Evaluating the Phase Diagram at finite Isospin and Baryon Chemical Potentials in NJL model

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Ref: Phys. Rev. D 82, 056006 (2010)

Outline

- Introduction: exotic pair states
- Theoretical work: NJL Model at finite chemical potential
- Phase diagram at finite chemical potential
- Summary

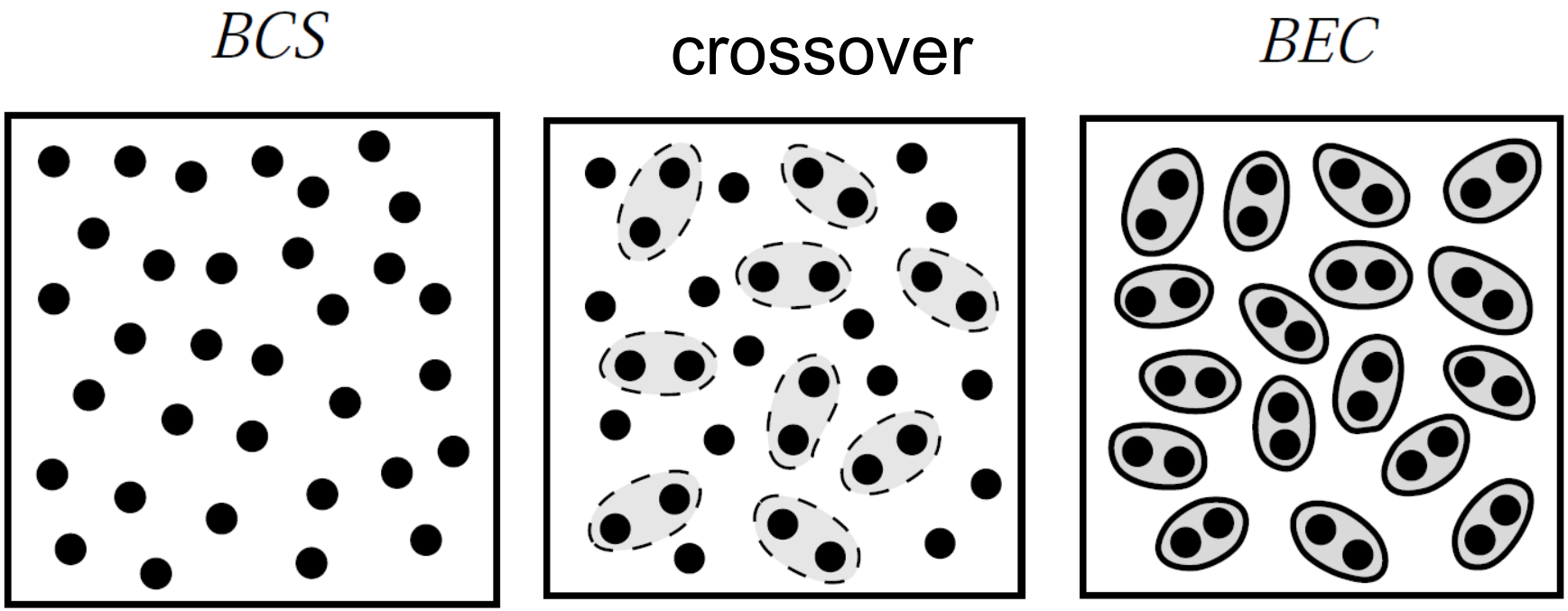
Introduction

- QCD at finite density possesses a rich phase structure:

1. Cold quark matter forms color superconductivity at high baryon density, for example, CFL, 2SC et al.
2. Pion superfluid occurs when the isospin chemical potential is larger than pion mass in vacuum. At low μ_I, the ground state is a superfluid pion condensate, but at high μ_I, it is a Fermi liquid with Cooper pairing. They are connected by BCS-BEC crossover.

BCS-BEC crossover

Qijin Chen et. al.: Physics Reports 412, 1-88 (2005)



In the BCS and BEC regimes: only one excitation spectrum with a single component. For the crossover case the excitations have two distinct components.

- In the standard BCS theory, the two fermionic species form a Cooper pair with same magnitude but opposite direction in momenta. But in real world, the two paired species have mismatched Fermi surface because of different chemical potential, unequal number densities or an external magnetic field et.al..
- Clogston limit: upper limit of mismatch.

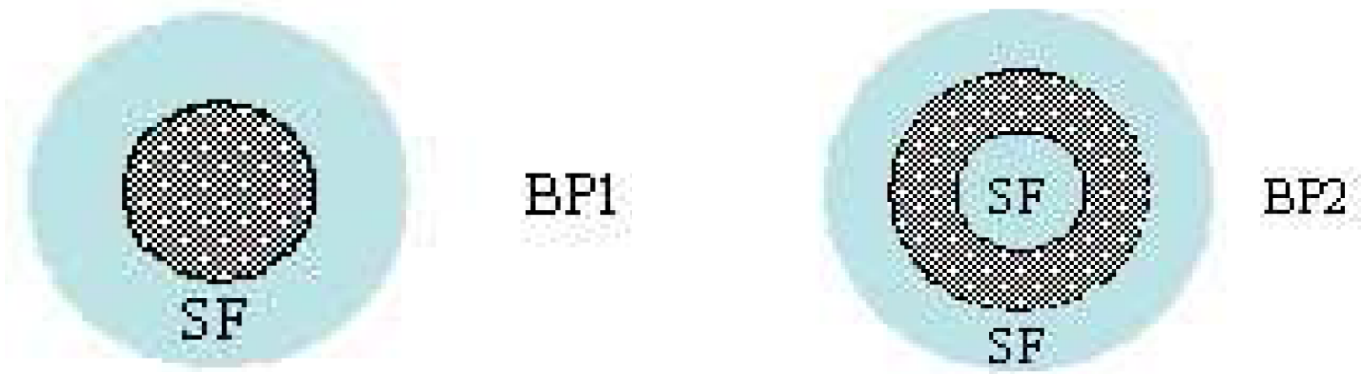
- What is the ground state in the asymmetric matter?

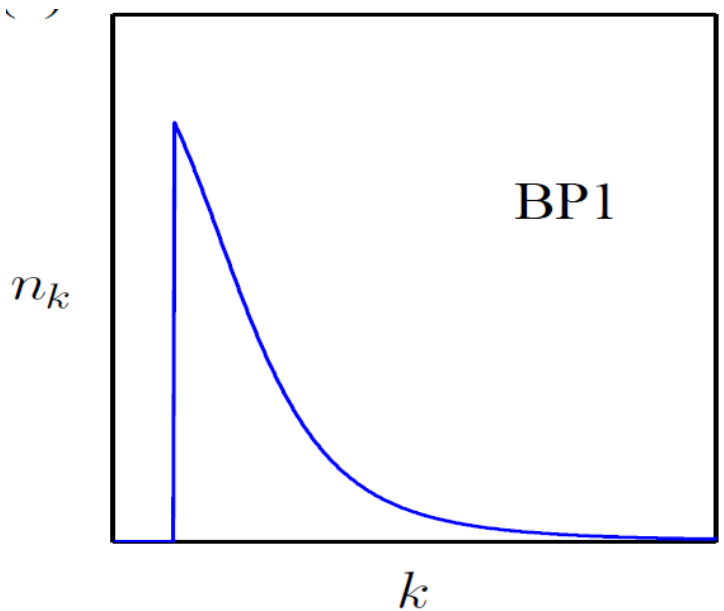
  This is a hot topic in both quark matter and condensed matter. Some candidates are proposed. We mainly discuss two of them: the Sarma phase and Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) phase.

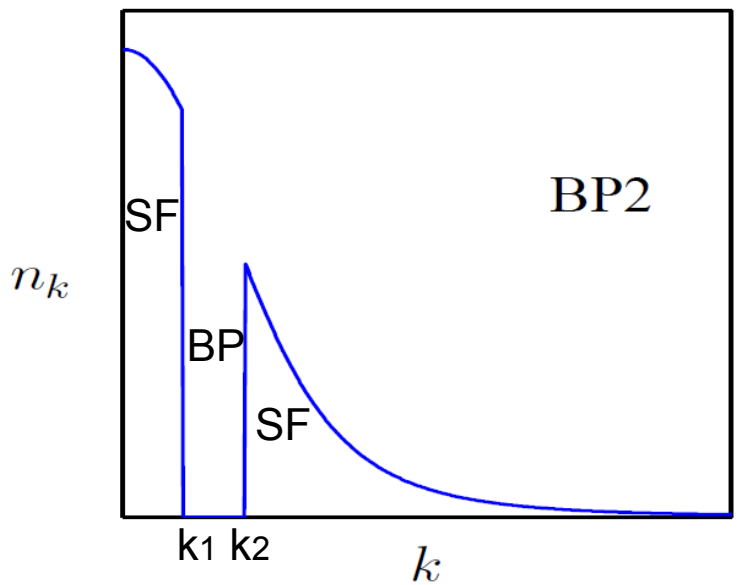
- The Breached Paired phase (also called Sarma phase)

Superfluid component is breached by normal one in the region $k_1 < k < k_2$ where gapless excitations happen.

W.Yi and L.M. Duan: Phys. Rev. Lett. 97, 120401 (2006)

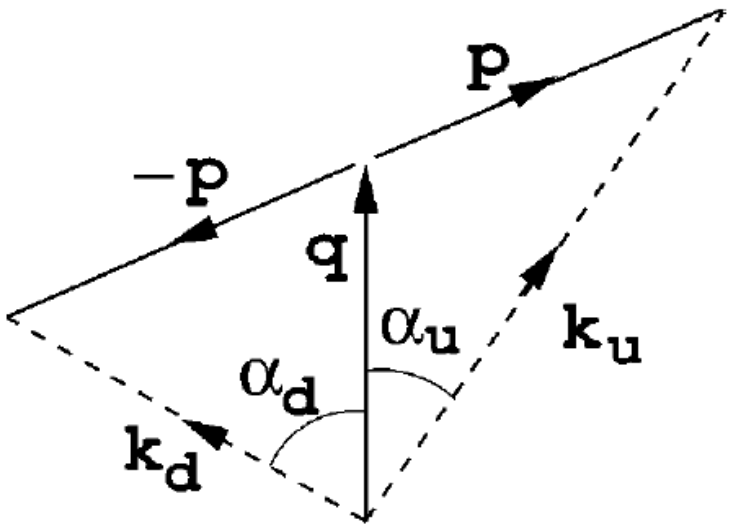


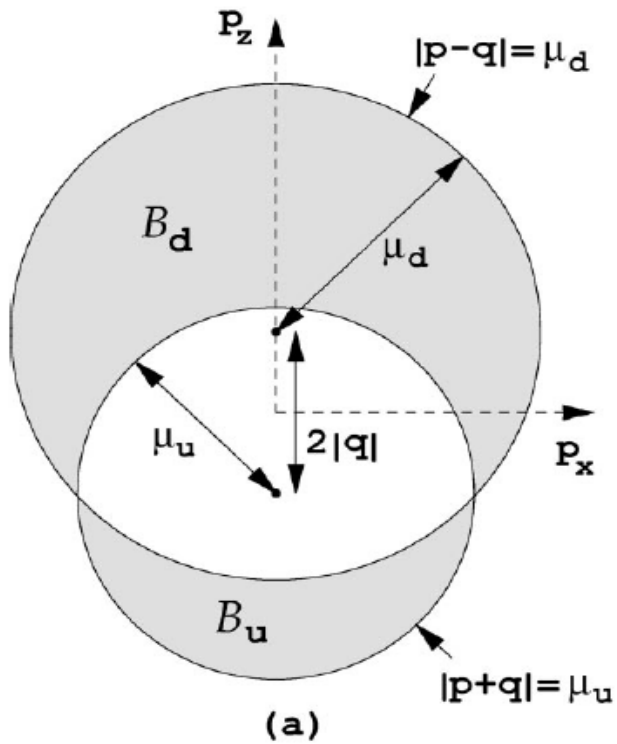




LOFF state is a spatially anisotropic ground state where the rotational symmetry and/or the translational symmetry are spontaneously broken. Each Cooper pair carries a total momentum $2q$.

M.Alford et. al. PRD63, 074016(2001)





Two Flavor Nambu–Jona-Lasinio Model

- The lagrangian density :

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m_0)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2\right]$$

- Chiral condensate : $\langle\bar{\psi}\psi\rangle = \sigma,$
- Pion condensates : $\langle\bar{\psi}i\gamma_5\tau_+\psi\rangle = \sqrt{2}\langle\bar{u}i\gamma_5 d\rangle = \pi^+ = \frac{\pi}{\sqrt{2}}e^{i\theta},$

$$\langle\bar{\psi}i\gamma_5\tau_-\psi\rangle = \sqrt{2}\langle\bar{d}i\gamma_5 u\rangle = \pi^- = \frac{\pi}{\sqrt{2}}e^{-i\theta}$$

- u and d quark potential : $\mu_u = \frac{\mu_B}{3} + \frac{\mu_I}{2}, \quad \mu_d = \frac{\mu_B}{3} - \frac{\mu_I}{2}$
- Partition function :

$$Z(T, \mu_I, \mu_B, V) = \int[d\bar{\psi}][d\psi]e^{-\int_0^{\beta}d\tau\int d^3\mathbf{x}\,\mathcal{L}_{mf}}$$

$$\mathcal{L}_{mf} = \bar{\psi}[i\gamma^{\mu}\partial_{\mu} - m + \hat{\mu}\gamma_0 - i\Delta\tau_1\gamma_5] - G(\sigma^2 + \pi^2)\psi$$

The thermodynamic potential:

$$\Omega = G(\sigma^2 + \pi^2) - 6\sum_{i=1}^{4}\int \frac{d^3\mathbf{p}}{(2\pi)^3} g(\omega_i(p)),$$

where $\omega_1(p) = E_\Delta^- + \mu, \qquad \omega_2(p) = E_\Delta^- - \mu,$

$$\omega_3(p) = E_\Delta^+ + \mu, \qquad \omega_4(p) = E_\Delta^+ - \mu$$

$$E_\Delta^\pm = \sqrt{(E_p^\pm)^2 + \Delta^2}, \qquad E_p^\pm = E_p \pm \mu_I/2,$$

$$E_p = \sqrt{p^2 + m^2}, \qquad \Delta = -2G\pi$$

and $g(x) = x/2 + T\ln(1 + e^{-x/T})$

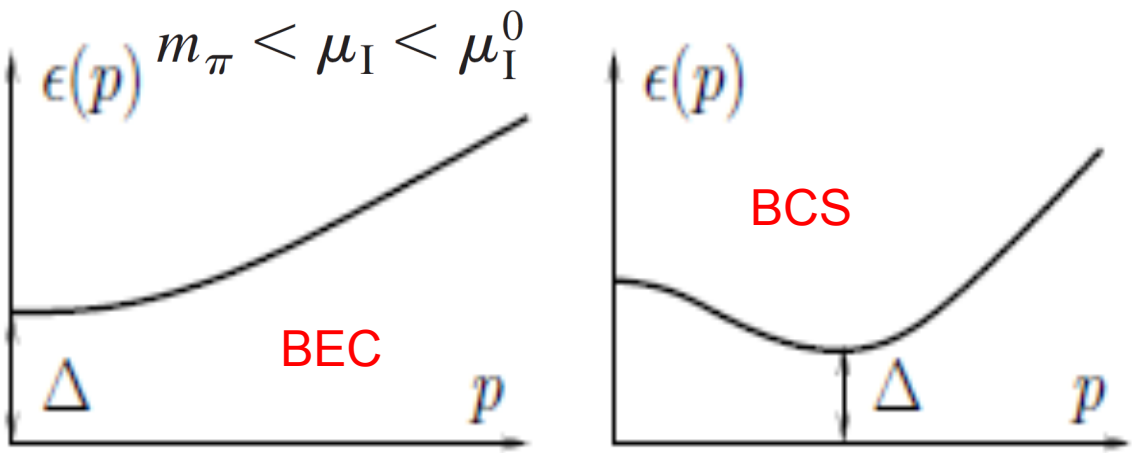
The gap equations for m and Δ :

$$\frac{\partial\Omega}{\partial m} = 0, \quad \frac{\partial\Omega}{\partial\Delta} = 0, \quad \frac{\partial^2\Omega}{\partial m^2} > 0, \quad \frac{\partial^2\Omega}{\partial\Delta^2} > 0.$$

- The explicit form of the gap equations:

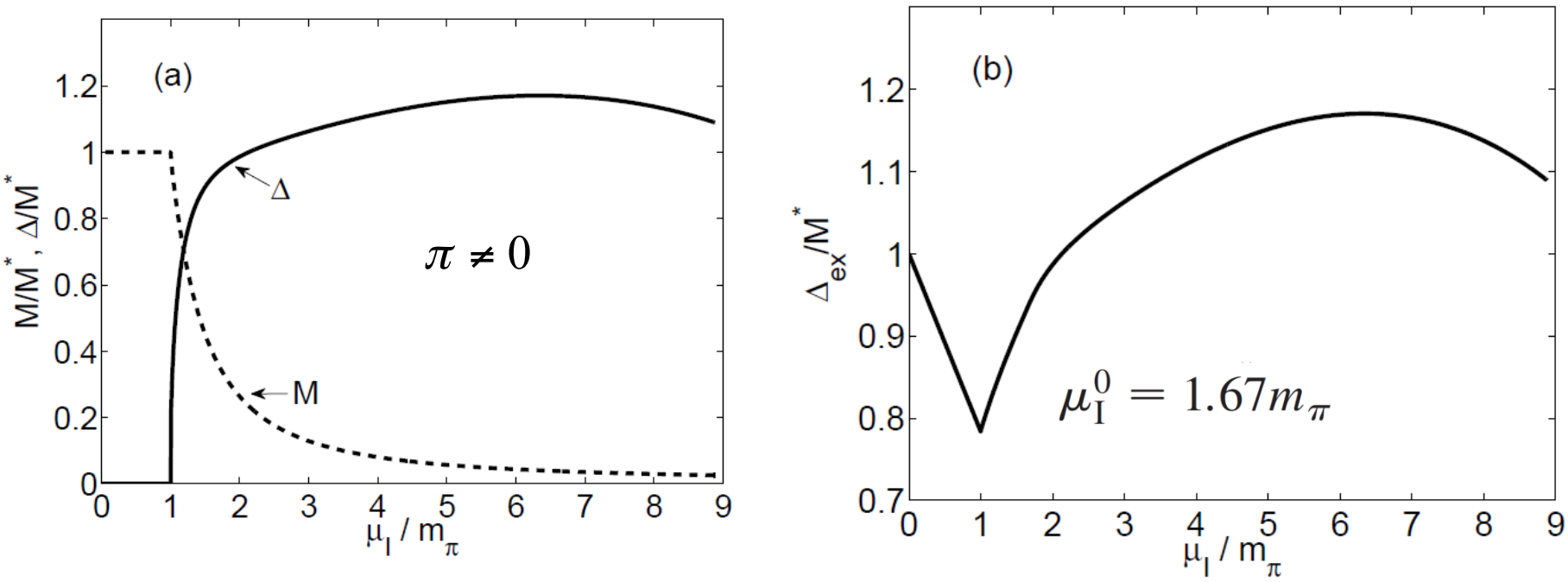
$$m - m_0 = 12G \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{m}{E_p} \left[\frac{E_p^-}{E_\Delta^-} (1 - f(\omega_1) - f(\omega_2)) + \frac{E_p^+}{E_\Delta^+} (1 - f(\omega_3) - f(\omega_4)) \right],$$

$$\Delta = 12G\Delta \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[\frac{1}{E_\Delta^-} (1 - f(\omega_1) - f(\omega_2)) + \frac{1}{E_\Delta^+} (1 - f(\omega_3) - f(\omega_4)) \right]$$



Single-particle excitation gap: $\Delta_{\rm ex}(\mu_{\rm I}) = \sqrt{\left(M_0 - \frac{\mu_{\rm I}}{2}\right)^2 \Theta\left(M_0 - \frac{\mu_{\rm I}}{2}\right) + \Delta_0^2}\,,$

where M_0 and Δ_0 stand for the real minimum of the grand potential at $\mu_{\rm B} = 0$.



(a) The isospin chemical potential $\mu_{\rm I}$ dependence of the dynamical quark mass M and the superfluid order parameter Δ (the M and Δ are in unit of the dynamical quark mass M^* in vacuum, the $\mu_{\rm I}$ is in unit m_π). (b) The isospin chemical potential $\mu_{\rm I}$ dependence of the single-particle excitation gap $\Delta_{\rm ex}$ (in unit M^*).

- Sarma phase at finite isospin and quark potential

From the conditions: $\omega_2(p) = 0$ and $\omega_4(p) = 0$

We obtain the gapless points:

$$p_1 = m\sqrt{\lambda_1^2 - 1}, \quad p_2 = m\sqrt{\lambda_2^2 - 1},$$

where

$$\lambda_1 = \frac{\mu_I/2 - \sqrt{\mu^2 - \Delta^2}}{m},$$

$$\lambda_2 = \frac{\mu_I/2 + \sqrt{\mu^2 - \Delta^2}}{m}.$$

Only in the case $\Delta < \mu$, there is the possibility to realize the Sarma phase. There are 3 types:

Type 1: $\lambda_1 > 0$ and $|\lambda_{1,2}| > 1$.

Only the branch ω_2 has two gapless nodes at p_1 and p_2

Type 2: $|\lambda_1| < 1$ and $|\lambda_2| > 1$.

Only the branch ω_2 has one gapless node at p_2 . This is in our case.

Type 3: $\lambda_1 < 0$ and $|\lambda_{1,2}| > 1$.

The branch ω_2 has one gapless node at p_2 , the ω_4 branch has one gapless node at p_1

- The LOFF phase

$$\langle \bar{\psi} i\gamma_5 \tau_+ \psi \rangle = \sqrt{2}\langle \bar{u} i\gamma_5 d \rangle = \pi^+ = \frac{\pi}{\sqrt{2}} e^{2i\mathbf{q}\cdot\mathbf{x}},$$

$$\langle \bar{\psi} i\gamma_5 \tau_- \psi \rangle = \sqrt{2}\langle \bar{d} i\gamma_5 u \rangle = \pi^- = \frac{\pi}{\sqrt{2}} e^{-2i\mathbf{q}\cdot\mathbf{x}}.$$

$$\chi_u(\mathbf{x}) = u(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}}, \quad \chi_d(\mathbf{x}) = d(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}}$$

$$\begin{aligned} \mathcal{L} &= \bar{\chi}\left[i\gamma^\mu \partial_\mu - m + \hat{\mu}\gamma_0 - \tau_3 \gamma\cdot\mathbf{q} - i\Delta\tau_1\gamma_5\right]\chi \\ &\quad -G(\sigma^2 + \pi^2), \end{aligned}$$

$$\Omega = G(\sigma^2 + \pi^2) - \frac{T}{V} \ln \det \mathcal{S}^{-1}.$$

$$\left[\det \mathcal{S}^{-1}\right]^{1/2} = \left[(p_0+\mu+\epsilon_-)^2-(\epsilon_+-\mu_I/2)^2-\Delta^2\right]\left[(p_0+\mu-\epsilon_-)^2-(\epsilon_++\mu_I/2)^2-\Delta^2\right]$$
$$+\,2\Delta^2\left[\mathbf{p}^2+m^2-\mathbf{q}^2-\sqrt{|\mathbf{p}+\mathbf{q}|^2+m^2}\sqrt{|\mathbf{p}-\mathbf{q}|^2+m^2}\right],$$

$$\Omega(T,\mu,\mu_I,\mathbf{q},m,\Delta) = G(\sigma^2+\pi^2)-3\sum_{i=1}^{4}\int\frac{d^3\mathbf{p}}{(2\pi)^3}(E_i(\mathbf{p},\mathbf{q})+2T\ln(1+\exp(-\beta E_i(\mathbf{p},\mathbf{q}))))$$
$$= \frac{(m_0-m)^2+\Delta^2}{4G}-3\sum_{i=1}^{4}\int\frac{d^3\mathbf{p}}{(2\pi)^3}(E_i(\mathbf{p},\mathbf{q})+2T\ln(1+\exp(-\beta E_i(\mathbf{p},\mathbf{q})))),$$

where
$$E_1(\mathbf{p},\mathbf{q}) = \sqrt{(\epsilon_+-\mu_I/2)^2+\Delta^2}+(\mu+\epsilon_-),$$
$$E_2(\mathbf{p},\mathbf{q}) = \sqrt{(\epsilon_+-\mu_I/2)^2+\Delta^2}-(\mu+\epsilon_-),$$
$$E_3(\mathbf{p},\mathbf{q}) = \sqrt{(\epsilon_++\mu_I/2)^2+\Delta^2}+(\mu-\epsilon_-),$$
$$E_4(\mathbf{p},\mathbf{q}) = \sqrt{(\epsilon_++\mu_I/2)^2+\Delta^2}-(\mu-\epsilon_-)$$
$$\epsilon_\pm = \frac{1}{2}\left(\sqrt{|\mathbf{p}+\mathbf{q}|^2+m^2}\pm\sqrt{|\mathbf{p}-\mathbf{q}|^2+m^2}\right).$$

Fukushima and Iida, PRD76, 054004 (2007);
J. O. Andersen and T. Brauner, PRD81, 096004 (2010).

$$\Omega_s(T,\mu_{\rm I},\mu_{\rm B},{\rm q}) = \Omega(T,\mu_{\rm I},\mu_{\rm B},{\rm q},M({\rm q}),\Delta({\rm q})) - \Omega(T,\mu_{\rm I},\mu_{\rm B},{\rm q},M({\rm q}),\Delta=0)$$

The unphysical term from q does not affect the gap equations for m and Δ:

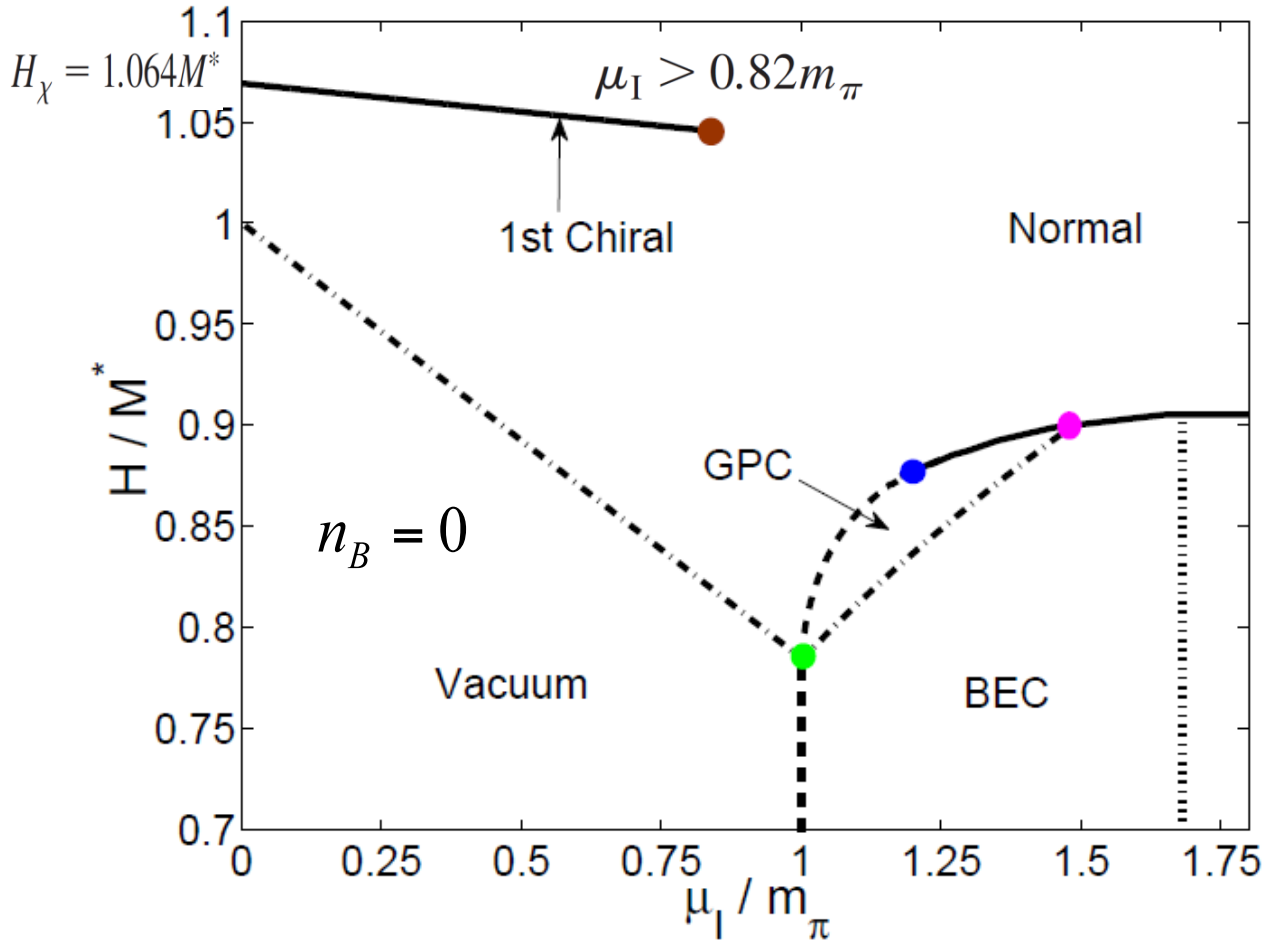
$$\frac{\partial\Omega}{\partial m}=0,\quad \frac{\partial\Omega}{\partial\Delta}=0,\quad \frac{\partial^2\Omega}{\partial m^2}>0,\quad \frac{\partial^2\Omega}{\partial\Delta^2}>0.$$

The optimal value of q obtained via minimizing $\Omega_{\rm sub}(T,\mu_{\rm I},\mu_{\rm B},{\rm q})$

$$\frac{\partial\Omega_s}{\partial q}=0,\frac{\partial^2\Omega_s}{\partial q^2}>0$$

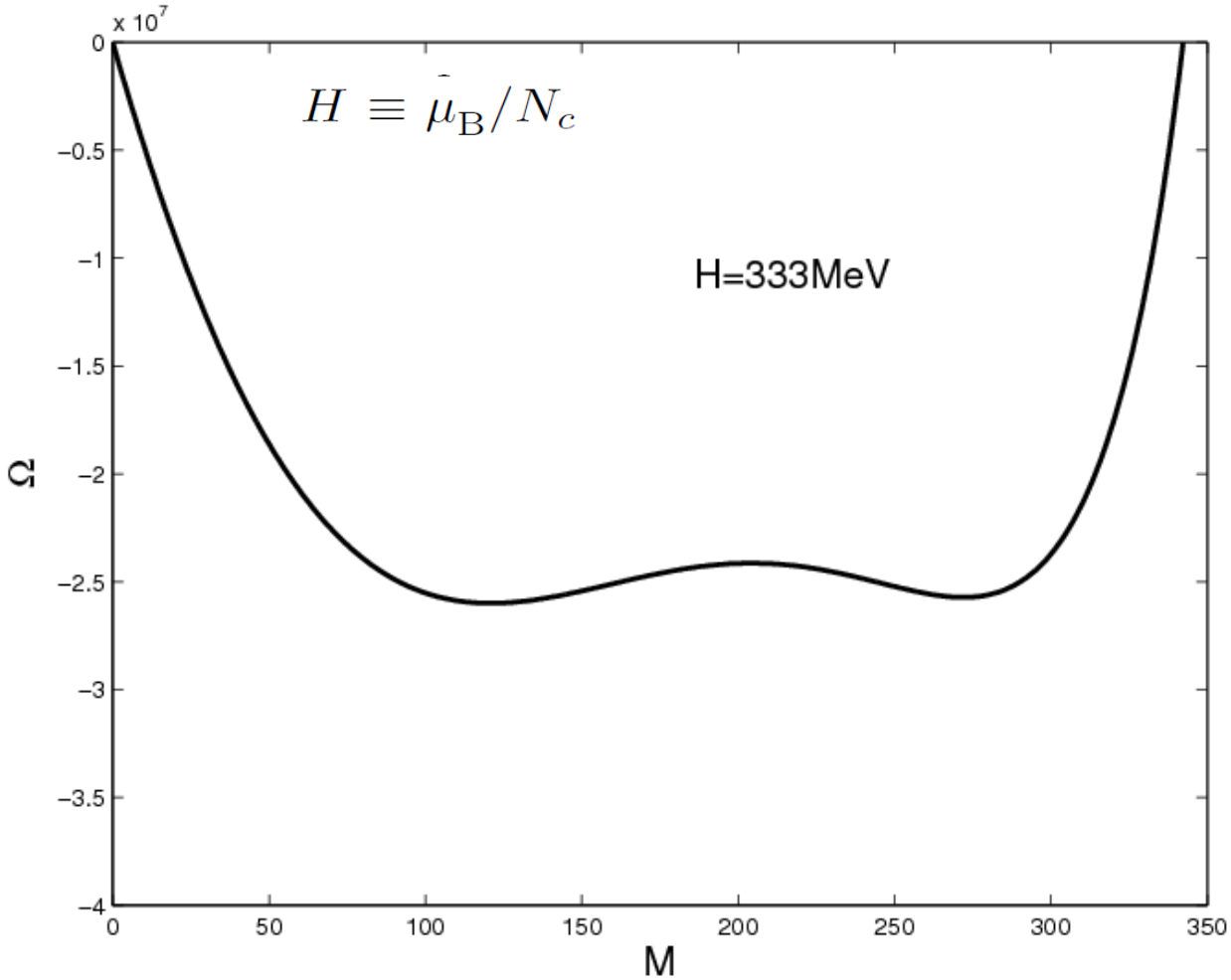
Phase Diagram:

- Phase transition in the low isospin chemical potential



quadruple point $(\mu_I, \mu_B) = (m_\pi, M_N - 1.5m_\pi)$

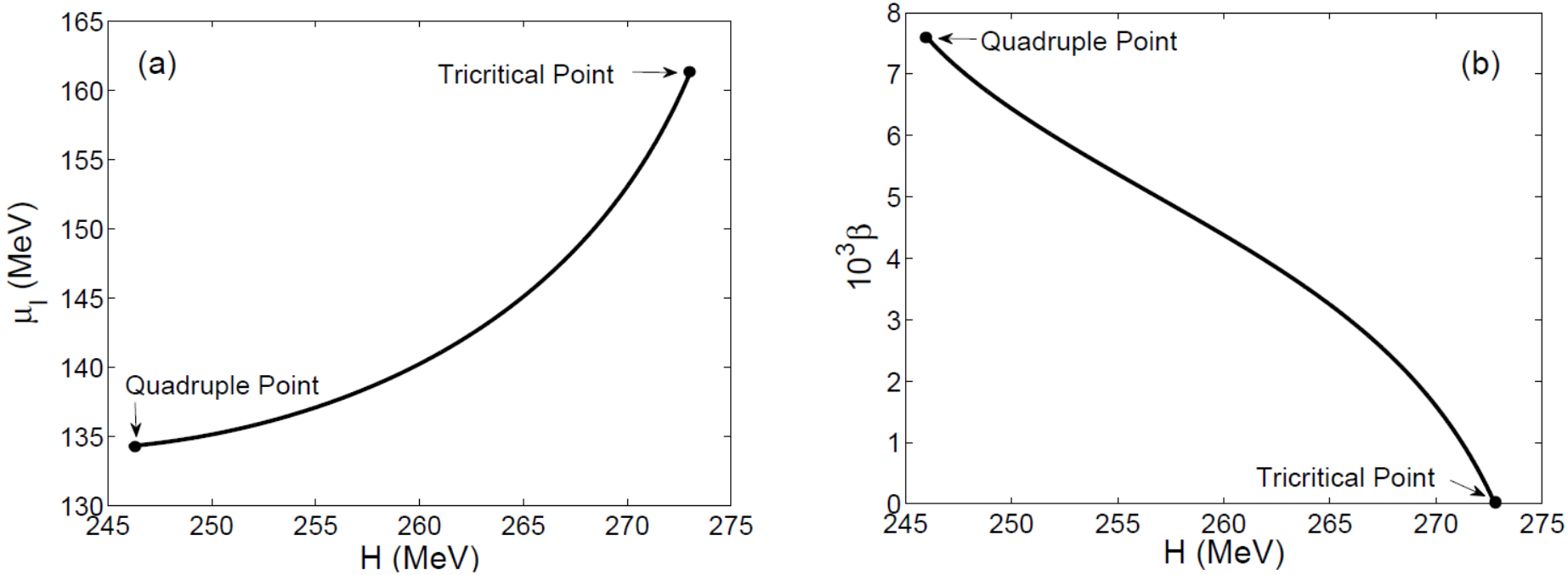
Chiral phase transition



Superuid-Normal Phase Transition and Tricritical Point

$$\Omega(\Delta^2, M(\Delta^2)) \;=\; \Omega(0, M(0)) + \frac{1}{2}\alpha\Delta^2 + \frac{1}{4}\beta\Delta^4 + \frac{1}{6}\gamma\Delta^6 \,.$$

$$\alpha = \beta = 0 \qquad (\mu_{\rm I}^{\rm tri}, H^{\rm tri}) = (160.5 \text{ MeV}, 272.8 \text{ MeV})$$

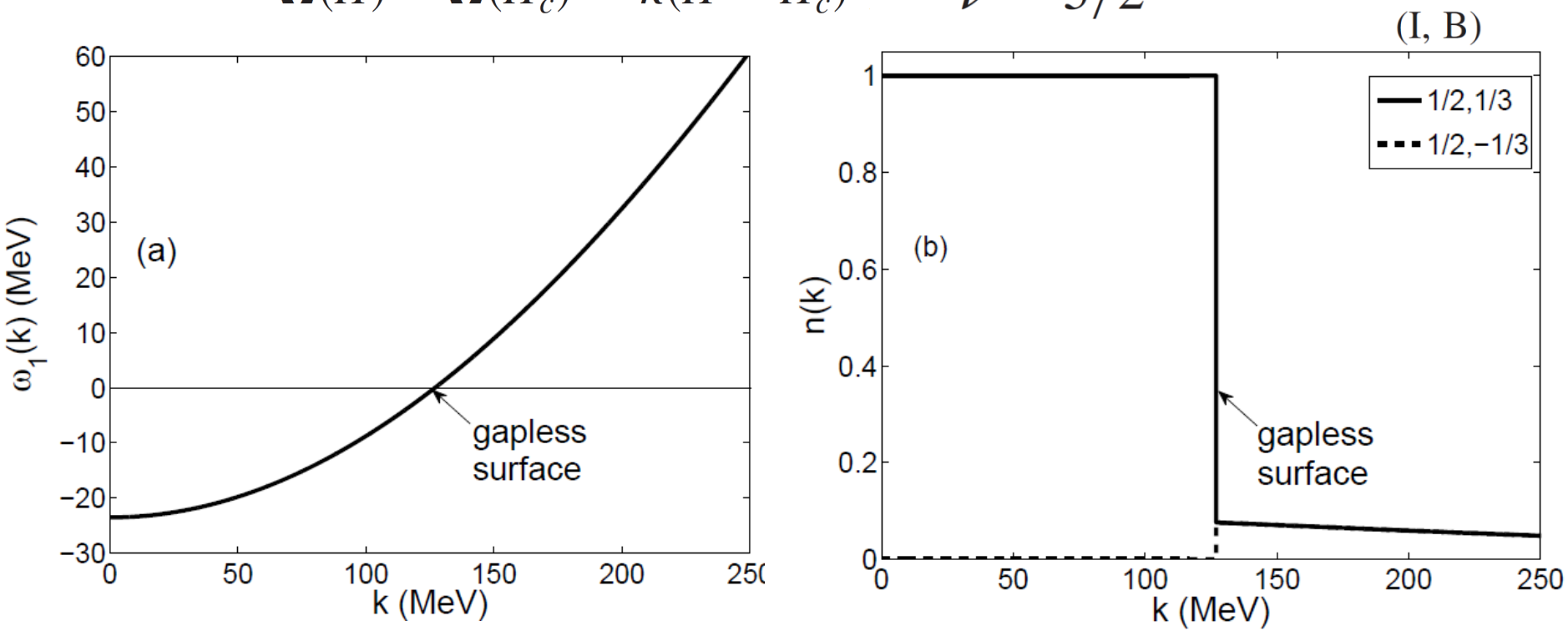


(a) Calculated result of the second order superfluid-normal phase boundary in Ginzburg-Landau theory. (b) The value of the coefficient β as a function of H along the second order phase transition line.

- Gapless pion condensate and topological quantum phase transition

$$C_V \propto e^{-\phi/T} \text{ at } H < H_c \qquad C_V \propto T \text{ at } H > H_c$$

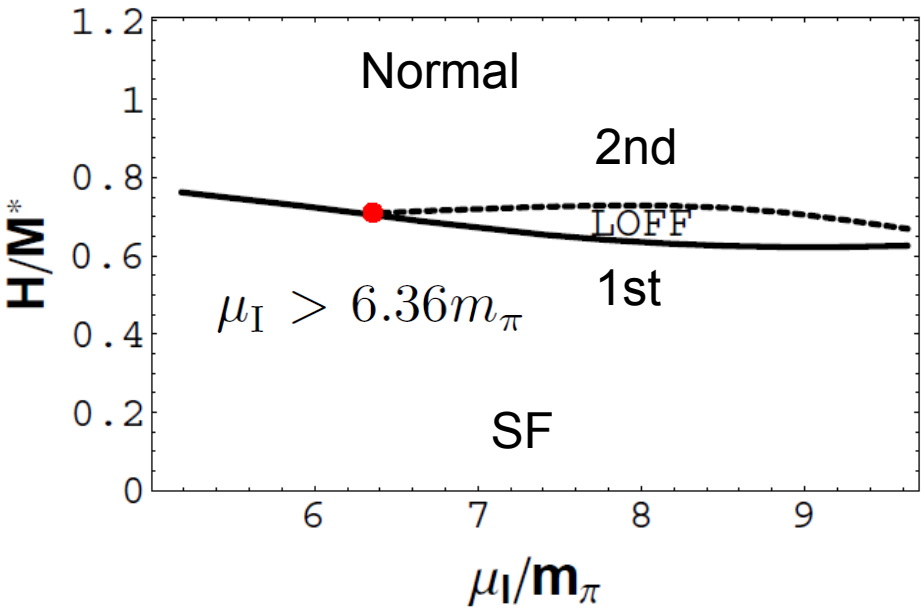
$$\Omega(H) - \Omega(H_c) = \kappa(H - H_c)^\nu \qquad \nu = 5/2$$

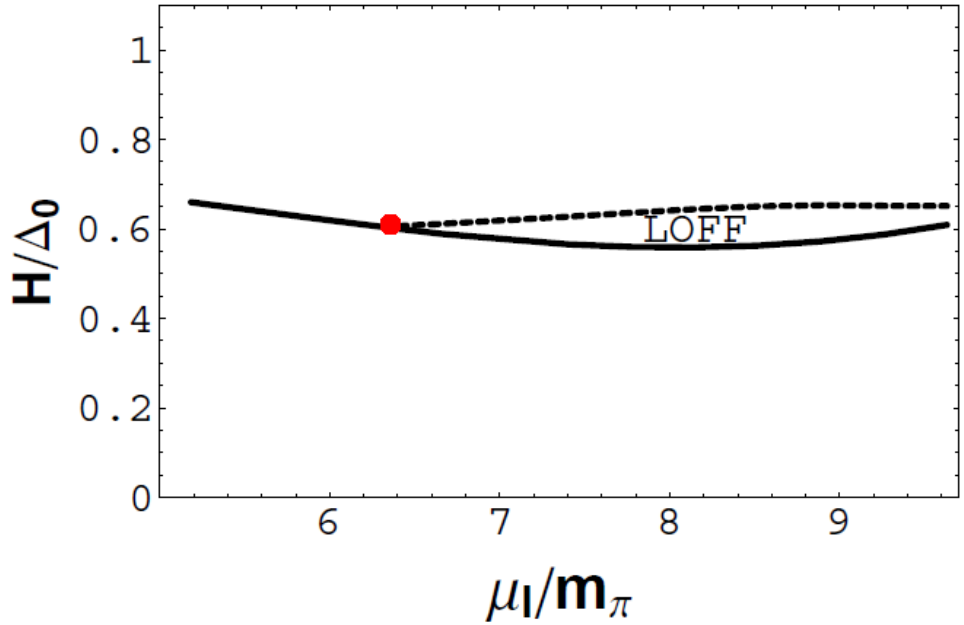


Calculated results of the lowest single-particle excitation $\omega_1(\mathbf{k})$ (panel (a)) and the corresponding particle occupation numbers $n(\mathbf{k})$ (panel (b)), at $\mu_I = 150$ MeV and $H = 265$ MeV. The numerical solutions for Δ and M are $\Delta = 140.7$ MeV and $M = 271.3$ MeV. The gapless surface is located at $k_2 \simeq 127$ MeV.

LOFF phase

The phase diagram at large isospin chemical potential



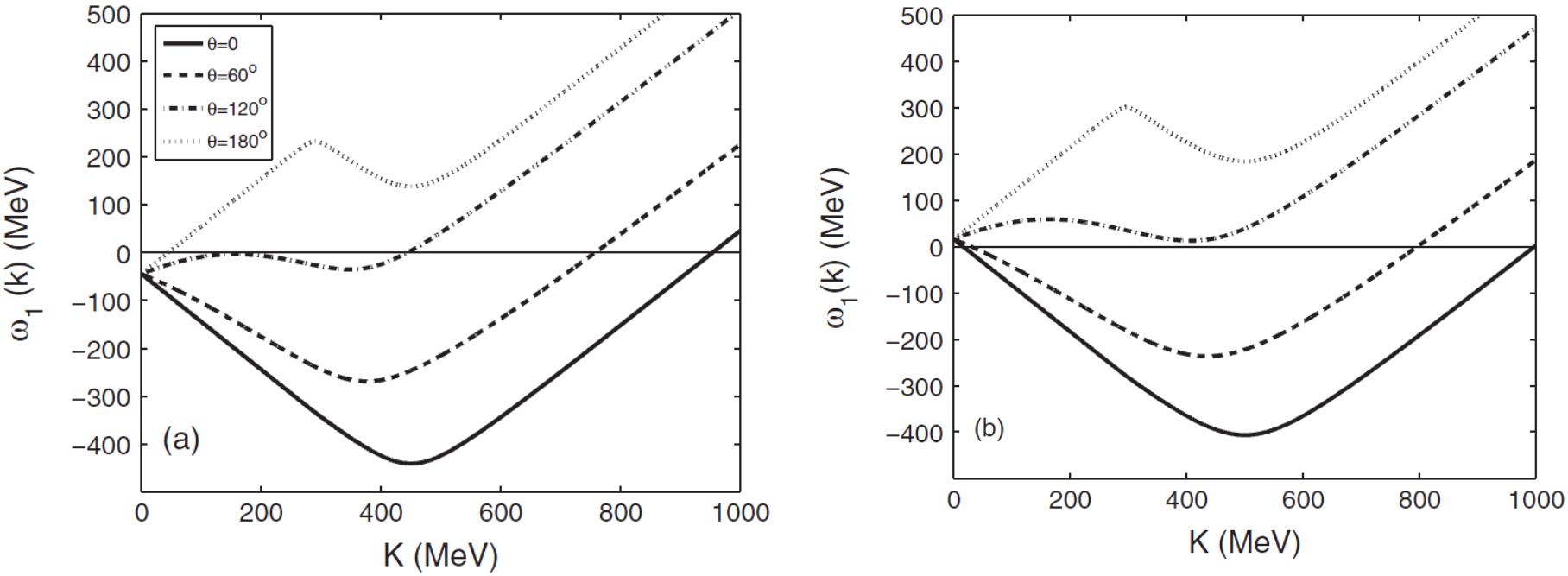


The LOFF window in the weak coupling case: $0.707\Delta_0 < H < 0.754\Delta_0$

Our calculation indicates the LOFF window in the strong coupling region.

Quasiparticle dispersions in LOFF phase:

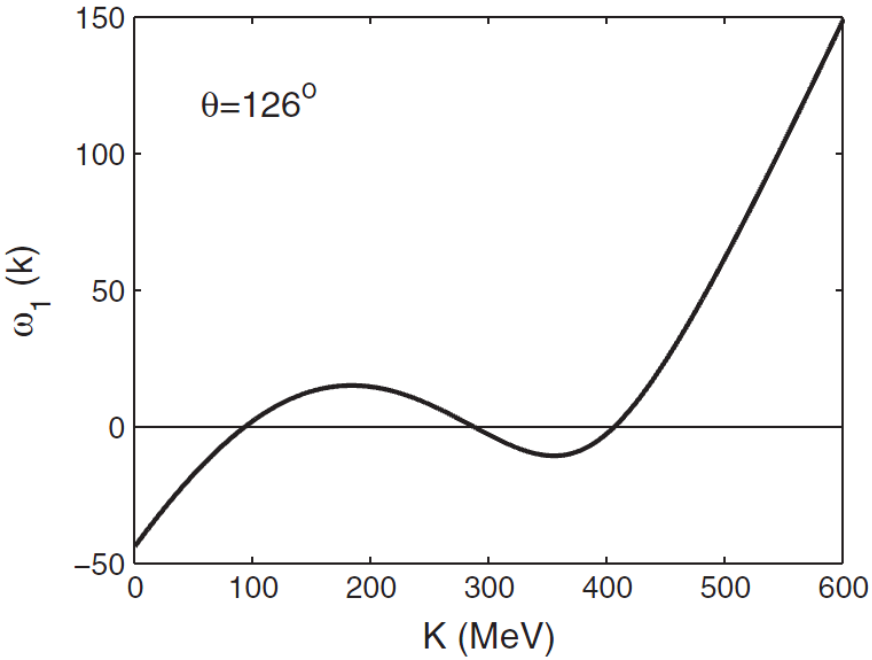
in the weak coupling case: $\omega_{1,2}(\mathbf{k}) = \pm\sqrt{\left(E_{\mathbf{k}} - \frac{\mu_{\mathrm{I}}}{2}\right)^2 + \Delta^2} - (H + q\cos\theta).$



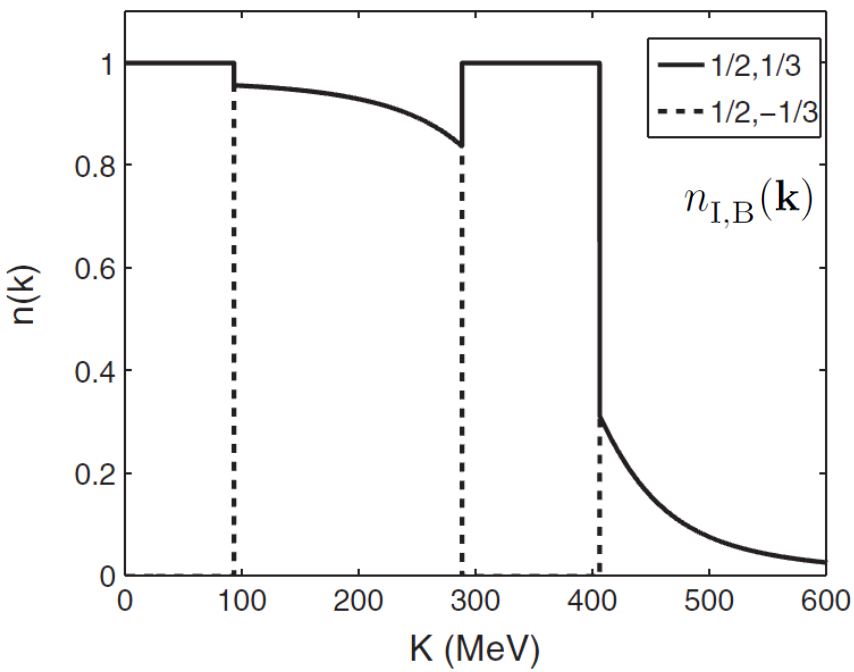
Calculated results of the dispersion relation for the quasiparticle branch $\omega_1(\mathbf{k})$ at several cases of the angle θ. Panel (a) for the case with $\mu_{\mathrm{I}} = 900$ MeV and $H = 217$ MeV, panel (b) for $\mu_{\mathrm{I}} = 1000$ MeV and $H = 210$ MeV.

Quasiparticle dispersions in LOFF phase:

Three gapless nodes

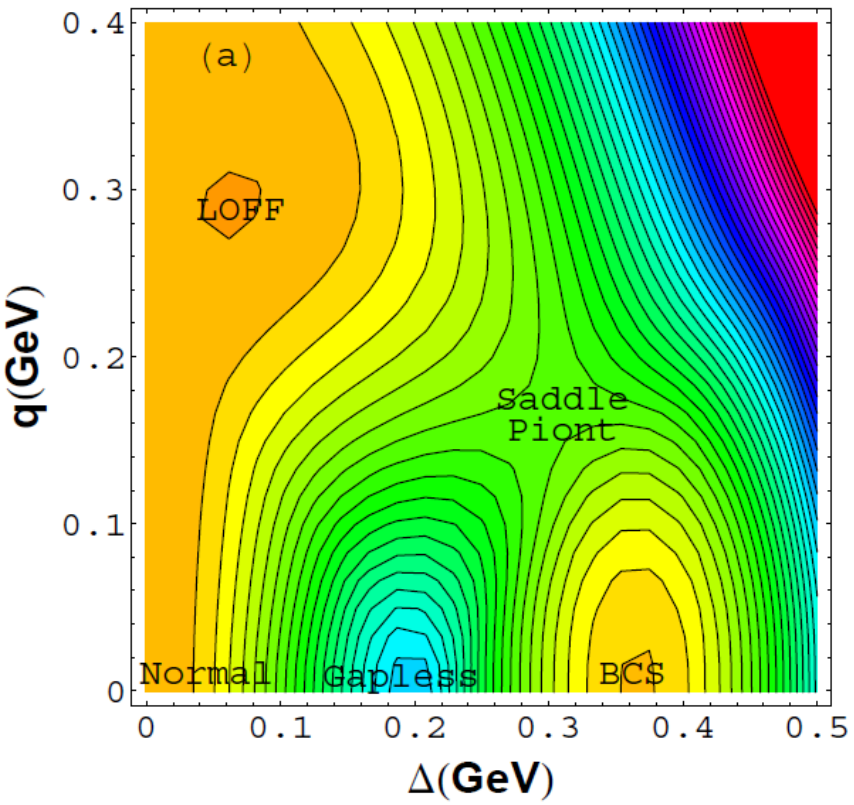


Two blocking regions

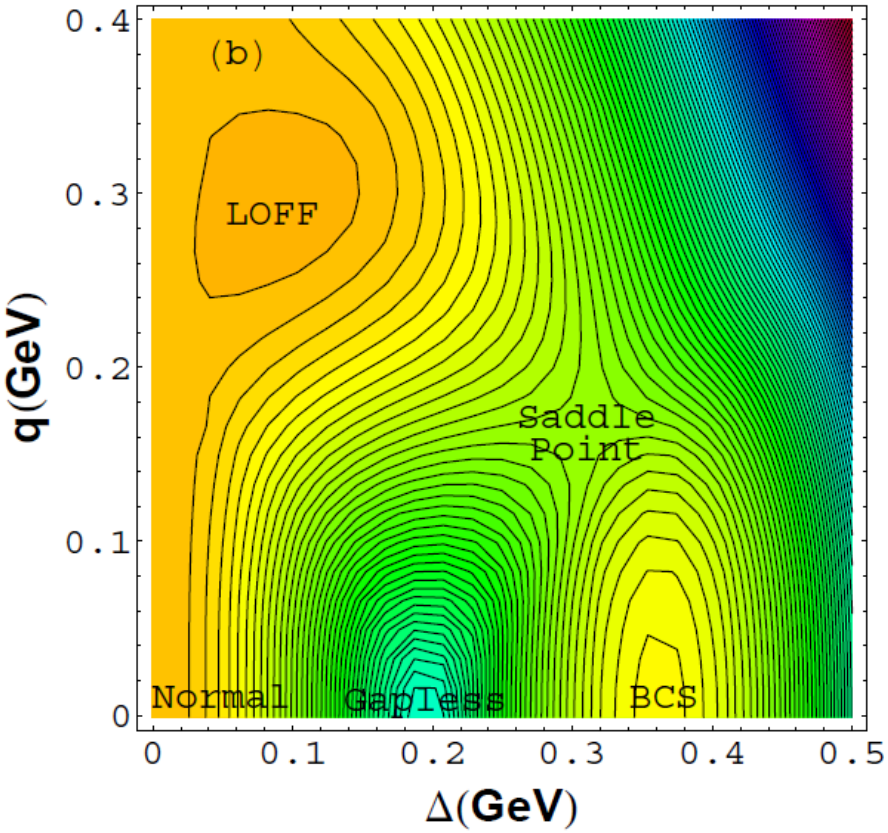


An example of the calculated dispersion relation for the quasiparticle branch $\omega_1(\mathbf{k})$ and the corresponding particle momentum distributions $n(\mathbf{k})$ at $\theta = 126^o$, and with $\mu_{\mathrm{I}} =$ 900 MeV and $H = 217$ MeV.

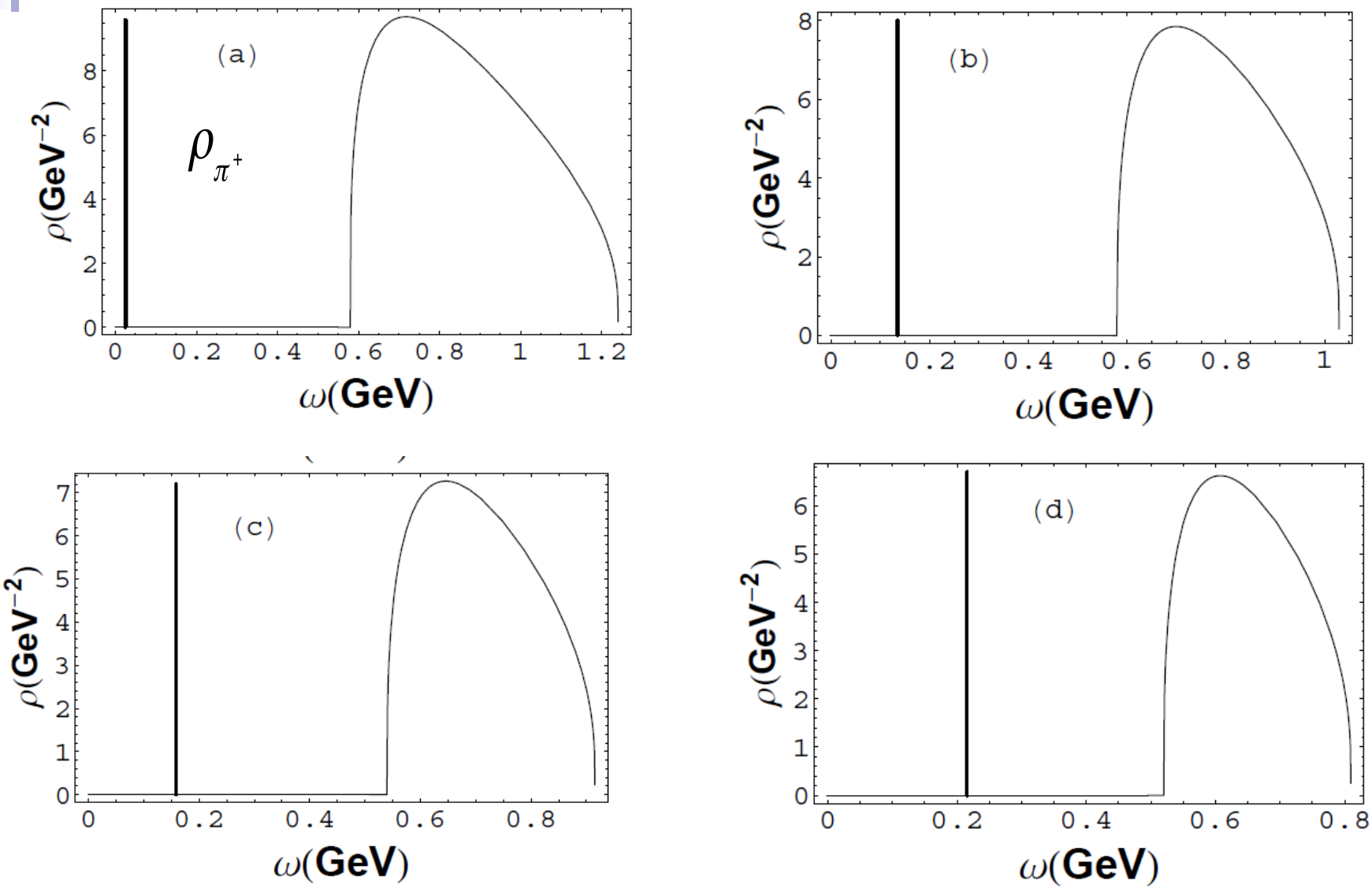
Contour plot of the thermodynamic potential $\Omega_{\rm sub}$



(a) at $\mu_I = 900$ MeV and $H = 217$ MeV



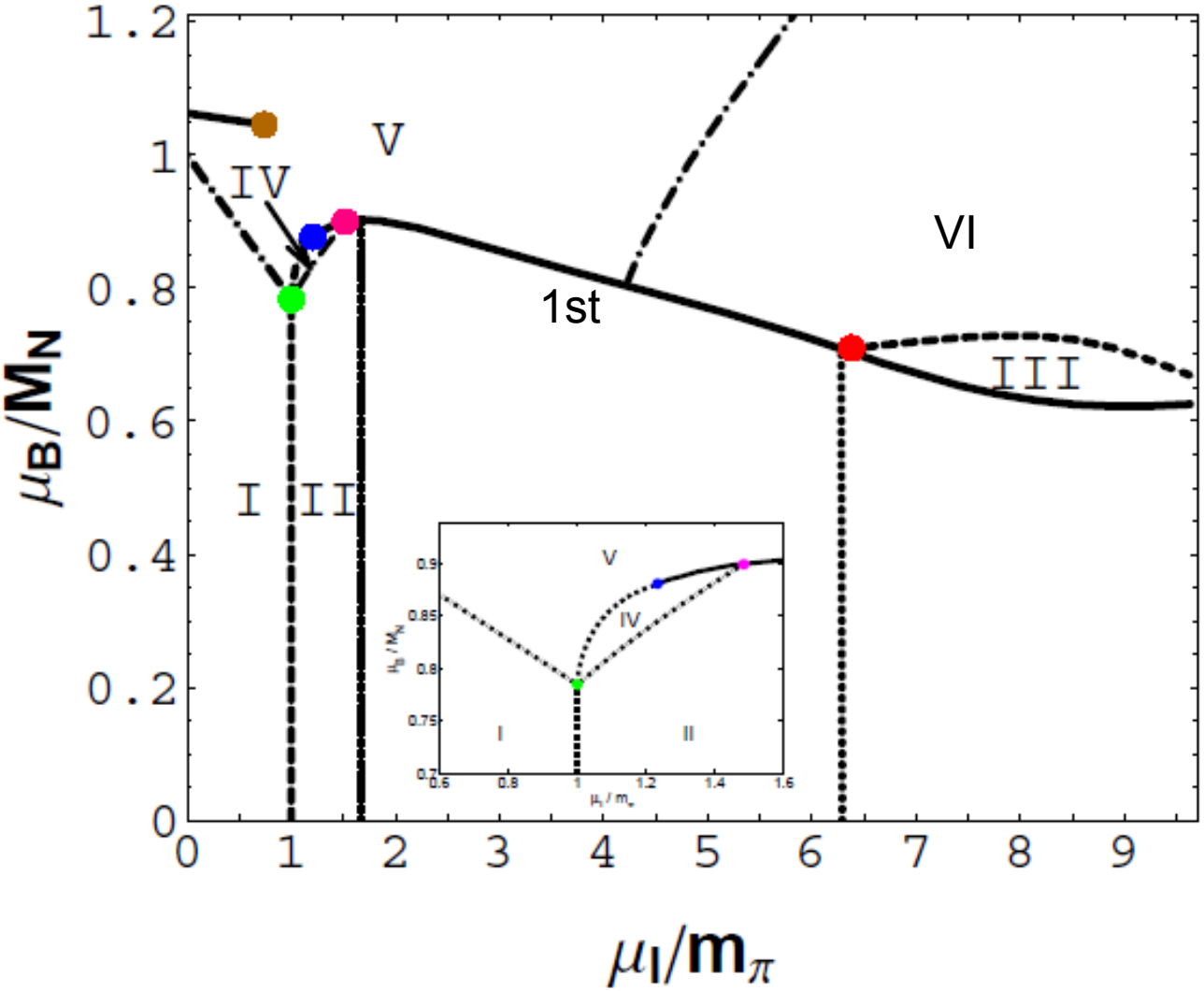
(b) at $\mu_I = 1000$ MeV and $H = 210$ MeV



Calculated spectral function $\rho(\omega, \mathbf{0})$ above the superfluid-normal transition line $H_{\mathrm{Normal}}(\mu_{\mathrm{I}})$ for several isospin chemical potentials (panels (a)-(d) are: (a) for $\mu_{\mathrm{I}} = 170$ MeV, $H = 290$ MeV, (b) for $\mu_{\mathrm{I}} = 300$ MeV, $H = 290$ MeV, (c) for $\mu_{\mathrm{I}} = 400$ MeV, $H = 270$ MeV and (d) for $\mu_{\mathrm{I}} = 500$ MeV, $H = 260$ MeV.

The $\mu_I - \mu_B$ phase diagram

$\mu_I/2 - H < M$



Summary

- In this work, we focus on the case with arbitrary isospin chemical potential μ_{I} and small baryon chemical potential $\mu_{\mathrm{B}} \leq \mu_{\mathrm{B}}^{\chi}$.

  The $\mu_{\mathrm{I}} - \mu_{\mathrm{B}}$ phase diagram shows a rich phase structure since the system undergoes a crossover from a Bose-Einstein condensate of charged pions to a BCS superfluid with condensed quark–antiquark Cooper pairs when μ_{I} increases at $\mu_{\mathrm{B}} = 0$, and a nonzero baryon chemical potential serves as a mismatch between the pairing species.
- We observe a gapless pion condensation phase near the quadruple point $(\mu_{\mathrm{I}}, \mu_{\mathrm{B}}) = (m_{\pi}, M_{\mathrm{N}} - 1.5 m_{\pi})$ The first order chiral phase transition becomes a smooth crossover when $\mu_{\mathrm{I}} > 0.82 m_{\pi}$.

- At very large isospin chemical potential $\mu_{\rm I} > 6.36 m_\pi$, an inhomogeneous LOFF superfluid phase appears in a window of $\mu_{\rm B}$, which should in principle exist for arbitrary large $\mu_{\rm I}$.

  Between the gapless and the LOFF phases, the pion superfluid phase and the normal quark matter phase are connected by a first order phase transition. In the normal phase above the superfluid domain, we find that charged pions are still bound states even though $\mu_{\rm I}$ becomes very large, which is quite different from that at finite temperature. Our phase diagram is in good agreement with that found in imbalanced cold atomic systems.

Thank you !