Gaussian functional method for chiral mesons in the linear sigma model

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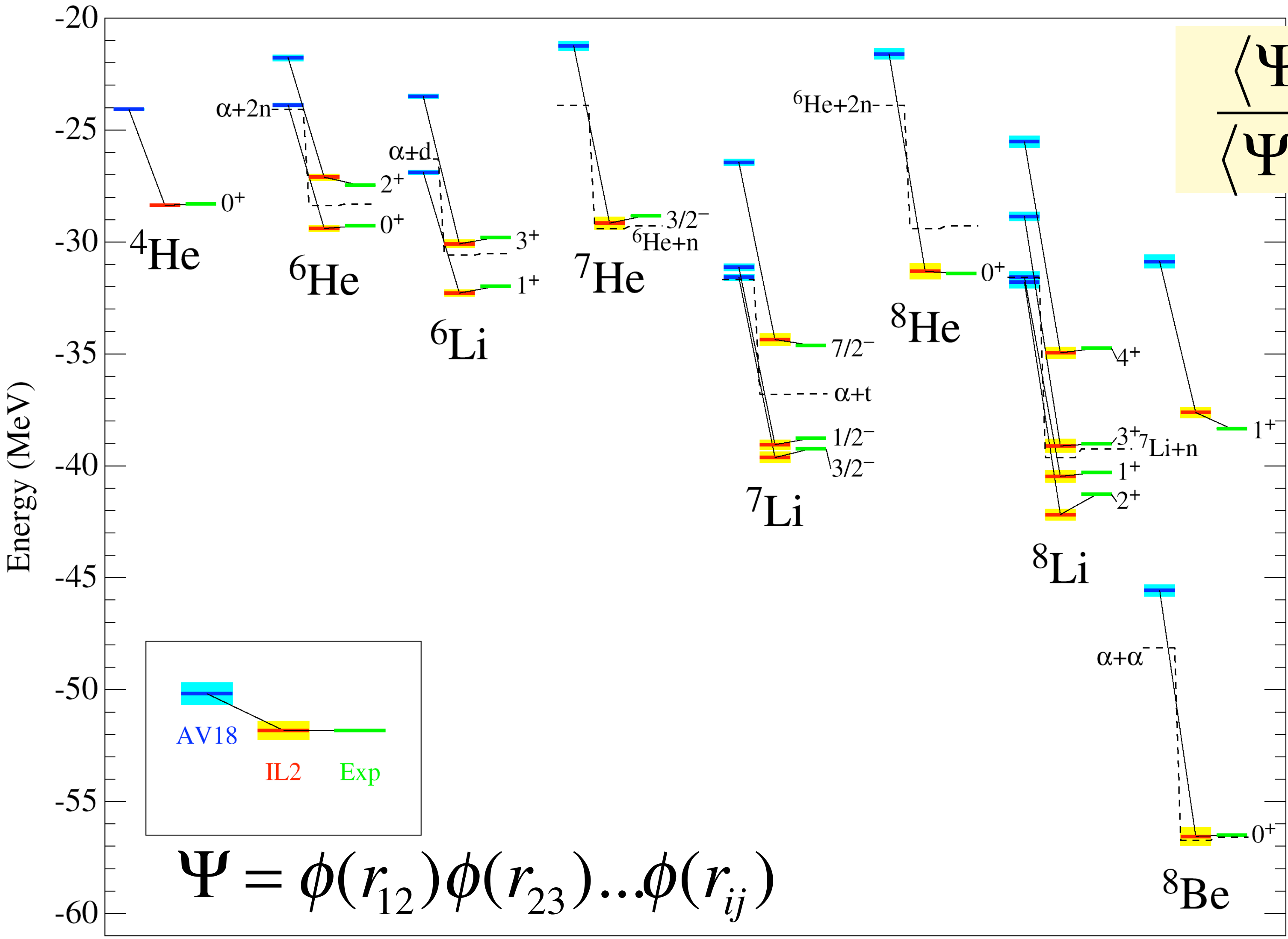
Many reasons to study pion and sigma mesons (chiral symmetry)

1. Mediator of strong interaction (Yukawa particle)
2. Play important role in Nuclear Physics
3. Nambu-Goldstone boson of chiral symmetry
4. Linear sigma model is a beautiful (simple) Lagrangian
5. Sigma meson is a Higgs boson in strong interaction
6. Non-linear sigma model is used phenomenologically
7. Renewed interest in linear sigma model
8. ⋯

Variational calculation of few body system with NN interaction

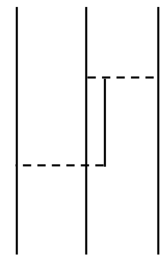
VMC+GFMC

$$\frac{\langle \Psi | V_\pi | \Psi \rangle}{\langle \Psi | V_{NN} | \Psi \rangle} \sim 80\%$$

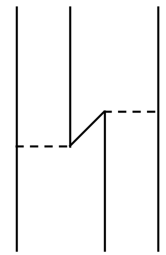


$$\Psi = \phi(r_{12})\phi(r_{23})\ldots\phi(r_{ij})$$

V_{NNN}



Fujita-Miyazawa



Relativistic

C. Pieper and R. B. Wiringa, Annu. Rev. Nucl. Part. Sci.51(2001)

Heavy nuclei (Super model)

Pion is key

Linear sigma model

beautiful non-perturbative Lagrangian

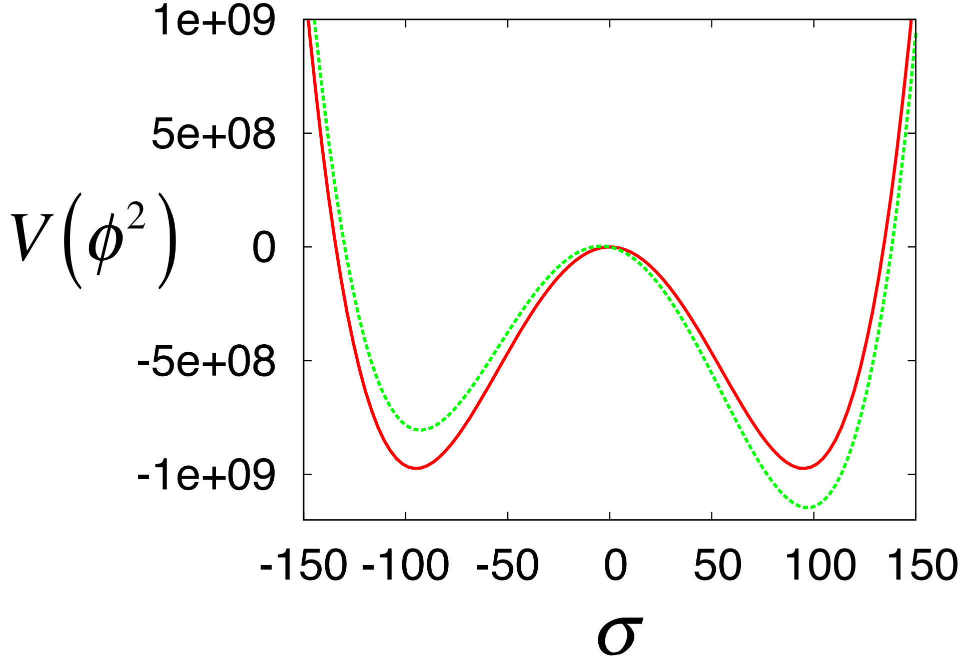
$$O(4): \qquad \phi = \left(\phi_0, \phi_1, \phi_2, \phi_3\right) = \left(\sigma, \vec{\pi}\right)$$

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - V\left(\phi^2\right)$$

$$V\left(\phi^2\right) = -\frac{1}{2}\mu_0^2 \phi^2 + \frac{1}{4}\lambda_0 \left(\phi^2\right)^2$$

Symmetry breaking term

$$L_{\chi SB} = -H_{\chi SB} = \varepsilon\sigma$$



Mean Field approximation

$$\sigma = 93 MeV$$

$$\vec{\pi} = 0$$

$$m_\pi = 139 MeV$$

$$\lambda_0 = 50$$

Gaussian Functional Method

Variational wave function (quantum fluctuation)

$$\Psi_0[\boldsymbol{\phi}] = \mathcal{N}\exp\left(-\frac{1}{4\hbar}\int d\boldsymbol{x}d\boldsymbol{y}[\phi_i(\boldsymbol{x}) - \langle\phi_i(\boldsymbol{x})\rangle]G^{-1}_{ij}(\boldsymbol{x},\boldsymbol{y})[\phi_j(\boldsymbol{y}) - \langle\phi_j(\boldsymbol{y})\rangle]\right)$$

$$G_{ij}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\delta_{ij}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{\boldsymbol{k}^2 + M_i^2}}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

$$\mathcal{H} = \int d\boldsymbol{y}\delta(\boldsymbol{y}-\boldsymbol{x})\left(-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi_i(\boldsymbol{x})\phi_i(\boldsymbol{y})} + \frac{1}{2}\nabla_{\boldsymbol{x}}\phi_i(\boldsymbol{x})\nabla_{\boldsymbol{y}}\phi_i(\boldsymbol{y}) + V(\boldsymbol{\phi}^2) + \mathcal{H}_{\chi SB}\right)$$

$$p_i(x) = \frac{\delta L}{\delta\partial_0\phi_i(x)} \qquad \left[p_i(x),\phi_j(y)\right]_t = \hbar\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$$

$$U\left(\langle\phi_i\rangle, M_i\right) = \langle\Psi_0[\phi]\|H|\Psi_0[\phi]\rangle$$

$$\int_{-\infty}^{\infty}e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

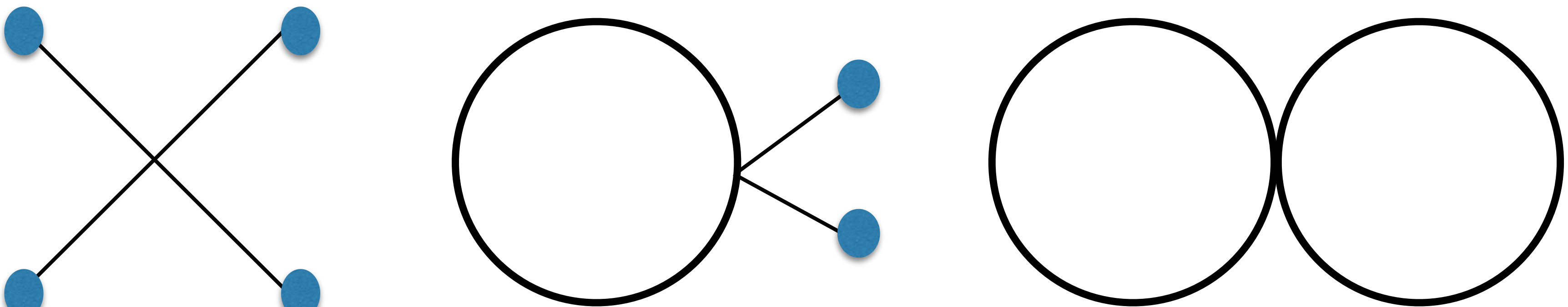
$$\int_{-\infty}^{\infty}x^2e^{-ax^2}dx = -\frac{d}{da}\int_{-\infty}^{\infty}e^{-ax^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

Ground state energy

$$U(M_i, \langle\phi_i\rangle) = -\varepsilon\langle\phi_0\rangle - \frac{1}{2}\mu_0^2\langle\phi\rangle^2 + \frac{\lambda_0}{4}[\langle\phi\rangle^2]^2 + \hbar\sum_i [I_1(M_i) - \frac{1}{2}\mu_0^2 I_0(M_i)$$
$$-\frac{1}{2}M_i^2 I_0(M_i)] + \frac{\lambda_0}{4}[6\hbar\sum_i \langle\phi_i\rangle^2 I_0(M_i) + 2\hbar\sum_{i\neq j}\langle\phi_i\rangle^2 I_0(M_j)$$
$$+3\hbar^2\sum_i I_0^2(M_i) + 2\hbar^2\sum_{i<j} I_0(M_i)I_0(M_j)],$$

$$I_0(M_i) = \frac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{\sqrt{\mathbf{k}^2+M_i^2}} = i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-M_i^2+i\epsilon},$$

$$I_1(M_i) = \frac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sqrt{\mathbf{k}^2+M_i^2} = -\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(k^2-M_i^2+i\epsilon\right)$$



CJT method

Cornwall Jackiew Tomboulis, Phys. Rev. 10 (1974) 2428

$$Z(J,K)=e^{(i/\hbar)W(J,K)}=\int d\phi \exp\left[\frac{i}{\hbar}\left(\int L(\phi(x))+\int \phi(x)J(x)+\frac{1}{2}\int \phi(x)K(x,y)\phi(y)\right)\right]$$

$$\frac{\delta W(J,K)}{\delta J(x)}=\langle \phi(x)\rangle=\varphi(x) \qquad \frac{\delta W(J,K)}{\delta K(x,y)}=\frac{1}{2}\left[\varphi(x)\varphi(y)+\hbar G(x,y)\right]$$

Effective action

$$\Gamma(\varphi,G)=W(J,K)-\int \varphi J-\frac{1}{2}\int \varphi K \varphi-\frac{1}{2}\hbar\int GK=-U(\varphi,G)\int d^4x$$

(Hartree-Fock method for bosons)

↑ Effective potential

Optimized expansion method

OE method : Okopinska, Ann. Phys. 228 (1993) 19: Phys. Lett. B375 (1996) 213

$$S[\phi,G]=S^{(0)}[\phi,G]+\varepsilon S^{(1)}[\phi,G]$$

$$=\int\frac{1}{2}\phi(x)G^{-1}(x,y)\phi(y)+\varepsilon\left[\int\frac{1}{2}\phi(x)[(-\partial^2+\mu^2)-G^{-1}(x,y)]\phi(y)+\int\frac{\lambda}{4}(\phi^2(x))^2\right]$$

$$\varepsilon=1$$

$$\phi(x)=\phi'(x)+\varphi$$

$$S[\phi'+\varphi,G]=S[\varphi,G]+\int\frac{1}{2}\phi'G^{-1}\phi'+\varepsilon\left[\int\frac{1}{2}\frac{\partial^2 S^{(1)}}{\partial\varphi^2}\phi'^2+\frac{1}{3!}\frac{\partial^3 S^{(1)}}{\partial\varphi^3}\phi'^3+\frac{1}{4!}\frac{\partial^4 S^{(1)}}{\partial\varphi^4}\phi'^4\right]$$

$$Z[\varphi,G]=e^{-S[\varphi,G]}\int d\phi'\exp\left[-\int\frac{1}{2}\phi'G^{-1}\phi'\right]\left[1-\varepsilon\left(\frac{1}{2}\frac{\partial^2 S^{(1)}}{\partial\varphi^2}\phi'^2+\frac{1}{3!}\frac{\partial^3 S^{(1)}}{\partial\varphi^3}\phi'^3+\frac{1}{4!}\frac{\partial^4 S^{(1)}}{\partial\varphi^4}\phi'^4\right)+O(\varepsilon^2)\right]$$

$$\Gamma[\varphi,G]=-U[\varphi,G]\int d^4x$$ Effective potential

Optimized expansion method for fermion and boson (Gellmann-Levy Lagrangian)

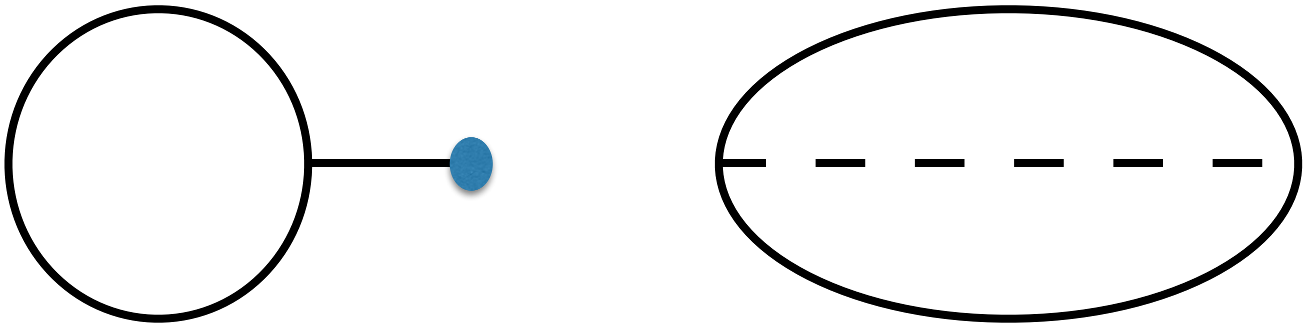
$$L = \bar{\psi}\left(i\gamma_\mu \partial^\mu - m - g\sigma - gi\gamma_5 \vec{\tau}\vec{\pi}\right)\psi + (Meson)$$

$$S_F(\phi,\psi) = S_F^{(0)}(M,\psi) + \varepsilon S_F^{(1)}(\phi,\psi)$$

$$= \int \bar{\psi}(i\gamma_\mu - M)\psi + \varepsilon \int \bar{\psi}(M - m - g\sigma - gi\gamma_5 \vec{\tau}\vec{\pi})\psi$$

$$U_{FB}(\varphi,G,G_F) = U(\varphi,G) - Tr\int \ln G_F^{-1}(M) + \left[m - M + g\varphi\right] Tr\int G_F$$

$$-\frac{1}{2} g^2 Tr\int G_F(M) G_F(M) G(M_i)$$

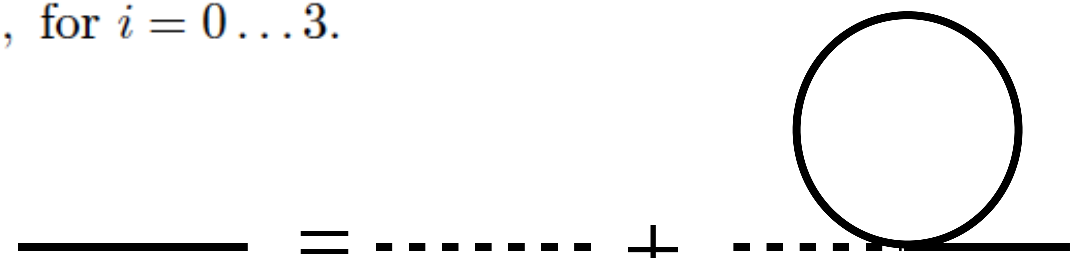


Hartree-Fock approximation

Gaussian functional method

Energy minimization

$$\left(\frac{\partial U(M_i, \langle \phi_i \rangle)}{\partial \langle \phi_i \rangle, M_i}\right)_{\min} = 0\,, \text{ for } i = 0 \ldots 3.$$



$$\begin{aligned}
\langle \phi_0 \rangle &= v\,, \\
\langle \phi_i \rangle &= 0 \text{ for } i = 1, 2, 3\,, \\
\mu_0^2 &= -\frac{\varepsilon}{v} + \lambda_0 \left[v^2 + 3\hbar I_0(M_\sigma) + 3\hbar I_0(M_\pi) \right]\,, \\
M_\sigma^2 &= -\mu_0^2 + \lambda_0 \left[3v^2 + 3\hbar I_0(M_\sigma) + 3\hbar I_0(M_\pi) \right] \\
M_\pi^2 &= -\mu_0^2 + \lambda_0 \left[v^2 + \hbar I_0(M_\sigma) + 5\hbar I_0(M_\pi) \right]\,.
\end{aligned}$$

Simplification

$$\begin{aligned}
M_\sigma^2 &= \frac{\varepsilon}{v} + 2\lambda_0 v^2\,, \\
M_\pi^2 &= \frac{\varepsilon}{v} + 2\lambda_0 \hbar \left[I_0(M_\pi) - I_0(M_\sigma) \right]
\end{aligned}$$

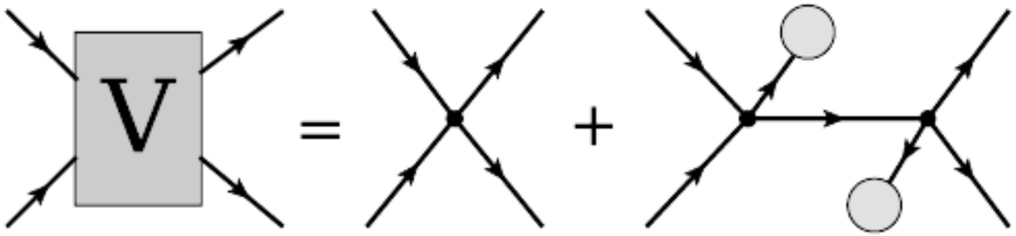
$$M_\pi \neq M_\sigma$$

Pion mass is not zero

Nambu-Goldstone theorem is not fulfilled.

Bethe-Salpeter equation

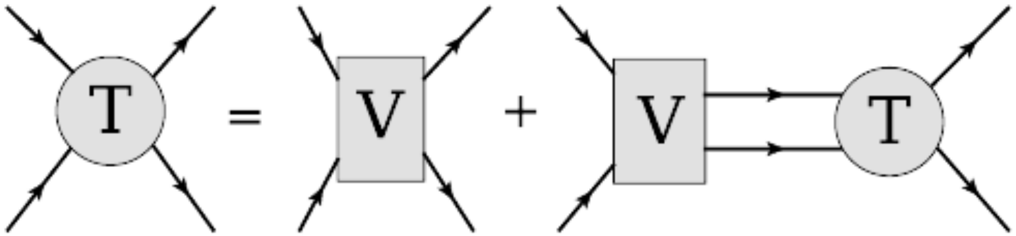
$$V_{\sigma\pi\to\sigma\pi}(s) = 2\lambda_0\left[1+\left(\frac{2\lambda_0 v^2}{s-M_\pi^2}\right)\right]$$



Physical meson mass

$$m_i^2 = \frac{\delta^2 U[\varphi, G]}{\delta\varphi_i^2}$$

$$T_{\sigma\pi\to\sigma\pi}(s) = V_{\sigma\pi\to\sigma\pi}(s) + V_{\sigma\pi\to\sigma\pi}(s)G_{\sigma\pi\to\sigma\pi}(s)T_{\sigma\pi\to\sigma\pi}(s)$$



$$T_{\sigma\pi\to\sigma\pi}(s) = \frac{V_{\sigma\pi\to\sigma\pi}(s)}{1 - V_{\sigma\pi\to\sigma\pi}(s)G_{\sigma\pi\to\sigma\pi}(s)}$$

Pole condition

$$1 - V(s)G(s) = 0$$

$$s = m_i^2$$

Nambu-Goldstone theorem

$$G_{\sigma\pi\to\sigma\pi}(p^2) \quad = \quad i\hbar\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-M_\sigma^2+i\epsilon]\,[(k-p)^2-M_\pi^2+i\epsilon]}$$

$$G_{\sigma\pi\to\sigma\pi}(s,T) \quad \xrightarrow{s\to 0} \quad \frac{I_0(M_\sigma^2,T)-I_0(M_\pi^2,T)}{M_\sigma^2-M_\pi^2}$$

$$\begin{aligned}1-VG(s=0) &= 1-2\lambda_0\left(1-\frac{2\lambda_0 v^2}{M_\pi^2}\right)\frac{I_0(M_\sigma)-I_0(M_\pi)}{M_\sigma^2-M_\pi^2}\\ &= 1+\left(1-\frac{M_\sigma^2}{M_\pi^2}\right)\frac{M_\pi^2}{M_\sigma^2-M_\pi^2}=0\end{aligned}$$

HF+RPA remove the center of mass motion

GF+BS recover the Nambu-Goldstone theorem

Linear σ model from quark-gluon dynamics

Kondo, Phys. Rev. D84 (2011) 061702

$$L_{QCD} = \bar{\psi}\left(i\gamma_{\mu}D^{\mu} - m\right)\psi + \frac{1}{4}F^{a}_{\mu\nu}F^{a\mu\nu}$$

$$A_{\mu}(x) = V_{\mu}(x) + \chi_{\mu}(x)$$

Cho-Fadeev-Niemi variable

$$V_{\mu}(x) = c_{\mu}(x)\vec{n}(x) + ig^{-1}[\vec{n}(x), \partial_{\mu}\vec{n}(x)]$$

$$\chi_{\mu}(x) = ig^{-1}[D_{\mu}[A]\vec{n}(x), \vec{n}(x)]$$

$$\int d\chi$$

(High energy mode)

$$L_{QCD} \to L_{DGL+NJL}(V, \psi) \quad (|p| < \Lambda)$$

$$\int dV\, d\bar{\psi}\, d\psi$$

bosonization

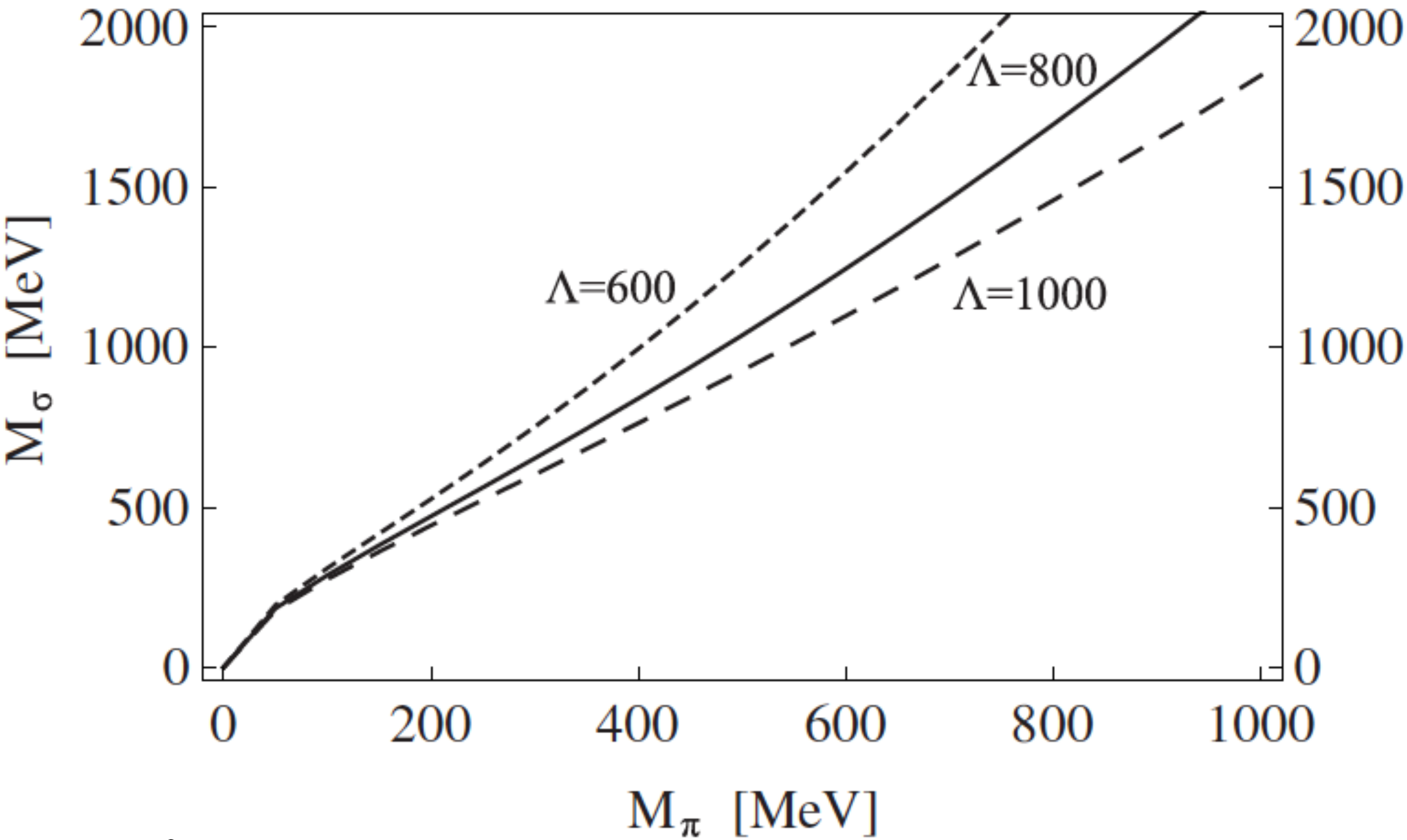
Low energy effective theory

$$\Lambda \sim 1GeV$$

$$L_{\sigma}[\phi_{meson}, \Psi_{fermion}] \quad (|p| < \Lambda)$$

Mesons and Fermions are composite

Results @ Gaussian functional method

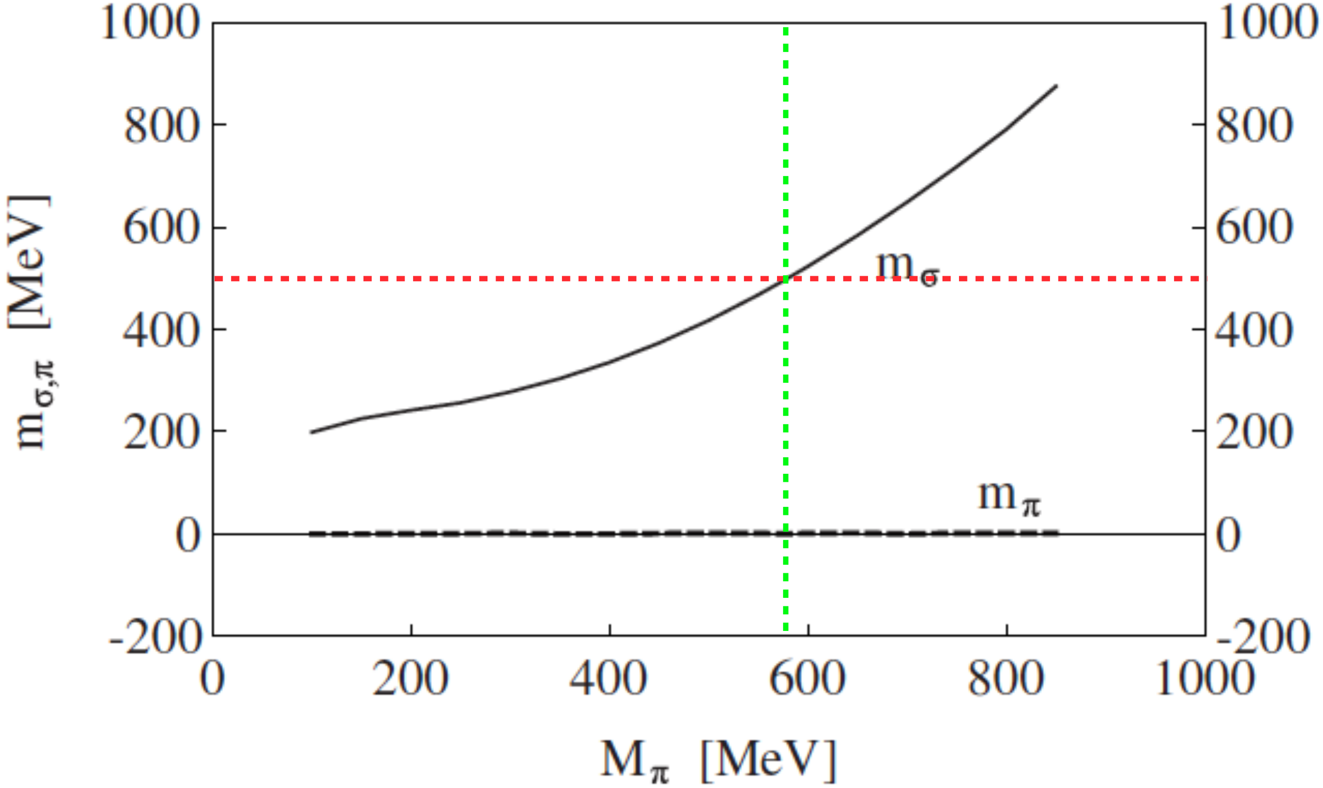


μ_0 λ_0

$v = 93 MeV$

Goldstone boson has finite mass!!

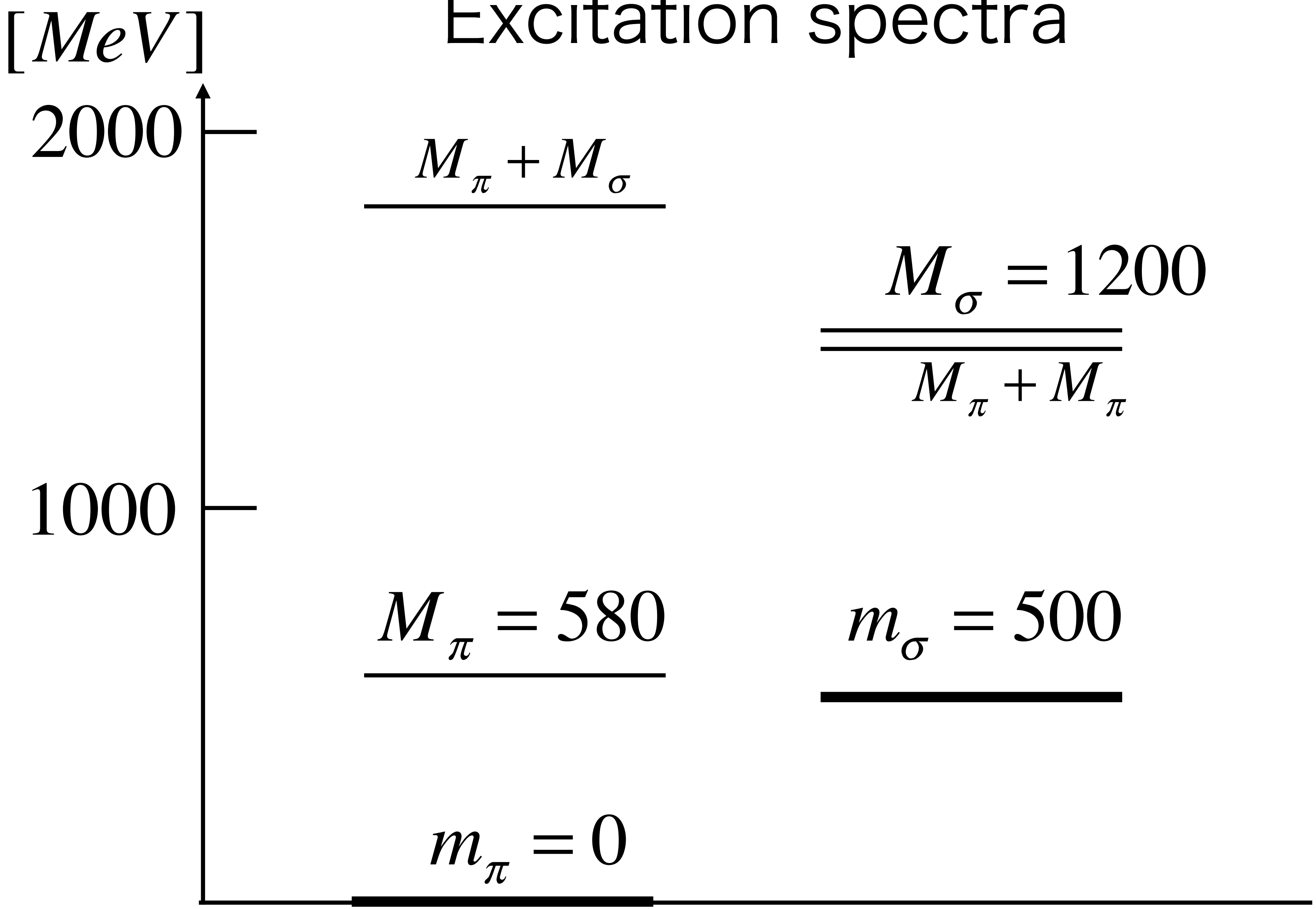
Numerical results @ BS method



$$\lambda_0 = 83.6\,,$$
$$\mu_0 = 1680\,\mathrm{MeV}$$
$$\Lambda = 800\,\mathrm{MeV}\,.$$

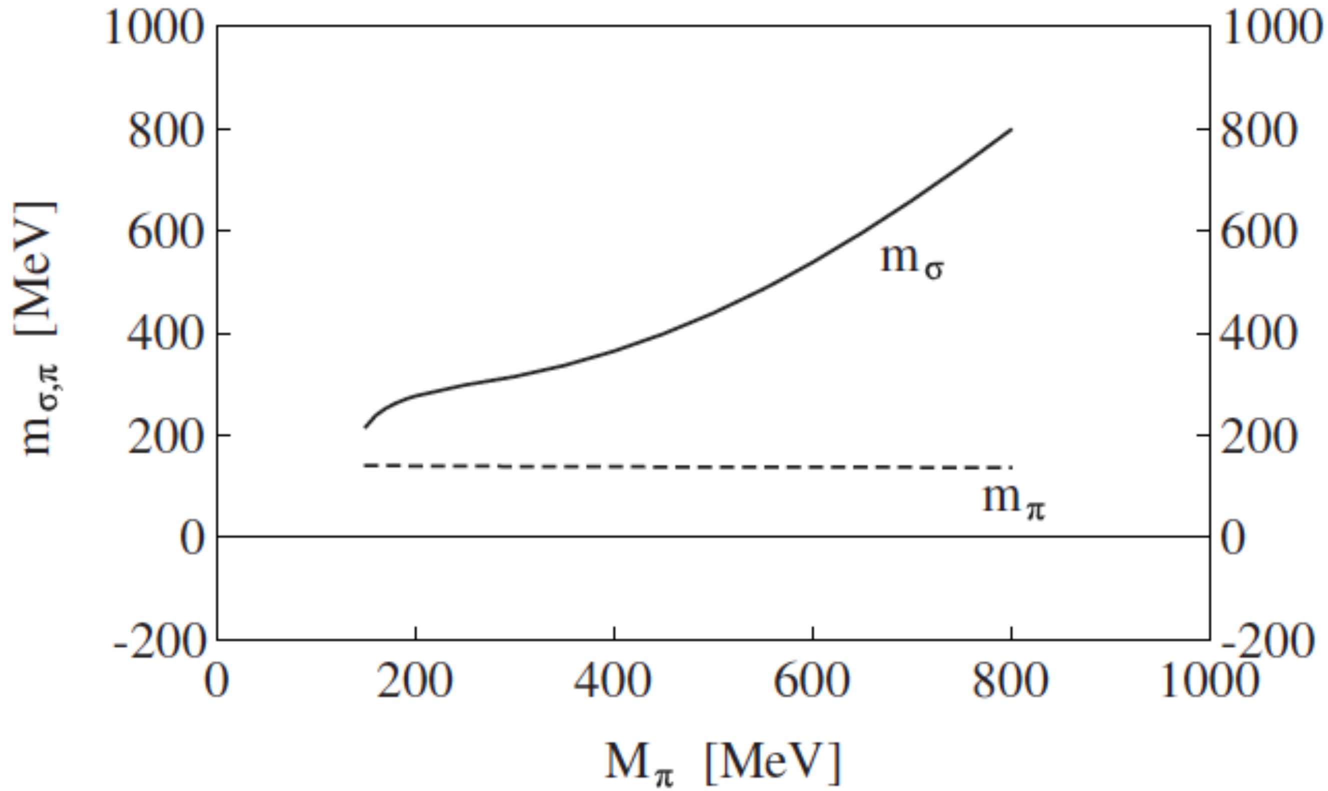
$$v(T=0) = f_\pi = 93.0\,\mathrm{MeV}\,,$$
$$M_\sigma(T=0) = 1200\,\mathrm{MeV}\,,$$
$$M_\pi(T=0) = 580\,\mathrm{MeV}\,,$$
$$m_\sigma(T=0) = 500\,\mathrm{MeV}\,,$$
$$m_\pi(T=0) = 0.0\,\mathrm{MeV}\,.$$

Excitation spectra



Sigma meson is 4 quark state

Finite pion mass



$$\begin{aligned} \varepsilon &= 142^2 \times 93.0\,\mathrm{MeV}^3 = 1.86 \times 10^6\,\mathrm{MeV}^3\,, \\ \lambda_0 &= 75.5\,, \\ \mu_0 &= 1610\,\mathrm{MeV}\,, \\ \Lambda &= 800\,\mathrm{MeV}\,. \end{aligned}$$

$$\begin{aligned} v(T=0) &= f_\pi = 93.0\,\mathrm{MeV} \\ M_\sigma(T=0) &= 1150\,\mathrm{MeV}\,, \\ M_\pi(T=0) &= 564\,\mathrm{MeV}\,, \\ m_\sigma(T=0) &= 500\,\mathrm{MeV}\,, \\ m_\pi(T=0) &= 138\,\mathrm{MeV}\,. \end{aligned}$$

Sigma meson is 4 quark state

PHYSICAL REVIEW D **81**, 114034 (2010)

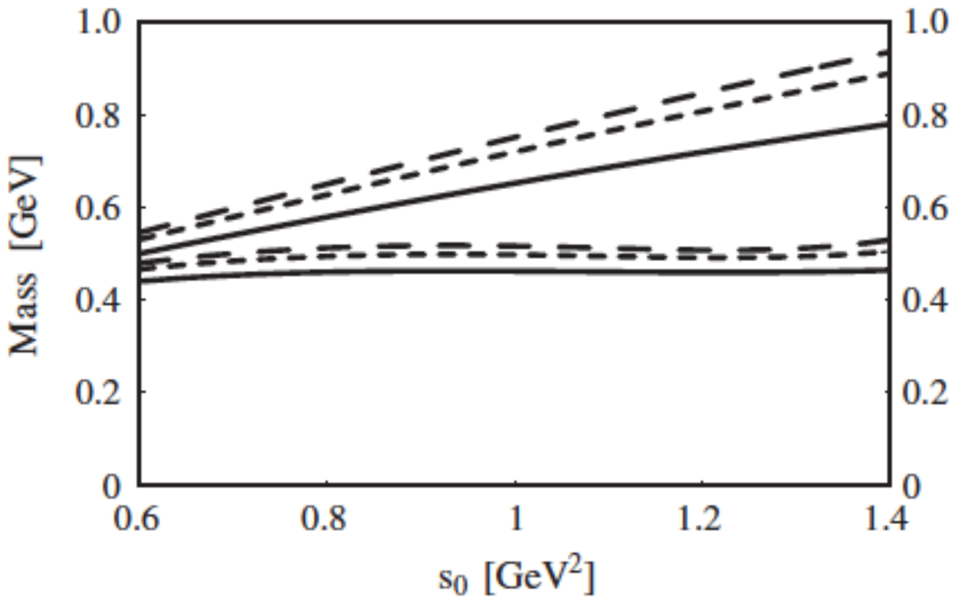
Light scalar meson $\sigma(600)$ in QCD sum rule with continuum

Hua-Xing Chen,[1,2,*] Atsushi Hosaka,[2,†] Hiroshi Toki,[2,‡] and Shi-Lin Zhu[1,§]

two quark current $J_2 = \bar{q}q$ higher than 1GeV

four quark current $J_4 = (\bar{q}q)^2$ lower than 1GeV

$$\rho(s) = f_Y^2 \delta(s - M_Y^2) + \rho_{\pi\pi}(s) + \rho_{\text{cont}}.$$



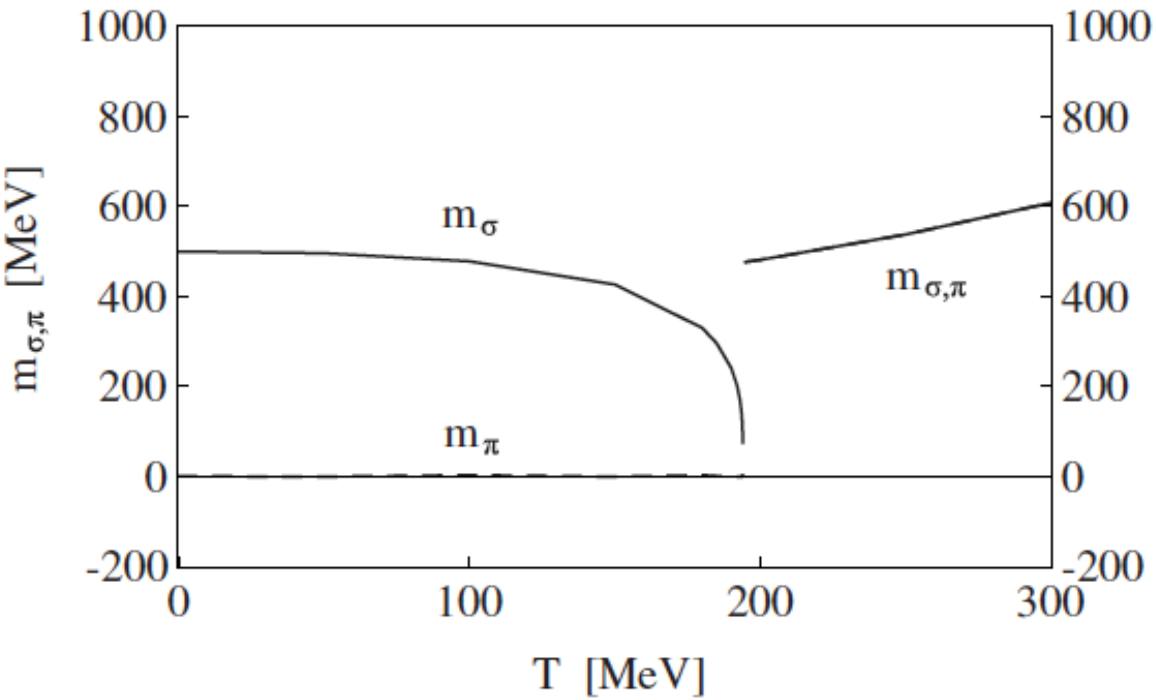
$\sigma(600) \sim 530\,MeV \pm 40\,MeV$

$\Gamma_{1/2} \sim 200\,MeV$

Finite temperature

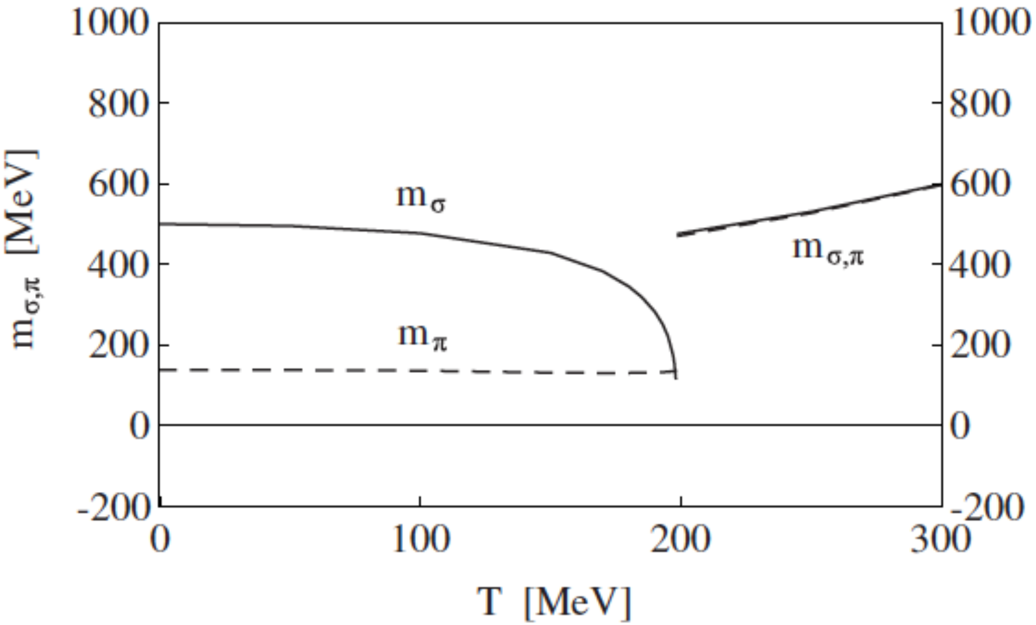
$$i\int \frac{d^4k}{(2\pi)^4} \to -T\sum_n \int \frac{d^3k}{(2\pi)^3}$$

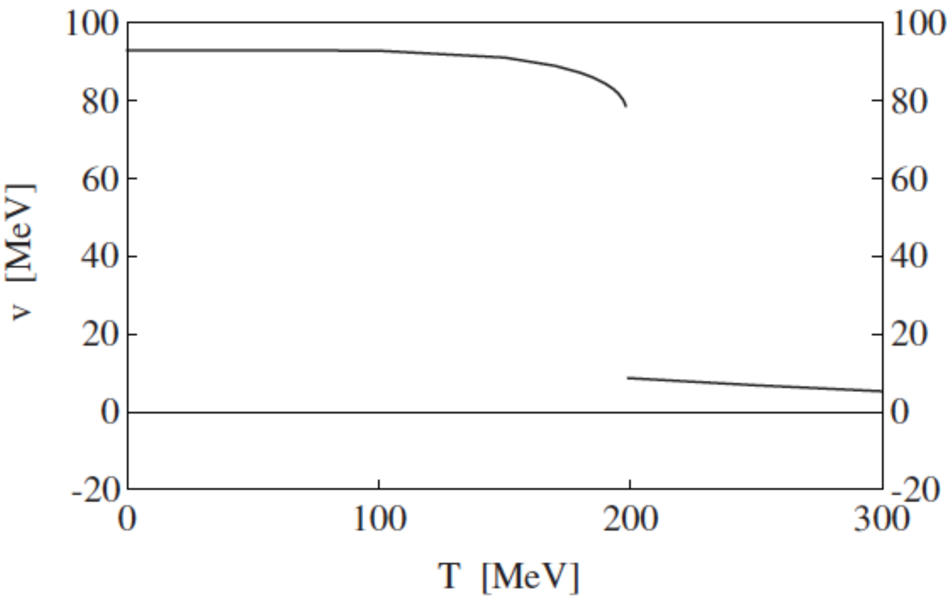
$$k^0 \to i\omega_n = i2\pi nT$$



First order phase transition vs. second order

Explicit symmetry breaking case at finite temperature





First order phase transition

Very similar to the case of non-linear sigma model

Order of phase transition

O(4) linear sigma model should be second order

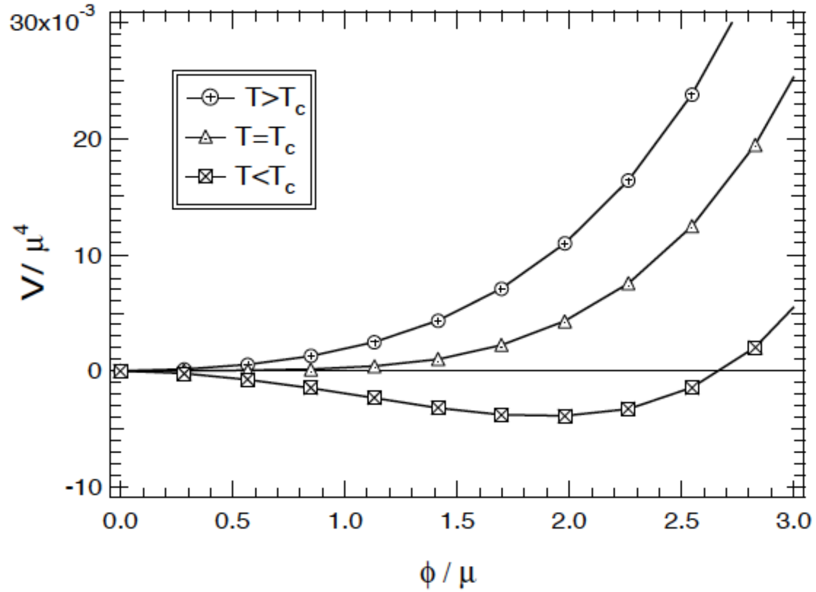
Ogure Sato, Prog. Theo. Phys. 102 (1999) 209

General discussion on phase transition

$$\Pi_\sigma = \frac{\partial^2 V(\sigma,\pi)}{\partial \sigma^2}$$

but assumption is made

$$\Pi_\pi = \frac{1}{\sigma}\frac{\partial V(\sigma,\pi)}{\partial \sigma}$$



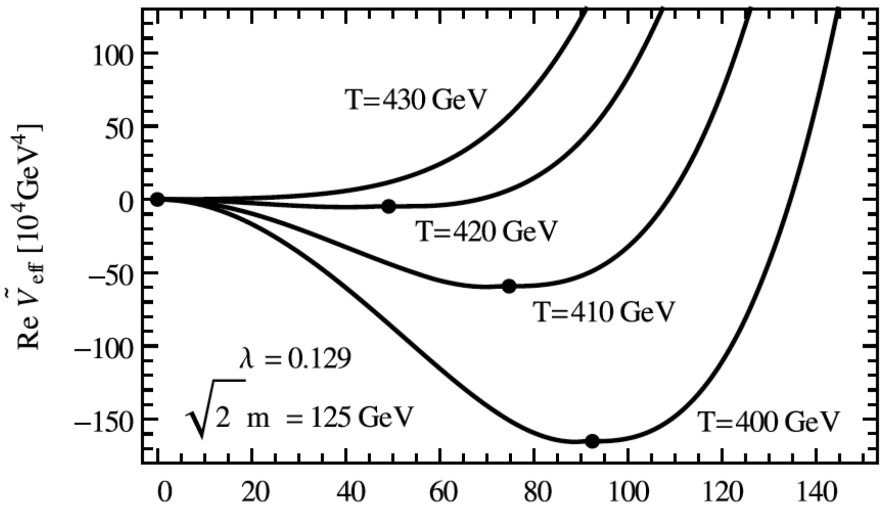
Pilaftsis Teresi, Nucl. Phys. B874 (2013) 594

Symmetry improved

CJT method

$$\varphi M_\pi^2 = 0$$

instead of $\dfrac{\partial V(\varphi)}{\partial \varphi} = 0$



Conclusion

- GF+BS approximation is used for linear sigma model (low energy effective theory)
  -correspond to HF+RPA in nuclear physics
- Sigma meson has 4 quark structure
- Phase transition is first order
  -'general' discussion leads to second order
- Want to relate to the non-linear sigma model
  Pion cloud: $N = e^{\frac{1}{2} i\gamma_5 \tau\cdot\pi / f_\pi} \psi$

Finite temperature (Nuclear Physics) with pions

Finite temperature

$$Z = \int d\phi(x) e^{i\int d^4x\, L(\phi(x))}$$

$$\Gamma[\varphi, G] = -U[\varphi, G]\int d^4x \qquad i\int \frac{d^4k}{(2\pi)^4} \to -T\sum_n \int \frac{d^3k}{(2\pi)^3}$$

$$U(\varphi, G : T) \qquad m(T)^2 = \frac{d^2U(\varphi, G : T)}{d\varphi^2}$$

We do not use the zero mass pion in thermodynamics

Second quantization

$$Z_J = \int d\phi(x)\, d\varphi(x)\, e^{i\int d^4x\,[L(\phi) + J(x)(\phi(x) - \varphi(x))]}$$