sum_{Q\neq 0}|Q|\left(\frac{(\Lambda^v_M)^2}{(\Lambda^v_M)^2+Q^2}\right)^4 , \quad (1b)$$

$\beta_{p-n}(m_\pi)$

Table: The magnetic polarizability β_{p-n} as a function of m_π, in units of $10^{-4}\ \mathrm{fm}^3$. δM_{sub}^{inel} is given by Eq. (1a).

m_π[GeV]	0.279	0.394	0.558	0.683
16^3		-0.246 ± 0.103	-0.258 ± 0.040	-0.294 ± 0.030
24^3	-0.316 ± 0.171	-0.134 ± 0.060	-0.202 ± 0.030	-0.298 ± 0.020

Table: The magnetic polarizability β_{p-n} as a function of m_π, in units of $10^{-4}\ \mathrm{fm}^3$. δM_{sub}^{inel} is given by Eq. (1b).

m_π[GeV]	0.279	0.394	0.558	0.683
16^3		-0.733 ± 0.307	-0.756 ± 0.118	-0.855 ± 0.087
24^3	-0.917 ± 0.498	-0.385 ± 0.172	-0.578 ± 0.087	-0.847 ± 0.057

Chiral extrapolation

The nucleon electromagnetic polarizabilities have been investigated in heavy baryon chiral perturbation theory V. Bernard, Phys. Rev. Lett. 67(1991)1515; Phys. Lett. B 319(1993)269.

The quantity β_{p-n} does not depend on the unknown low energy constants c_2 and c^+. The $1/m_\pi$ terms in β_p and β_n will cancel each other.

Finally we get

$$\beta_{p-n}(m_\pi) = c_l \ln \frac{m_\pi}{M_N} + c_0 + c_1 \frac{m_\pi}{M_N} ,$$

with the model independent coefficient $c_l = 2.51 \times 10^{-4}\ \mathrm{fm}^3$ fixed by chiral perturbation theory.

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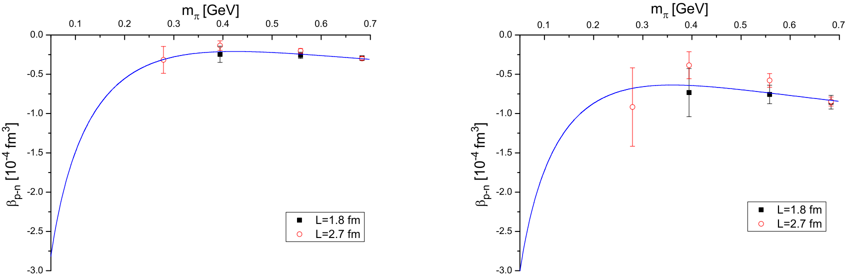

Figure: (left): β_{p-n} given by fitting Tab. 4; (right): β_{p-n} given by fitting Tab. 5

Physical result

Table: Fitted parameters and extrapolated β_{p-n} at physical pion mass, in unit of $10^{-4}\ \mathrm{fm}^3$.

	c_0	c_1	$\chi^2_{d.o.f}$	β^{phy}_{p-n}
cubic	4.83 ± 0.12	-6.88 ± 0.27	$8.19/(7-2) = 1.64$	-0.98 ± 0.12
quartic	4.68 ± 0.34	-7.69 ± 0.78	$8.68/(7-2) = 1.74$	-1.25 ± 0.36

We make a conservative estimate by taking the average value of cubic and quartic results,

$$\beta_{p-n} = (-1.12 \pm 0.40) \times 10^{-4}\ \mathrm{fm}^3 .$$

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In continuous limit, the inelastic subtraction term contributes to the electromagnetic p-n mass splitting as

$$\delta M_{sub}^{inel}|_{p-n} = -\frac{3\beta_{p-n}}{8\pi}\int_0^\infty dQ^2 Q^2 \left(\frac{(\Lambda_M^v)^2}{(\Lambda_M^v)^2 + Q^2}\right)^3 = 0.30 \pm 0.04 \ \mathrm{MeV} \ , \quad (2a)$$

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Again, we take the average value of the above two results,

$$\delta M_{sub}^{inel}|_{p-n} = 0.21 \pm 0.11 \ \mathrm{MeV} \ .$$

Final result

Combining with the precisely determined part of WLCM analysis,

$$(\delta M^{el} + \delta M^{inel} + \delta M^{el}_{sub})|_{p-n} = 0.83 \pm 0.03 \ \mathrm{MeV} \ .$$

We finally obtain the total electromagnetic contribution to the proton-neutron mass splitting,

$$\delta M^{\gamma}_{p-n} = 1.04 \pm 0.11 \ \mathrm{MeV} \ .$$

Summary

- We make a combined analysis of WLCM formalism with lattice simulations.
- The isovector nucleon magnetic polarizability β_{p-n} is extracted as a function of pion mass.
- At physical pion mass,

$$\begin{aligned} \beta_{p-n} &= (-1.12 \pm 0.40) \times 10^{-4}\ \mathrm{fm}^3 \\ \delta M^{\gamma}_{p-n} &= 1.04 \pm 0.11\ \mathrm{MeV} \end{aligned}$$

Thanks for your patience