Electromagnetic Mass Splitting of Proton and Neutron

X. G. Wang
In Collaboration with A. W. Thomas and R. D. Young

CSSM, University of Adelaide

Apr. 11, 2014
Physical interest

The physical proton-neutron mass splitting has been measured extremely precisely,

\[ M_p - M_n = -1.2933322(4) \text{ MeV}. \]

Separation of the electromagnetic and \( u - d \) quark mass difference contribution to the p-n mass difference is of enormous interest

\[ M_p - M_n = (M_p - M_n)|_{QED} + (M_p - M_n)|_{QCD} \]
The physical proton-neutron mass splitting has been measured extremely precisely,

\[ M_p - M_n = -1.2933322(4) \text{ MeV}. \]

Separation of the electromagnetic and $u - d$ quark mass difference contribution to the p-n mass difference is of enormous interest

\[ M_p - M_n = (M_p - M_n)\vert_{QED} + (M_p - M_n)\vert_{QCD} \]

$SU(3) \chi PT \oplus$ Lattice data on octet baryon masses

|                | $(M_p - M_n)|_{QCD}$ [MeV] |
|----------------|---------------------------|
| PACS-CS        | $-2.9 \pm 0.4$            |
| QCDSF-UKQCD    | $-2.4 \pm 0.3$            |

**Table:** Proton and neutron mass difference due to QCD interactions, corresponding to $R = \frac{m_u}{m_d} = 0.553 \pm 0.043$. 
The electromagnetic self-energy of the nucleon,

\[ \delta M^\gamma = \frac{i}{2M} \frac{\alpha}{(2\pi)^3} \int_R d^4q \frac{T_{\mu\nu}^\mu(p, q)}{q^2 + i\epsilon}, \]

where \( T_{\mu\nu} \) is the spin averaged forward Compton scattering tensor

\[ T_{\mu\nu} = \frac{i}{2} \sum_\sigma \int d^4xe^{iq\cdot x} \langle p\sigma| T\{J_\mu(x)J_\nu(0)\}|p\sigma\rangle, \]

\[ T_{\mu\nu}(\nu, q^2) = \]

\[ -(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) T_1(\nu, q^2) + \frac{1}{M^2} (p_\mu - \frac{p\cdot q}{q^2} q_\mu) (p_\nu - \frac{p\cdot q}{q^2} q_\nu) T_2(\nu, q^2) \]
The electromagnetic self-energy of the nucleon,

\[
\delta M^\gamma = \frac{i}{2M} \frac{\alpha}{(2\pi)^3} \int_R d^4q \frac{T^\mu_\mu(p, q)}{q^2 + i\epsilon},
\]

where \( T^\mu_\nu \) is the spin averaged forward Compton scattering tensor

\[
T^\mu_\nu = \frac{i}{2} \sum_\sigma \int d^4x e^{iq \cdot x} \langle p\sigma| T\{J^\mu(x)J^\nu(0)\}|p\sigma \rangle,
\]

\[
T^\mu_\nu(\nu, q^2) = -(g^\mu_\nu - \frac{q^\mu q_\nu}{q^2}) T_1(\nu, q^2) + \frac{1}{M^2}(p^\mu - \frac{p \cdot q}{q^2}q^\mu)(p^\nu - \frac{p \cdot q}{q^2}q^\nu) T_2(\nu, q^2).
\]
The imaginary part of a forward scattering amplitude arises from a sum of contributions from all possible intermediate states.
Establish unsubtracted dispersion representations for $T_1(\nu, Q^2)$ and $T_2(\nu, Q^2)$ in respect to $\nu$

Separation between elastic and inelastic contributions

$$T_i = T_i^{el} + T_i^{inel}$$

where

$$T_i^{el} \sim G_{E,M} \quad T_i^{inel} \sim F_{1,2}(\nu, Q^2)$$
The dominate contribution comes from the elastic intermediate state exchange,

\[ \delta M^\gamma \approx \delta M^\gamma_{el} = 0.76 \pm 0.30 \text{ MeV} \]

Renormalization

\[ T_i(\nu, Q^2) \sim 1/Q^2 \ (Q^2 \to \infty) \]

Possible Subtraction

\[ T_1(\nu, Q^2): \text{once subtraction} \]
\[ T_2(\nu, Q^2): \text{no subtraction} \]
Recently, Walker-Loud \textit{et al.}(WLCM) reanalyzed EM mass splitting of proton and neutron with

- Subtracted dispersion representation

\[ \delta M^\gamma = \delta M^{el} + \delta M^{inel} + \delta M^{el}_{sub} + \delta M^{inel}_{sub} + \delta \tilde{M}^{ct}. \]

- Modern knowledge on $G_{E,M}$ and $F_{1,2}(x, Q^2)$ from the latest accurate experiments.

Elastic contribution

\[
\delta M^{el} = \frac{\alpha}{\pi} \int_0^{\Lambda^0} dQ \left\{ \frac{3\sqrt{\tau_{el}} G_M^2}{2(1 + \tau_{el})} + \frac{[G_E^2 - 2\tau_{el} G_M^2]}{1 + \tau_{el}} \times \left[ 1 + \tau_{el}\right]^{3/2} - \tau_{el}^{3/2} - \frac{3}{2} \sqrt{\tau_{el}} \right\},
\]

with \( \tau_{el} = Q^2 / 4M^2 \).

\( G_E, G_M \): EM Form Factors using Kelly parametrization

\[
\delta M^{el}|_{p-n} = 1.39(02) \text{ MeV}
\]
Inelastic contribution

\[
\delta M^{\text{inel}} = \frac{\alpha}{\pi} \int_{\nu}^{\Lambda_0} dQ \int_{\nu_{th}}^{\infty} \left\{ \frac{3 F_1(\nu, Q^2)}{M} \left[ \frac{\tau^{3/2} - \tau \sqrt{1 + \tau} + \sqrt{\tau/2}}{\tau} \right] \\
+ \frac{F_2(\nu, Q^2)}{\nu} \left[ (1 + \tau)^{3/2} - \tau^{3/2} - \frac{3}{2} \sqrt{\tau} \right] \right\}
\]

with \( \tau = \nu^2 / Q^2 \).

\( F_1, F_2 \): Structure functions from Bosted and Christy


\[
\delta M^{\text{inel}} |_{p-n} = 0.057(16) \text{ MeV}
\]
elastic subtraction term

\[ \delta M_{\text{sub}}^{el} = -\frac{3\alpha}{16\pi M} \int_0^{\Lambda_0^2} dQ^2 [2G_M^2(Q^2) - 2F_1^2(Q^2)] = -0.62(02) \text{ MeV} \]

inelastic subtraction term

\[ \delta M_{\text{sub}}^{inel} = -\frac{3\beta}{8\pi} \int_0^{\Lambda_0^2} dQ^2 Q^2 \left( \frac{m_0^2}{m_0^2 + Q^2} \right)^2 \]

where \( m_0^2 = 0.71 \text{ GeV}^2 \) and \( \beta_{p-n} = (-1 \pm 1) \times 10^{-4} \text{ fm}^3 \)

taken from H.W. Griesshammer, et al., Prog. Part. Nucl. Phys. 67(2012)841,

\[ \delta M_{\text{sub}}^{inel} |_{p-n} = 0.47(47) \text{ MeV} \]
counter term

\[ \delta \tilde{M}_{p-n}^{ct} = 3 \alpha \ln \left( \frac{\Lambda_0^2}{\Lambda_1^2} \right) \frac{e_u^2 m_u - e_d^2 m_d}{8 \pi M \delta} \langle p | \delta (\bar{u} u - \bar{d} d) | p \rangle \]

where \( \delta = (m_d - m_u)/2. \)

In QCD, \( m_u, d \sim \delta \), so this contribution is numerically second order in isospin breaking, \( O(\alpha \delta) \).

With \( \Lambda_1^2 = 100 \text{ GeV}^2 \) and \( \Lambda_0^2 = 2 \text{ GeV}^2 \),

\[ |\delta \tilde{M}_{p-n}^{ct}| < 0.02 \text{ MeV}. \]
Finally, the WLCM analysis led to a significantly larger numerical value for the electromagnetic contribution to the p-n mass difference but with a rather large error,

\[ \delta M^\gamma |_{p-n} = 1.30(03)(47) \text{ MeV}, \]
Figure: The p-n mass difference due to QED interactions. $16^3$ (squares) and $24^3$ (circles) lattice sizes. The lattice cutoff $a^{-1} \approx 1.78$ GeV.
| Lattice Size | $(M_p - M_n)|_{QED}$  | $(M_p - M_n)|_{QCD}$  | $(M_p - M_n)$ |
|--------------|------------------------|------------------------|----------------|
| $16^3$       | 0.33(11)               | -2.265(70)             | -1.93(12)      |
| $24^3$       | 0.383(68)              | -2.51(14)              | -2.13(16)      |

**Table:** Proton and neutron mass difference due to QED and QCD interactions, in unit of MeV.

\[
(M_p - M_n)_{|QED} \quad (M_p - M_n)_{|QCD} \quad (M_p - M_n)
\]

\[
\begin{array}{ccc}
1.59(30)(35) & -2.28(25)(7) & -0.68(39)(36)
\end{array}
\]

Table: Proton and neutron mass difference due to QED and QCD interactions, in units of MeV. These results are obtained by extrapolating to infinite volume limit.

Finite volume correction for EM contributions should be significant!

\[
(M_p - M_n)_{\text{QED}} \quad (M_p - M_n)_{\text{QCD}} \quad (M_p - M_n)
\]

1.59(30)(35) \quad -2.28(25)(7) \quad -0.68(39)(36)

Table: Proton and neutron mass difference due to QED and QCD interactions, in units of MeV. These results are obtained by extrapolating to infinite volume limit.

Finite volume correction for EM contributions should be significant!
Our work

- We make a combined analysis of WLCM formalism with lattice simulations.

\[ WLCM(L, m_\pi) = LQCD(L, m_\pi) \]

- Provide constraint on \( \delta M_{\text{inel}}^{\text{sub}} \)
- Extract the isovector nucleon magnetic polarizability \( \beta_{p-n} \)
- Reduce the uncertainty of \( \delta M_{p-n}^{\gamma} \)
\[ \delta M_{\text{el}}(L) = \frac{2\pi \alpha}{L^3} \sum_{|Q| \neq 0} \frac{1}{Q^2} \left\{ \frac{3\sqrt{\tau_{el}} G^2_M}{2(1 + \tau_{el})} + \frac{[G_E^2 - 2\tau_{el} G^2_M]}{1 + \tau_{el}} \right\} \times \left[ 1 + \tau_{el} \right]^{3/2} - \tau_{el}^{3/2} - \frac{3}{2}\sqrt{\tau_{el}} \right\} \Bigg) , \]

\[ \delta M_{\text{sub}}(L) = -\frac{3\alpha\pi}{4M L^3} \sum_{Q \neq 0} \frac{1}{|Q|} [2G^2_M - 2F^2_1] \]

The integral over continuous variable \( \vec{Q} \) is replaced by a sum over the discrete values,

\[ \vec{Q} = \frac{2\pi}{L} \vec{n} , \quad \vec{n} \in \mathbb{Z}^3 . \]
In order to evaluate Eqs. (1) as a function of quark, or equivalently pion mass, we use a parametrization of lattice data for the nucleon isovector and isoscalar form factors introduced in J.D. Ashley et al., Eur. Phys. J. A 19(2004)9,

\[
G_{M}^{v,s} = \mu_{v,s}(m_{\pi}) \frac{1 + Q^{2}/(\Lambda_{M}^{v,s})^{2}}{(1 + Q^{2}/(\Lambda_{M}^{v,s})^{2})^{2}} ,
\]
\[
G_{E}^{v,s} = \frac{1}{(1 + Q^{2}/(\Lambda_{E}^{v,s})^{2})^{2}} .
\]

The electromagnetic form factors of proton and neutron can be reconstructed by

\[
G^{p}(Q^{2}, m_{\pi}) = \frac{1}{2}(G^{s} + G^{v}) , \quad G^{n}(Q^{2}, m_{\pi}) = \frac{1}{2}(G^{s} - G^{v}) .
\]
In order to evaluate Eqs. (1) as a function of quark, or equivalently pion mass, we use a parametrization of lattice data for the nucleon isovector and isoscalar form factors introduced in J.D. Ashley et al., Eur. Phys. J. A 19(2004)9,

\[ G_{M}^{v,s} = \frac{\mu_{v,s}(m_{\pi})}{(1 + Q^2/(\Lambda_{M}^{v,s})^2)^2}, \]
\[ G_{E}^{v,s} = \frac{1}{(1 + Q^2/(\Lambda_{E}^{v,s})^2)^2}. \]

The electromagnetic form factors of proton and neutron can be reconstructed by

\[ G^{p}(Q^2, m_{\pi}) = \frac{1}{2}(G^{s} + G^{v}), \quad G^{n}(Q^2, m_{\pi}) = \frac{1}{2}(G^{s} - G^{v}). \]
Following D.B. Leinweber, Phys. Rev. D 60(1999)034014, one can use a Padé approximant to parametrize the magnetic moments

$$\mu_i(m_\pi) = \frac{\mu_0}{1 - \frac{\chi_i}{\mu_0} m_\pi + cm_\pi^2},$$

- $\chi_v = -8.82$ and $\chi_s = 0$ determined model independently from chiral perturbation theory
- $\mu_0$ and $c$ are determined by fitting lattice data M. Gockeler et al. (QCDSF Collaboration), Phys. Rev. D 71(2005)034508
The dipole masses of isovector magnetic and electric form factors are parametrized as

\[
\left(\Lambda_{M}^{v}\right)^2 = \frac{12(1 + A_1 m_{\pi}^2)}{A_0 + \frac{\chi_1}{m_{\pi}} \frac{2}{\pi} \arctan(\mu/m_{\pi}) + \frac{\chi_2}{2} \ln\left(\frac{m_{\pi}^2}{m_{\pi}^2 + \mu^2}\right)} ,
\]

\[
\left(\Lambda_{E}^{v}\right)^2 = \frac{12(1 + B_1 m_{\pi}^2)}{B_0 + \frac{\chi_2}{2} \ln\left(\frac{m_{\pi}^2}{m_{\pi}^2 + \mu^2}\right)} ,
\]

where

\[
\chi_1 = \frac{g_A^2 m_N}{8\pi f_{\pi}^2 \kappa_v} , \quad \chi_2 = -\frac{5g_A^2 + 1}{8\pi^2 f_{\pi}^2} ,
\]

with \(g_A = 1.27\) the axial coupling constant and \(f_{\pi} = 93\text{MeV}\) the pion decay constant. \(m_N = 940\text{MeV}\) is the nucleon mass and \(\kappa_v = 4.2\) is the isovector anomalous magnetic moment of the nucleon (in the chiral limit).
The isoscalar dipole masses are observed to be roughly linear in $m_{\pi}^2$,

$$(\Lambda_{E,M}^s)^2 = a_{E,M} + b_{E,M} m_{\pi}^2.$$ 

The parameters are determined by fitting lattice data from QCDSF Collaboration M. Gockeler et al. (QCDSF Collaboration), Phys. Rev. D 71(2005)034508

$$A_0 = 8.65, \quad A_1 = 0.28,$$
$$B_0 = 11.71, \quad B_1 = 0.72,$$
$$a_E = 1.09, \quad b_E = 0.85,$$
$$a_M = 1.09, \quad b_M = 0.68,$$

and the arbitrary scale parameter $\mu = 0.14$ GeV.
Figure: Total elastic contribution to nucleon mass splitting at finite volume. Lattice data are taken from Blum, PRD 82(2010)094508, $16^3$ (squares) and $24^3$ (circles) lattice sizes.
The discrepancy between the curves and lattice data may be compensated by including the inelastic contribution.

\[ \delta M^{inel} = 0.057 \text{ MeV} \quad (\text{WLCM}) \]

\[
\delta M_{sub}^{inel}(L) = -\frac{3\pi\beta_{p-n}}{2} \frac{1}{L^3} \sum_{Q \neq 0} |Q| \left( \frac{(\Lambda_M^{\nu})^2}{(\Lambda_M^{\nu})^2 + Q^2} \right)^3 , \quad (1a)
\]

\[
\delta M_{sub}^{inel}(L) = -\frac{3\pi\beta_{p-n}}{2} \frac{1}{L^3} \sum_{Q \neq 0} |Q| \left( \frac{(\Lambda_M^{\nu})^2}{(\Lambda_M^{\nu})^2 + Q^2} \right)^4 , \quad (1b)
\]
Table: The magnetic polarizability $\beta_{p-n}$ as a function of $m_\pi$, in units of $10^{-4}$ fm$^3$. $\delta M_{sub}^{inel}$ is given by Eq. (1a).

<table>
<thead>
<tr>
<th>$m_\pi$ [GeV]</th>
<th>0.279</th>
<th>0.394</th>
<th>0.558</th>
<th>0.683</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16^3$</td>
<td>−0.246 $\pm$ 0.103</td>
<td>−0.258 $\pm$ 0.040</td>
<td>−0.294 $\pm$ 0.030</td>
<td></td>
</tr>
<tr>
<td>$24^3$</td>
<td>−0.316 $\pm$ 0.171</td>
<td>−0.134 $\pm$ 0.060</td>
<td>−0.202 $\pm$ 0.030</td>
<td>−0.298 $\pm$ 0.020</td>
</tr>
</tbody>
</table>

Table: The magnetic polarizability $\beta_{p-n}$ as a function of $m_\pi$, in units of $10^{-4}$ fm$^3$. $\delta M_{sub}^{inel}$ is given by Eq. (1b).

<table>
<thead>
<tr>
<th>$m_\pi$ [GeV]</th>
<th>0.279</th>
<th>0.394</th>
<th>0.558</th>
<th>0.683</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16^3$</td>
<td>−0.733 $\pm$ 0.307</td>
<td>−0.756 $\pm$ 0.118</td>
<td>−0.855 $\pm$ 0.087</td>
<td></td>
</tr>
<tr>
<td>$24^3$</td>
<td>−0.917 $\pm$ 0.498</td>
<td>−0.385 $\pm$ 0.172</td>
<td>−0.578 $\pm$ 0.087</td>
<td>−0.847 $\pm$ 0.057</td>
</tr>
</tbody>
</table>
Chiral extrapolation


The quantity $\beta_{p-n}$ does not depend on the unknown low energy constants $c_2$ and $c^+$. The $1/m_\pi$ terms in $\beta_p$ and $\beta_n$ will cancel each other.

Finally we get

$$\beta_{p-n}(m_\pi) = c_I \ln \frac{m_\pi}{M_N} + c_0 + c_1 \frac{m_\pi}{M_N},$$

with the model independent coefficient $c_I = 2.51 \times 10^{-4}$ fm$^3$ fixed by chiral perturbation theory.

The quantity $\beta_{p-n}$ does not depend on the unknown low energy constants $c_2$ and $c^+$. The $1/m_\pi$ terms in $\beta_p$ and $\beta_n$ will cancel each other.

Finally we get

$$\beta_{p-n}(m_\pi) = c_I \ln \frac{m_\pi}{M_N} + c_0 + c_1 \frac{m_\pi}{M_N},$$

with the model independent coefficient $c_I = 2.51 \times 10^{-4} \text{ fm}^3$ fixed by chiral perturbation theory.

The quantity $\beta_{p-n}$ does not depend on the unknown low energy constants $c_2$ and $c^+$. The $1/m_\pi$ terms in $\beta_p$ and $\beta_n$ will cancel each other.

Finally we get

$$\beta_{p-n}(m_\pi) = c_l \ln \frac{m_\pi}{M_N} + c_0 + c_1 \frac{m_\pi}{M_N},$$

with the model independent coefficient $c_l = 2.51 \times 10^{-4}$ fm$^3$ fixed by chiral perturbation theory.
Figure: (left): $\beta_{p-n}$ given by fitting Tab. 4; (right): $\beta_{p-n}$ given by fitting Tab. 5
Table: Fitted parameters and extrapolated $\beta_{p-n}$ at physical pion mass, in unit of $10^{-4}$ fm$^3$.

<table>
<thead>
<tr>
<th></th>
<th>$c_0$</th>
<th>$c_1$</th>
<th>$\chi^2_{d.o.f}$</th>
<th>$\beta_{p-n}^{\text{phy}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubic</td>
<td>4.83 ± 0.12</td>
<td>−6.88 ± 0.27</td>
<td>8.19/(7 − 2) = 1.64</td>
<td>−0.98 ± 0.12</td>
</tr>
<tr>
<td>quartic</td>
<td>4.68 ± 0.34</td>
<td>−7.69 ± 0.78</td>
<td>8.68/(7 − 2) = 1.74</td>
<td>−1.25 ± 0.36</td>
</tr>
</tbody>
</table>

We make a conservative estimate by taking the average value of cubic and quartic results,

$$\beta_{p-n} = (−1.12 ± 0.40) \times 10^{-4} \text{ fm}^3.$$
### Fitted parameters and extrapolated $\beta_{p-n}$ at physical pion mass, in unit of $10^{-4}$ fm$^3$.

<table>
<thead>
<tr>
<th></th>
<th>$c_0$</th>
<th>$c_1$</th>
<th>$\chi^2_{d.o.f}$</th>
<th>$\beta_{p-n}^{\text{phy}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cubic</td>
<td>4.83 ± 0.12</td>
<td>−6.88 ± 0.27</td>
<td>8.19/(7 − 2) = 1.64</td>
<td>−0.98 ± 0.12</td>
</tr>
<tr>
<td>quartic</td>
<td>4.68 ± 0.34</td>
<td>−7.69 ± 0.78</td>
<td>8.68/(7 − 2) = 1.74</td>
<td>−1.25 ± 0.36</td>
</tr>
</tbody>
</table>

We make a conservative estimate by taking the average value of cubic and quartic results,

$$\beta_{p-n} = (-1.12 \pm 0.40) \times 10^{-4} \text{ fm}^3.$$
In continuous limit, the inelastic subtraction term contributes to the electromagnetic p-n mass splitting as

\[
\delta M_{\text{sub}}^{\text{inel}} |_{p-n} = - \frac{3 \beta \rho_{p-n}}{8 \pi} \int_0^\infty dQ^2 Q^2 \left( \frac{(\Lambda^\nu_M)^2}{(\Lambda^\nu_M)^2 + Q^2} \right)^3
\]

\[
= 0.30 \pm 0.04 \text{ MeV} ,
\]

\[
\delta M_{\text{sub}}^{\text{inel}} |_{p-n} = - \frac{3 \beta \rho_{p-n}}{8 \pi} \int_0^\infty dQ^2 Q^2 \left( \frac{(\Lambda^\nu_M)^2}{(\Lambda^\nu_M)^2 + Q^2} \right)^4
\]

\[
= 0.12 \pm 0.04 \text{ MeV} .
\]

Again, we take the average value of the above two results,

\[
\delta M_{\text{sub}}^{\text{inel}} |_{p-n} = 0.21 \pm 0.11 \text{ MeV} .
\]
Combining with the precisely determined part of WLCM analysis,

\[
(\delta M^e + \delta M^{inel} + \delta M^{el}_{sub})|_{p-n} = 0.83 \pm 0.03 \text{ MeV}.
\]

We finally obtain the total electromagnetic contribution to the proton-neutron mass splitting,

\[
\delta M^\gamma_{p-n} = 1.04 \pm 0.11 \text{ MeV}.
\]
We make a combined analysis of WLCM formalism with lattice simulations.

The isovector nucleon magnetic polarizability $\beta_{p-n}$ is extracted as a function of pion mass.

At physical pion mass,

$$\beta_{p-n} = (-1.12 \pm 0.40) \times 10^{-4} \text{ fm}^3$$

$$\delta M_{\gamma}^{p-n} = 1.04 \pm 0.11 \text{ MeV}$$
Thanks for your patience