QCD Relics of Astrophysical Relevance

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Acknowledgments

This set of results is due to the effort of many authors among whom I cite:

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Plan of the Talk

We start with a digression about Effective-Actions and Generalized Nambu Jona-Lasinio formalisms. In the Heavy Quark limit, they are shown to coincide. We obtain an NJL derivation of the known result of QCD sum rules.
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- Next, we summarize the highlights of the GNJL formalism, namely how to use Valatin-Bogoliubov pseudo-unitary transformations to obtain exact solution for fermion condensates. A concrete example of a Mass-Gap equation is given. Multiple solutions—one for each Landau level—are given.
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- We briefly discuss the phenomenological implication of the Mass-Gap equation in the low energy domain of hadronic physics and show how it unifies, in the same vision, so diverse phenomena like N-N repulsive cores, pion masses, hadronic scattering, notably $\pi - \pi$ scattering lengths.
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- Next, we summarize the highlights of the GNJL formalism, namely how to use Valatin-Bogoliubov pseudo-unitary transformations to obtain exact solution for fermion condensates. A concrete example of a Mass-Gap equation is given. Multiple solutions-one for each Landau level-are given.

- We briefly discuss the phenomenological implication of the Mass-Gap equation in the low energy domain of hadronic physics and show how it unifies, in the same vision, so diverse phenomena like N-N repulsive cores, pion masses, hadronic scattering, notably $\pi - \pi$ scattering lengths.

- Finally we discuss the possibility of a gravity stabilized domains—extra solutions of mass gap equation. A Tolman-Oppenheimer-Volkoff calculation of gravitationally stable such domains will be presented. They are shown to be dark
Effective Action–A simple example: Small loops


\[ S = \int \bar{\psi} (\gamma_\mu \partial_\mu + m) \psi + \frac{1}{2} \int_{x,y} j^a_\mu (x) \left< g^2 A^a_\mu (x) A^b_\nu (y) \right> j^b_\nu (y), \]

where \( j^a_\mu \equiv \bar{\psi} \gamma_\mu T^a \psi \) and \( \left< g^2 A^a_\mu (x) A^b_\nu (y) \right> \simeq \frac{1}{4} \lambda y \rho \left< g^2 F^a_\mu \lambda F^b_\nu \rho \right> \).
**Effective Action - A simple example: Small loops**


\[
S = \int_x \bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + \frac{1}{2} \int_{x,y} j^a_\mu(x) \langle g^2 A^a_\mu(x)A^b_\nu(y) \rangle j^b_\nu(y),
\]

where \( j^a_\mu \equiv \bar{\psi} \gamma_\mu T^a \psi \) and \( \langle g^2 A^a_\mu(x)A^b_\nu(y) \rangle \approx \frac{1}{4} x_\lambda y_\rho \langle g^2 F^a_\mu \lambda F^b_\nu \rho \rangle \).

- Bosonization of the four-quark interaction

\[
\langle g^2 A^a_\mu(x)A^b_\nu(y) \rangle = x_\lambda y_\rho \cdot \frac{\langle G^2 \rangle}{48(N_c^2 - 1)} \cdot \delta^{ab}(\delta_\mu_\nu \delta_\lambda_\rho - \delta_\mu_\rho \delta_\lambda_\nu)
\]

goess via an auxiliary Abelian-like field \( A^a_\mu(x) = \frac{1}{2} x_\nu n^a \mathcal{F}_\nu \mu \), where \( \mathcal{F}_\nu \mu \) and \( n^a \) are constant, and \( n^a n^b = \delta^{ab} \).
Effective Action-A simple example: Small loops


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S = \int_x \bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + \frac{1}{2} \int_{x,y} j_\mu^a(x) \langle g^2 A_\mu^a(x) A_\nu^b(y) \rangle j_\nu^b(y),
\]

where \( j_\mu^a \equiv \bar{\psi}\gamma_\mu T^a \psi \) and \( \langle g^2 A_\mu^a(x) A_\nu^b(y) \rangle \simeq \frac{1}{4} x y \langle g^2 F^a_\mu \delta F^b_\nu \rangle \).

- Bosonization of the four-quark interaction

\[
\langle g^2 A_\mu^a(x) A_\nu^b(y) \rangle = x y \delta^{ab} \left( \delta_{\mu\nu} \delta_{\lambda\rho} - \delta_{\mu\rho} \delta_{\lambda\nu} \right)
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goes via an auxiliary Abelian-like field \( A_\mu^a(x) = \frac{1}{2} x_\nu n^a F^a_\mu \), where \( F^a_\mu \) and \( n^a \) are constant, and \( n^a n^b = \delta^{ab} \).

- The resulting one-loop Euler–Heisenberg–Schwinger Lagrangian in the \( A_\mu^a \)-field,

\[
\text{tr} \ln \frac{\gamma_\mu D_\mu [A] + M}{\gamma_\mu \partial_\mu + M} = -2N_f \cdot \text{tr} (T^a T^a) \cdot \int_0^\infty ds \frac{e^{-M^2 s}}{(4\pi^2 s)^2} \left[ a b s^2 \cot(a s) \coth(b s) - 1 \right],
\]

where \( a^2 - b^2 = E^2 - H^2 \), \( a b = |EH| \), can be expanded at large \( M \) in the number of external \( A_\mu^a \)-lines \( \Rightarrow \) an NJL-based derivation of \( \langle \bar{\psi}\psi \rangle \text{heavy} \); \( N_c=3 = -N_f \cdot \frac{\alpha_s \langle (F^a_{\mu\nu})^2 \rangle}{12\pi M} \).
In the infinite current quark mass limit both approaches: the Generalized Nambu, Jona-Lasinio (GNJL) and Effective Action coincide.
Going beyond the Gaussian approximation

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Going outside this limit requires the relaxation of either the constancy of $F_{\nu\mu}$ or (and) to abandon the gaussian approximation in the cumulant expansion. We have the following picture
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- GNJL
- Effective Action
  - Heavy Quark Limit
  - Chiral Limit
  - $m$
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\[
\begin{align*}
\text{GNJL} & \quad \text{Effective Action} \\
\langle g^2 A^a_\mu(x) A^b_\nu(y) \rangle & \quad \mathcal{F}_{\nu\mu} = \text{const.} \\
\text{One Gluon} & \quad \text{Small loops Area Square Law} \\
\text{Heavy Quark Limit} & \quad \text{Chiral Limit}
\end{align*}
\]
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- **GNJL**
  - One Gluon: $\langle g^2 A^a_\mu(x) A^b_\nu(y) \rangle$
  - Confining Ansatz: $K_{\mu\nu}(x, y)$

- **Effective Action**
  - Small loops: $F_{\nu\mu} = \text{const.}$ Area Square Law
  - Large loops: $F_{\nu\mu} \neq \text{const.}$ Area Law

- **Heavy Quark Limit**: $m \rightarrow 0$

- **Chiral Limit**: $m \rightarrow 0$
Going beyond the Gaussian approximation

In the infinite current quark mass limit both approaches: the Generalized Nambu, Jona-Lasinio (GNJL) and Effective Action coincide.

Going outside this limit requires the relaxation of either the constancy of $F_{\nu\mu}$ or (and) to abandon the gaussian approximation in the cumulant expansion. We have the following picture:

- The GNJL approach leads to a small loop expansion with $F_{\nu\mu} = \text{const.}$
- The Effective Action approach leads to a large loop expansion with $F_{\nu\mu} \neq \text{const.}$

The diagrams illustrate the transition from the heavy quark limit to the chiral limit, with intermediate states representing different approximations.

- One Gluon: $\langle g^2 A_\mu^a(x) A_\nu^b(y) \rangle$
- Confining Ansatz: $K_{\mu\nu}(x, y)$
- 2nd. Quantization: Surface Dynamics
- Heavy Quark Limit
- Small loops Area Square Law
- Large loops Area Law
- Chiral Limit
Going beyond the Gaussian approximation

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Going outside this limit requires the relaxation of either the constancy of $F_{\nu\mu}$ or (and) to abandon the gaussian approximation in the cumulant expansion. We have the following picture
The two most fundamental nonperturbative phenomena in QCD are SCSB and confinement. Are they interrelated?
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In Nambu–Jona-Lasinio–type models with confinement:

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\langle \bar{\psi} \psi \rangle \propto -g^2 \langle (F_{\mu \nu}^a)^2 \rangle \times \text{(vacuum correlation length)}
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In Nambu–Jona-Lasinio–type models with confinement:

\[ \langle \bar{\psi}\psi \rangle \propto -g^2 (F^a_{\mu\nu})^2 \times \text{(vacuum correlation length)} \]


The starting idea is to get \( \langle \bar{\psi}\psi \rangle \) from the one-loop quark effective action:

\[ \langle \bar{\psi}\psi \rangle = -\frac{\partial}{\partial m} \langle \Gamma[A^a_{\mu}] \rangle, \]

assuming for the Wilson loop entering \( \langle \Gamma[A^a_{\mu}] \rangle \) an area law.
NJL-Linking SCSB with confinement: A general strategy.

\begin{equation*}
\left\langle \Gamma[A^a_{\mu}] \right\rangle = -(2S + 1)N_f \int_0^\infty \frac{ds}{s} e^{-m^2 s} \int P \mathcal{D}z_\mu \int A \mathcal{D}\psi_\mu e^{-\int_0^s d\tau \left( \frac{1}{4} \dot{z}^2_\mu + \frac{1}{2} \psi_\mu \dot{\psi}_\mu \right)} \times \left\{ \text{tr} \mathcal{P} \exp \left[ ig \int_0^s d\tau T^a \left( A^a_{\mu} \dot{z}_\mu - \psi_\mu \psi_\nu F^a_{\mu\nu} \right) \right] \right\} - N_c \right\}.
\end{equation*}
\[ \langle \Gamma[A^a_{\mu}] \rangle = -(2S + 1) N_f \int_0^\infty \frac{ds}{s} e^{-m^2 s} \int P \mathcal{D}z \int A \mathcal{D}\psi \mu e^{-\int_0^s d\tau \left( \frac{1}{4} \dot{z}^2_{\mu} + \frac{1}{2} \psi_{\mu} \dot{\psi}_{\mu} \right)} \times \]

\[ \times \left\{ \left\langle \text{tr P} \exp \left[ ig \int_0^s d\tau T^a (A^a_{\mu} \dot{z}_{\mu} - \psi_{\mu} \dot{\psi}_{\nu} F^a_{\mu\nu}) \right] \right\rangle - N_c \right\} . \]

Only when \( \int P \mathcal{D}z \int A \mathcal{D}\psi [\cdots] \xrightarrow{\text{const} \sqrt{s}} \) at \( s \to \infty \), we have a finite quark condensate in the chiral limit:

\[ \langle \overline{\psi} \psi \rangle \propto \frac{\partial}{\partial m} \int_0^\infty \frac{ds}{s} e^{-m^2 s} \cdot \frac{\text{const}}{\sqrt{s}} = -2\sqrt{\pi} \cdot \text{const} \]

(T. Banks and A. Casher, '80).
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\[
\langle \Gamma[A^a_{\mu}] \rangle = -(2S + 1) N_f \int_0^\infty \frac{ds}{s} e^{-m^2 s} \int_P Dz_\mu \int_A D\psi_\mu e^{-\int_0^s d\tau \left( \frac{1}{4} \dot{z}^2_\mu + \frac{1}{2} \psi_\mu \dot{\psi}_\mu \right)} \times
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The second idea: Parametrize via \( z_\mu(\tau) \) the minimal area \( S_{\text{min}} \), entering the area law:

\[
\langle W[z_\mu] \rangle = \left\langle \text{tr } \mathcal{P} \exp \left( ig \int_0^s d\tau T^a A^a_\mu \dot{z}_\mu \right) \right\rangle \to N_c \cdot e^{-\sigma(s) \cdot S_{\text{min}}}.
\]
Linking SCSB with confinement: A general strategy.

Find an ansatz for $S_{\text{min}}[z_{\mu}]$ so to enable the analytic calculation of $\left\langle \Gamma[A_{\mu}^a]\right\rangle$, and impose the $\int_{P} \mathcal{D}z_{\mu} \int_{A} \mathcal{D}\psi_{\mu} \cdots \rightarrow 1/\sqrt{s}$ asymptotic behavior $\Rightarrow \sigma(s)$. 
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- A cone-shaped surface in 3D can be generalized to 4D as

$$S_{3d} = \frac{1}{2} \int_0^s d\tau |z \times \dot{z}| \rightarrow S_{4d} = \frac{1}{2\sqrt{2}} \int_0^s d\tau |\varepsilon_{\mu\nu\lambda\rho} z_\lambda \dot{z}_\rho| \geq \frac{1}{4\sqrt{3}} |\Sigma_{\mu\nu}| := S_{\text{min}}[z_\mu],$$

where $\Sigma_{\mu\nu}(s) = \varepsilon_{\mu\nu\lambda\rho} \int_0^s d\tau z_\lambda \dot{z}_\rho.$
Linking SCSB with confinement: A general strategy.

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- The simple exponential ansatz for the Wilson loop at all distances,
  
  $$\langle W[z_\mu] \rangle = N_c \cdot e^{-\tilde{\sigma} |\Sigma_{\mu\nu}|}, \text{ where } \tilde{\sigma}(s) = \frac{\sigma(s)}{4\sqrt{3}},$$
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yields for $\langle \Gamma[A^a_\mu] \rangle$ the Euler–Heisenberg–Schwinger Lagrangian in an auxiliary constant Abelian field $B_{\mu\nu}$, to be averaged with the weight $1/\left(1 + \frac{B_{\mu\nu}^2}{4\tilde{\sigma}^2}\right)^{7/2}$.
Linking SCSB with confinement: A general strategy.

The quark condensate becomes:

$$\langle \bar{\psi} \psi \rangle = -\frac{3N_f}{4\pi^2} \cdot m \int_0^\infty ds \ e^{-m^2 s} \cdot \tilde{\sigma}^2 \cdot f[A(s)],$$

where

$$A(s) \equiv \frac{1}{2\tilde{\sigma}^2 s^2} \quad \text{and} \quad f[A] = \frac{(\sqrt{1+A} - 1)^4 \cdot (5A + 4\sqrt{1+A} + 6)}{(1+A)^{5/2}}.$$
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We obtain \( \tilde{\sigma}(s) \) from the condition

\[ \tilde{\sigma}^2 \cdot f[A] \to \frac{\sigma_0^{3/2}}{\sqrt{s}} \quad \text{at} \quad s \to \infty, \quad \text{where} \quad \sigma_0 = \text{const}. \]

Then, we obtain the chiral condensate:

\[ \langle \bar{\psi} \psi \rangle \simeq -\frac{3N_f}{4\pi^{3/2}} \cdot \sigma_0^{3/2} = -\mathcal{N}, \quad \mathcal{N} = (250 \text{ MeV})^3 \Rightarrow \quad m \gtrsim 2\sqrt{\pi} \left( \frac{\mathcal{N}}{3N_f G_{\text{max}}} \right)^{1/3}, \]
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It is possible to obtain reasonable values for the constituent quark mass \( m = 460 \text{ MeV} \) while reproducing, at the same time, the heavy-quark limit of the squared area law.
The Hamiltonian of a relativistic fermion in an external field $A_\mu$ has the following form in 2+1 dimensions:

$$H = \int d^2 x \bar{\psi}(x) \left[ -i \gamma^j D_j + m \right] \psi(x),$$
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- This constitutes a demonstration model for the more complicated 3+1 dimensions
**NJL** - A simple example: Vacuum Structure in Strong Magnetic fields

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$$H = \int d^2 x \bar{\psi}(x) \left( -i \gamma^j D_j + m \right) \psi(x),$$

Choose $A_\mu = -B y \delta_{\mu 1}$, where $B > 0$ is the magnetic field strength.

This constitutes a demonstration model for the more complicated 3+1 dimensions.

Let us use a Bogoliubov-Valatin transformation to obtain the known results:

$$B \langle 0 | \psi^\dagger(x) \psi(x) | 0 \rangle_B = -\frac{|eB|}{2\pi},$$

$$E_n = \sqrt{m^2 + 2 n |eB|}$$

with $n$ standing for the Landau levels.
We need just three steps to construct the wave-function of a particle in a magnetic field. From,

$$\psi(x) = \sum_p \frac{1}{\sqrt{L_x L_y}} \left\{ u(p) \, a_p + v(p) \, b^\dagger_p \right\} e^{ip \cdot x}$$
A simple case for Valatin-Bogoliubov Transformations

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\[ u(p) = \sqrt{\frac{E_p + m}{2E_p}} \begin{bmatrix} 1 \\ \frac{p_y - ip_x}{E_p + m} \end{bmatrix} ; \quad v(p) = \sqrt{\frac{E_p + m}{2E_p}} \begin{bmatrix} -\frac{p_y + ip_x}{E_p + m} \\ 1 \end{bmatrix} \]

\[ \{ a_p^\dagger, a_{p'} \} = \{ b_p^\dagger, b_{p'} \} = \delta_{p_x p'_x} \delta_{p_y p'_y}, \quad E_p = \sqrt{m^2 + |p|^2}. \]

The \( u \) and \( v \) spinors are the solutions of the Dirac equation for positive and negative energy respectively. (with \( \cos \phi = \sqrt{\frac{E_p + m}{2E_p}}, \quad \sin \phi = \sqrt{\frac{E_p - m}{2E_p}} \)).
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\[ \{ a^\dagger_p, a_p \} = \{ b^\dagger_p, b_p \} = \delta_{p_x p'_x} \delta_{p_y p'_y}, \quad E_p = \sqrt{m^2 + |p|^2}. \]

The \( u \) and \( v \) spinors are the solutions of the Dirac equation for positive and negative energy respectively. (with \( \cos \phi = \sqrt{\frac{E_p + m}{2E_p}}, \ \sin \phi = \sqrt{\frac{E_p - m}{2E_p}}. \))

Step 1: perform the following canonical transformation,

\[ \begin{bmatrix} \tilde{a}_p \\ \tilde{b}^\dagger_{-p} \end{bmatrix} = R_\phi(p) \begin{bmatrix} a_p \\ b^\dagger_{-p} \end{bmatrix} \quad; \quad \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = R^*_\phi(p) \begin{bmatrix} u(p) \\ v(p) \end{bmatrix} \]

\[ R_\phi(p) = \begin{bmatrix} \cos \phi & -\sin \phi (\hat{p}_y + i\hat{p}_x) \\ \sin \phi (\hat{p}_y - i\hat{p}_x) & \cos \phi \end{bmatrix}, \quad \hat{p} = \frac{p}{|p|} \]
A simple example of a non-trivial vacuum

The vacuum associated to the new operators $\tilde{a}$ and $\tilde{b}$ is given by

$$|\tilde{0}\rangle = S|0\rangle = \prod_p (\cos \phi + \sin \phi a_p^\dagger b_{-p}^\dagger)|0\rangle$$

$$\tilde{a}_p|\tilde{0}\rangle = 0 \quad , \quad \tilde{b}_p|\tilde{0}\rangle = 0$$
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We should think of $\psi(x) = \sum_{p} \frac{1}{\sqrt{L_x L_y}} \left\{ u(p) a_{p} + v(p) b_{-p}^\dagger \right\} e^{ip \cdot x}$ as an inner product between the Hilbert space spanned by the spinors $\{u, v\}$ and the Fock space generated by $\{a, b\}$

It is invariant under V-B transformations: any rotation in the Fock space must engender a counter-rotation in the Hilbert space.
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It is invariant under V-B transformations: any rotation in the Fock space must engender a counter-rotation in the Hilbert space.

Choose $\phi$ as to ensure that the new spinors $\tilde{u}$ and $\tilde{v}$ are momentum independent:

$$\tilde{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that all the momentum dependence of $\psi$ is stored in

$$\{\tilde{a}_p, \tilde{b}_p\} = S\{\hat{a}, \hat{b}\} S$$
Use the Landau level representation

\[ e^{ip_y y} = e^{-i\ell^2 p_x p_y} \sqrt{2\pi} \sum_{n=0}^{\infty} i^n \omega_n(\xi) \omega_n(\ell p_y) \]

\[ \omega_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x) \]

\[ l = \sqrt{|eB|}, \quad \xi = \frac{y}{\ell} + lp_x \]
Landau Levels

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\[ l = \sqrt{|eB|} \quad \xi = \frac{y}{l} + l p_x \]

The wave function can now be written in the following way:

\[ \psi(x) = \sum_{n_{px}} \frac{1}{\sqrt{\ell L_x}} \left\{ \hat{u}_{n_{px}}(y) \hat{a}_{n_{px}} + \hat{v}_{n_{px}}(y) \hat{b}_{n_{px}}^\dagger \right\} e^{ip_x x} \]

\[
\begin{bmatrix}
\hat{a}_{n_{px}} \\
\hat{b}_{n_{px}}^\dagger
\end{bmatrix} = \sum_{p_y} \frac{i^n \sqrt{2\pi \ell}}{\sqrt{L_y}} \begin{bmatrix}
\omega_n(\ell p_y) & 0 \\
0 & -\omega_{n-1}(\ell p_y)
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_{p} \\
\tilde{b}_{p}^\dagger
\end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{u}_{n_{px}}(y) \\
\hat{v}_{n_{px}}(y)
\end{bmatrix} = \begin{bmatrix}
\omega_n(\xi) & 0 \\
0 & i\omega_{n-1}(\xi)
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\begin{bmatrix}
\tilde{u} \\
\tilde{v}
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\]
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The wave function can now be written in the following way:

\[ \psi(x) = \sum_{n p_x} \frac{1}{\sqrt{L_y}} \left\{ \hat{u}_{n p_x}(y) \hat{a}_{n p_x} + \hat{v}_{n p_x}(y) \hat{b}^\dagger_{n-p_x} \right\} e^{ip_x x} \]

The new operators satisfy the anticommutation relations:

\[ \{ a_{n p_x}^\dagger, a_{n' p'_x} \} = \{ b_{n p_x}^\dagger, b_{n' p'_x} \} = \delta_{nn'} \delta_{pp'} \]

The vacuum is invariant under this change of basis, i.e., \( \hat{a}_{n p_x} |\tilde{0}\rangle = 0 \), \( \hat{b}_{n p_x} |\tilde{0}\rangle = 0 \)
An example of Mass Gap Equation

There are several approaches one can use:

1-consider the Ward identity or;
2-get rid of anomalous Bogoliubov terms or;
3-Derive it as the condition for the vacuum energy to be a minimum or;
4-use a Dyson equation for the fermion propagator,

Here we use 2. We have with

\[ \cos \theta_n = \sqrt{\frac{E_n + m}{2E_n}} \quad \sin \theta_n = \sqrt{\frac{E_n - m}{2E_n}} \]
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Third step \( R_{\theta_n} = \begin{bmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{bmatrix} \)
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Third step \( R_{\theta_n} = \begin{bmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{bmatrix} \)

Setting the the anomalous terms in the Hamiltonian to zero finds \( \theta_n \). We can obtain the following mass gap equations,

\[
\begin{align*}
(\ell \, m \cos \theta_0 + \sin \theta_2/\sqrt{2}) \sin \theta_0 &= 0, \quad n = 0, \\
\ell \, m \sin 2\theta_n - \sqrt{2n} \cos 2\theta_n &= 0, \quad n > 0,
\end{align*}
\]

For any \( n \) have the following solution: \( \tan 2\theta_n = \frac{\sqrt{2n|eB|}}{m}, \quad E_n = \sqrt{m^2 + 2n|eB|} \)
Let us construct the vacuum state in a magnetic field $|0\rangle_B$, annihilated by the operators $a_{np_x}$ and $b_{np_x}$:

$$a_{np_x}|0\rangle_B = 0, \quad b_{np_x}|0\rangle_B = 0$$
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- $|0\rangle_B = \prod_{n \neq p_x} (\cos \theta_n + \sin \theta_n \hat{a}^\dagger_{np_x} \hat{b}^\dagger_{n-p_x}) |\tilde{0}\rangle$
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Finally we come to the problem of dynamical symmetry breaking in the presence of the magnetic field. We obtain

$$\langle 0 | \psi^{\dagger}(x) \psi(x) | 0 \rangle_B = -\frac{|eB|}{2\pi}$$

A fermion condensate occurs even in the absence of any additional interaction between fermions. This is an inherent property of the 2+1 dimensional Dirac theory in an external magnetic field.
Vacuum Condensates and 3+1

Let us construct the vacuum state in a magnetic field $|0\rangle_B$, annihilated by the operators $a_{npx}$ and $b_{npx}$:

- $a_{npx} |0\rangle_B = 0$, $b_{npx} |0\rangle_B = 0$

- $|0\rangle_B = \prod_{npx} (\cos \theta_n + \sin \theta_n \hat{a}_{npx}^{\dagger} \hat{b}_{n-px}^{\dagger}) |\tilde{0}\rangle$

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A fermion condensate occurs even in the absence of any additional interaction between fermions. This is an inherent property of the 2+1 dimensional Dirac theory in an external magnetic field.

In 3+1 Dimensions with quartic interactions we can perform these very same 3-steps.
A class of Hamiltonians

Consider now the simplest Hamiltonian containing the ladder-Dyson-Schwinger machinery for chiral symmetry.

In any case most of the results presented here do not depend on the kernel choice

\[
H = \int \, d^3x \, q^+(x) \left( -i \bar{\alpha} \cdot \vec{\nabla} \right) q(x) + \int \frac{d^3x \, d^3y}{2} J^a_\mu(x) K^{ab}_{\mu\nu}(x - y) J^b_\nu(y)
\]
A class of Hamiltonians

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With,

- \[ J^a_\mu (x) = \bar{q}(x) \gamma_\mu \frac{\lambda^a}{2} q(x) \]
- \[ K_{\mu \nu}^{ab} (x - y) = \delta^{ab} K_{\mu \nu} (|x - y|) \]
A class of Hamiltonians

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\[ H = \int d^3x \ q^+(x) \left( -i \vec{\alpha} \cdot \vec{\nabla} \right) q(x) + \int \frac{d^3x \ d^3y}{2} J_\mu^a(x) K_{\mu\nu}^{ab}(x-y) J_\nu^b(y) \]

With,

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- \( K_{\mu\nu}^{ab}(x-y) = \delta^{ab} K_{\mu\nu}(|x-y|) \)

This class of Hamiltonians has rich phenomenological consequences enabling us to study a variety of hadronic phenomena controlled by global symmetries

- Reproduces in a non-trivial manner the low energy properties of pion physics like, for instance, \( \pi - \pi \) Weinberg results for the scattering lengths together with Oakes-Renner, Goldberger-Treiman....
- Possesses the mechanism of pole-doubling in what concerns scalar decays (Unitarization).
Bogoliubov Transformations

We can rotate the creation and annihilation Fock space operators. It is canonical!

\[ |\tilde{0}> = Exp \left\{ \hat{Q}_0^+ - \hat{Q}_0 \right\} |0> \]

\[ \hat{Q}_0^+(\Phi) = \sum_{cf} \int d^3p \Phi(p) M_{ss'}(\theta, \phi) \hat{b}_{cs}^+(\vec{p}) \hat{d}_{cs'}^+(-\vec{p}) \]
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With, the \(^3P_0\) Coupling (Parity +):

\[ M_{ss'} (\theta, \phi) = -\sqrt{8\pi} \sum_{m_l m_s} \left[ \begin{array}{cc} 1 & 1 \\ m_l & m_s \end{array} \right] \times \left[ \begin{array}{cc} 1/2 & 1/2 \\ s & s' \end{array} \right] \frac{1}{2}^{m_s} y_{l m_l} (\theta, \phi) \]
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With, the \( ^3P_0 \) Coupling (Parity +):

\[ M_{ss'}(\theta, \phi) = -\sqrt{8\pi} \sum_{m_l m_s} \begin{pmatrix} 1 & 1 & 0 \\ m_l & m_s & 0 \end{pmatrix} \times \begin{pmatrix} 1/2 & 1/2 & 1 \\ s & s' & m_s \end{pmatrix} y_{1m_l}(\theta, \phi) \]

The functions \( \Phi(p) \) classify the infinite set of possible Fock spaces:
Fock Space

The Fock space operators transform like, \( \tilde{b}_{cfs}(p) = S \hat{b}_{cfs} S^{-1} \), so that,
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\[
\begin{bmatrix}
\tilde{b} \\
\tilde{d}^+
\end{bmatrix}_s = \begin{bmatrix}
\cos \phi & -\sin \phi M_{ss'} \\
\sin \phi M_{ss'}^* & \cos \phi
\end{bmatrix}
\begin{bmatrix}
\hat{b} \\
\hat{d}^+
\end{bmatrix}_{s'}
\]
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\begin{bmatrix}
\hat{b} \\
\hat{d}^+
\end{bmatrix}_{s'}
\]

Then we can consider the fermion field \( \Psi_{fc}(\vec{x}) \) as an inner product between the Hilbert space spanned by the spinors \( \{u,v\} \) and the Fock space spanned by the operators \( \{\hat{b},\hat{d}\} \):

\[
\Psi_{fc}(\vec{x}) = \int d^3p \left[ u_s(p) b_{cfs}(\vec{p}) + u_s(p) d_{cfs}^+(\vec{-p}) \right] e^{i\vec{p} \cdot \vec{x}}
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\sin \phi M_{ss}' & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\hat{b} \\
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\]

So that requiring invariance of \( \Psi_{fc}(\vec{x}) \) under the Fock space rotations, is tantamount to require a counter-rotation of the spinors \( u \) and \( v \),

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi M_{ss}' \\
\sin \phi M_{ss}' & \cos \phi
\end{pmatrix} \begin{pmatrix}
u \\
v
\end{pmatrix}
\]
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Then we can consider the fermion field $\Psi_{fc}(\vec{x})$ as an inner product between the Hilbert space spanned by the spinors $\{u,v\}$ and the Fock space spanned by the operators $\{\hat{b},\hat{d}\}$:

$$
\Psi_{fc}(\vec{x}) = \int d^3 p \left[ u_s(p) b_{cfs}(\vec{p}) + u_s(p) d_{cfs}^+(\vec{-p}) \right] e^{i \vec{p} \cdot \vec{x}}
$$

So that requiring invariance of $\Psi_{fc}(\vec{x})$ under the Fock space rotations, is tantamount to require a counter-rotation of the spinors $u$ and $v$,

$$
\begin{bmatrix}
u \\
u
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & -\sin \phi M^*_{ss'} \\
\sin \phi M_{ss'} & \cos \phi
\end{bmatrix}
\begin{bmatrix}
u \\
u
\end{bmatrix}
$$

The $\{u,v\}$, contain now the information on the angle $\phi(p)$. 

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Renormalized fermion propagators

Variation in $\phi$ is the same as to cut the fermion propagators $S_\phi$. 
Renormalized fermion propagators

Variation in $\phi$ is the same as to cut the fermion propagators $S_\phi$, 

$$\delta_\phi \left[ S_\phi \right] + (\text{diagram}) = 0$$

$$\langle \text{diagram} \rangle + \text{diagram} \Rightarrow H_2^A = 0$$
Renormalized fermion propagators

Variation in $\phi$ is the same as to cut the fermion propagators $S_\phi$,

$$\delta \phi \left[ \begin{array}{c} \circled{S_\phi} \\ \circ \end{array} \right] + \left[ \begin{array}{c} \circ \end{array} \right] = 0$$

$$\Gamma_\mu(p, p') = \gamma_\mu + i \int \frac{d^4 q}{(2\pi)^4} K(q) \Omega S(p' + q) \Gamma_\mu(p' + q, p + q) \Omega S(p' + q)$$

$$i(p - p')^\mu \Gamma_\mu(p, p') = S^{-1}(p') - S^{-1}(p)$$

With the full propagator being,

$$S = S_0 + S_0 \Sigma S_0 + S_0 \Sigma S_0 \Sigma S_0 + \ldots = S_0 + S_0 \Sigma S_0$$

$$\Sigma = \left[ \begin{array}{c} \circ \end{array} \right] + \left[ \begin{array}{c} \circ \end{array} \right] + \ldots = S$$

$$S = S_0 + S_0 \Sigma S_0, \quad S^{-1}(p_0, \vec{p}) = S_0^{-1}(p_0, \vec{p}) - \Sigma(\vec{p})$$
Renormalized fermion propagators

Variation in $\phi$ is the same as to cut the fermion propagators $S_\phi$,

\[
\delta_\phi \left[ \begin{array}{c} \bigcirc & S_\phi \\ \text{scissors} \end{array} \right] + \left[ \begin{array}{c} \bigcirc \\ \text{scissors} \end{array} \right] = 0
\]

\[
\left\langle \begin{array}{c} \text{scissors} \\ \bigcirc \\ \text{scissors} \end{array} \right\rangle + \left\langle \begin{array}{c} \text{scissors} \\ \bigcirc \bigcirc \bigcirc \bigcirc \end{array} \right\rangle \rightarrow H_2^A = 0
\]

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- $i(p - p')^\mu \Gamma_\mu(p, p') = S^{-1}(p') - S^{-1}(p)$
- With the full propagator being,

\[
S = S_0 + \sum S_0 + \sum S_0 S_0 + \sum S_0 S_0 S_0 + \cdots = S_0 + S_0 \sum S
\]

\[
\sum = S_0 + \sum + \cdots = S
\]

\[
S = S_0 + S_0 \sum S, \quad S^{-1}(p_0, \vec{p}) = S_0^{-1}(p_0, \vec{p}) - \sum(\vec{p})
\]
Quasi-Classical regime and $\hbar$ expansions

The equation for the mass operator $\Sigma$ is non-linear,

$$i\Sigma(p) = \hbar \int \frac{d^4k}{(2\pi\hbar)^4} V(p - k)\gamma_0 \frac{1}{S^{-1}_0(k_0,k) - \Sigma(k)\gamma_0},$$
Quasi-Classical regime and $\hbar$ expansions

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$$i\Sigma(\vec{p}) = \hbar \int \frac{d^4k}{(2\pi\hbar)^4} V(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k,\vec{k}) - \Sigma(\vec{k})} \gamma_0,$$

- the Fourier transform of the linear potential $\sigma|\vec{x}|$ is (with L. Glozman and A. Nefediev):

$$V(\vec{p}) = \int d^3 x e^{i\frac{\vec{p}\vec{x}}{\hbar}} \sigma|\vec{x}| = -\frac{8\pi\sigma\hbar^4}{p^4} = \hbar^4 \tilde{V}(\vec{p}),$$

where $\tilde{V}(\vec{p})$ does not contain $\hbar$. 
Quasi-Classical regime and $\hbar$ expansions

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  \[
i\Sigma(\vec{p}) = \hbar \int \frac{d^4 k}{(2\pi \hbar)^4} V(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k_0, \vec{k}) - \Sigma(\vec{k}) \gamma_0},
  \]

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  \[
  V(\vec{p}) = \int d^3 x e^{i \frac{\vec{p} \cdot \vec{x}}{\hbar}} \sigma|\vec{x}| = -\frac{8\pi \sigma \hbar^4}{p^4} = \hbar^4 \tilde{V}(\vec{p}),
  \]
  where $\tilde{V}(\vec{p})$ does not contain $\hbar$.

- we find: $i\Sigma(\vec{p}) = \hbar \int \frac{d^4 k}{(2\pi)^4} \tilde{V}(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k_0, \vec{k}) - \Sigma(\vec{k}) \gamma_0}$.

Each loop brings an extra power of $\hbar$. 

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$$i\Sigma(\vec{p}) = \hbar \int \frac{d^4 k}{(2\pi\hbar)^4} V(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k_0,\vec{k}) - \Sigma(\vec{k})} \gamma_0,$$

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where $\tilde{V}(\vec{p})$ does not contain $\hbar$.

we find: $i\Sigma(\vec{p}) = \hbar \int \frac{d^4 k}{(2\pi)^4} \tilde{V}(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k_0,\vec{k}) - \Sigma(\vec{k})} \gamma_0.$

Each loop brings an extra power of $\hbar$.

Parametrize $\Sigma(\vec{p})$, in the form, $\Sigma(\vec{p}) = [A_p - m] + (\vec{\gamma} \hat{p}) [B_p - p],

to obtain the dressed–quark Green’s function: $S^{-1}(\vec{p}, p_0) = \gamma_0 p_0 - (\vec{\gamma} \hat{p}) B_p - A_p.$
Quasi-Classical regime and $\hbar$ expansions

The equation for the mass operator $\Sigma$ is non-linear,

$$i\Sigma(\vec{p}) = \hbar \int \frac{d^4k}{(2\pi\hbar)^4} V(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k_0, \vec{k}) - \Sigma(\vec{k}) \gamma_0},$$

the Fourier transform of the linear potential $\sigma|\vec{x}|$ is (with L. Glozman and A. Nefediev):

$$V(\vec{p}) = \int d^3x e^{i \frac{\vec{p}}{\hbar} \cdot \vec{x}} \sigma|\vec{x}| = -\frac{8\pi\sigma\hbar^4}{p^4} = \hbar^4 \tilde{V}(\vec{p}),$$

where $\tilde{V}(\vec{p})$ does not contain $\hbar$.

we find: $i\Sigma(\vec{p}) = \hbar \int \frac{d^4k}{(2\pi)^4} \tilde{V}(\vec{p} - \vec{k}) \gamma_0 \frac{1}{S_0^{-1}(k_0, \vec{k}) - \Sigma(\vec{k}) \gamma_0}.$

Each loop brings an extra power of $\hbar$.

Parametrize $\Sigma(\vec{p})$, in the form, $\Sigma(\vec{p}) = [A_p - m] + (\vec{\gamma}\hat{\vec{p}})[B_p - p]$,

to obtain the dressed–quark Green’s function: $S^{-1}(\vec{p}, p_0) = \gamma_0 p_0 - (\vec{\gamma}\hat{\vec{p}})B_p - A_p$.

The functions $A_p$ and $B_p$ represent the scalar part and the space–vectorial part of the effective Dirac operator.

Finally $\tan \varphi_p = \frac{A_p}{B_p}$

$\varphi_p \to \infty \to 0$: only the vectorial part survives
$\varphi_p \to 0 \to \pi/2$: only the scalar part survives
Breakdown of the expansion for $\varphi_p$ in powers of $\hbar$

The mass gap equation $A_p \cos \varphi_p - B_p \sin \varphi_p = 0$, with

$$A_p = m + \frac{\hbar}{2} \int \frac{d^3 k}{(2\pi)^3} \tilde{V}(\vec{p} - \vec{k}) \sin \varphi_k, \quad B_p = p + \frac{\hbar}{2} \int \frac{d^3 k}{(2\pi)^3} (\hat{p} \hat{k}) \tilde{V}(\vec{p} - \vec{k}) \cos \varphi_k$$
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- For free particles the chiral angle reduces to the free Foldy angle, $\varphi_p^{(0)} = \arctan \frac{m}{p}$, which diagonalizes the free Dirac Hamiltonian $H = \vec{\alpha} \vec{p} + \beta m$
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1-Introduce dimensionless variables in the integral, $\vec{p} = \mu c \vec{x}$ and $\vec{k} = \mu c \vec{y}$;

2-define $\mu$ such that the resulting equation does not contain any scale at all. We have $\mu = \sqrt{\sigma \hbar c}/c^2$ and expand $\varphi_p$ in low–momentum,
  \[
  \varphi_{p_{p\to0}} \approx \frac{\pi}{2} - \text{const} \frac{pc}{\mu c^2} + \ldots = \frac{\pi}{2} - \text{const} \frac{pc}{\sqrt{\sigma \hbar c}} + \ldots.
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- We cannot build an action $S$ out of the string tension $\sigma$ and the speed of light $c$ to obtain an expansion $\varphi_p = \frac{\hbar}{3} \times f_1(p) + \frac{\hbar^2}{3^2} \times f_2(p) + \ldots$
\( \hbar \) expansions: \( m \neq 0 \)

With \( m \neq 0 \) things change: the classical action \( S \sim \frac{m^2 c^3}{\sigma} \); we have an expansion parameter \( \frac{\sigma \hbar c}{(mc)^2} \) and the mass–gap admits a solution in the form of a “perturbative” series in powers of \( \hbar \),

\[
\varphi_p = \sum_{n=0}^{\infty} \left( \frac{\sigma \hbar c}{(mc)^2} \right)^n \tilde{f}_n \left( \frac{p}{mc} \right).
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\textbf{$\hbar$ expansions: $m \neq 0$}

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To sum it up: we have two different regimes according to the parameter $m/\sqrt{\sigma}$: The spontaneous breaking of chiral symmetry is relevant for $m \ll \sqrt{\sigma}$, with heavy quark physics relevant for the opposite
Summary on V.B. transformations

The true vacuum, with the minimal vacuum energy, contains an infinite set of strongly correlated $^3P_0$ quark-antiquark pairs

$$|0[\varphi]\rangle = S_\varphi |0\rangle_0 = e^{Q_0^+ - Q_0} |0\rangle_0$$
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\[ w_{0p} = \cos^4 \frac{\varphi_p}{2}, \; w_{1p} = 2 \sin^2 \frac{\varphi_p}{2} \cos^2 \frac{\varphi_p}{2}, \; w_{2p} = \sin^4 \frac{\varphi_p}{2} \]
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- $\langle 0[\varphi]|0[\varphi]\rangle = \prod_p (w_{0p} + w_{1p} + w_{2p}) = 1, \langle 0[\varphi]|0\rangle_0 = \exp \left[ V \sum_p \ln \left( \cos^2 \frac{\varphi_p}{2} \right) \right] \xrightarrow{V \to \infty} 0$
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- The V.B. transformations constitute an Abelian Group:

$$S_\varphi \begin{bmatrix} b^\dagger \\ d \end{bmatrix} S_{-\varphi} \rightarrow \mathcal{R}[\varphi] \begin{bmatrix} b^\dagger \\ d \end{bmatrix} \quad \mathcal{R}[\varphi] \mathcal{R}[\tilde{\varphi}] = \mathcal{R}[\varphi + \tilde{\varphi}].$$
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The true vacuum, with the minimal vacuum energy, contains an infinite set of strongly correlated \( ^3P_0 \) quark-antiquark pairs

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\[
S_{\varphi} \begin{bmatrix} b^\dagger \\ d \end{bmatrix} S_{-\varphi} \rightarrow R[\varphi] \begin{bmatrix} b^\dagger \\ d \end{bmatrix}, \quad R[\varphi] R[\tilde{\varphi}] = R[\varphi + \tilde{\varphi}].
\]

Regardless of the particular form of the \( K_{\mu \nu}(x, y) \) these class of models have phenomenologically nice features:

1. For massless quarks it possesses a massless pion. (As an instance of the Mass Gap)

2. It is at least qualitatively successful in describing hadronic scattering, namely the issue of \( \pi - \pi \) scattering: (The Adler zeros)
The pion: An example of Mass Gap

P. Bicudo, S. Cotanch, F. Llanes-Estrada, P. Maris, JEFTR, A. Szczepaniak

FIG. 1. Pion Salpeter equation. In terms of the Dirac matrices $\beta$ and $\bar{\alpha}$, the projection operators for the quark propagator, with momentum $\vec{k}$, are $\Lambda^\pm = (1 \pm \sin(\phi)\beta \pm \cos(\phi)\bar{\alpha} \cdot \vec{k})/2$, and denoted in the figure by $\{+, -\}$. Note that $\Phi^\pm$ is consistent with the normalization condition, Eq. (8), and should contain the cluster propagators obtained after integrating the quark propagator energy, $E_q$. This is the reason the propagator cuts are displayed in the figure. Two such cluster propagators are needed for the two $\Phi^i$'s but only one is generated per integration loop. This necessitates multiplying and dividing the diagrams by the missing cluster propagator leading to the factors $\pm m_\pi + E_q + E_{\bar{q}}$ appearing in the diagram.
Exotic versus non-exotic: $\pi-\pi$ scattering

A pictorial description of the Weinberg formula for the scattering lengths of pions. We have,
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\[ \text{Adler Zero} = \text{Mass Gap equation} + (\text{Exotic} + \text{Non-Exotic} = 0) \]
Macroscopic properties of a replica-filled domain

The full internal-energy density \( \varepsilon(T) = \varepsilon_{\text{vac}}(T) + \varepsilon_h(T) \). of a macroscopic domain whose quantum states are built on top of a replica goes as follows.
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$\varepsilon_{\text{vac}}(T) = \varepsilon_R(T) - \varepsilon_0(T)$: the difference of the internal-energy density of the excited vacuum, corresponding to a replica state inside the domain, and the internal-energy density of the unexcited vacuum outside the domain.
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- The internal-energy density of the unexcited vacuum has the form

$$\varepsilon_{0}(T) = \frac{1}{4} \left[ -\frac{b}{32\pi^{2}} \langle G^{2} \rangle_{T} + (m_{u} + m_{d}) \langle \bar{\psi}\psi \rangle_{T} \right]$$
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\]

- Use the known temperature-dependent gluonic and chiral condensates, which at temperatures \( T \ll m_\pi \) of interest read

\[
\langle G^2 \rangle_T = \langle G^2 \rangle - \frac{24m_\pi^2 T}{b} S_1 \left( \frac{m_\pi}{T} \right), \quad \langle \bar{\psi}\psi \rangle_T = \langle \bar{\psi}\psi \rangle \left[ 1 - \frac{3m_\pi T}{4\pi^2 f_\pi^2} S_1 \left( \frac{m_\pi}{T} \right) \right].
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$$

- Using the Gell-Mann–Oakes–Renner relation we have

$$
\varepsilon_0(T) = \varepsilon_0(0) + \frac{3m_\pi^3 T}{8\pi^2} S_1 \left( \frac{m_\pi}{T} \right),
$$

where we used the expansion

$$
S_\nu(x) \equiv \sum_{n=1}^{\infty} \frac{K_\nu(n x)}{n^\nu}.
$$
Macroscopic properties of a replica-filled domain

For the internal-energy density of the excited vacuum (GMOR rules), we must have an expression similar:

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\]

\[
\varepsilon_{\text{vac}}(T) = \varepsilon + \frac{3T}{8\pi^2} \left[ m_{\pi R}^3 S_1 \left( \frac{m_{\pi R}}{T} \right) - m_{\pi}^3 S_1 \left( \frac{m_{\pi}}{T} \right) \right],
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Neglect the outer temperature with respect to the temperature inside the domain (neglect the outer pressure). Then the pressure of the relativistic pionic gas inside the domain reads:

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p_h(T) = \frac{3m^2_{\pi R} T^2}{2\pi^2} S_2 \left( \frac{m_{\pi R}}{T} \right).
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Use standard thermodynamics \( \varepsilon_h(T) = T \frac{dp_h(T)}{dT} - p_h(T). \) to obtain at temperatures \( T \ll m_{\pi} \)

\[ \varepsilon(T) \simeq \varepsilon - \frac{3Tm^3_{\pi}}{8\pi^2} K_1 \left( \frac{m_{\pi}}{T} \right). \]
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$$p_{h}(T) = \frac{3m_{\pi R}^2 T^2}{2\pi^2} S_2 \left( \frac{m_{\pi R}}{T} \right).$$

Use standard thermodynamics $\varepsilon_h(T) = T \frac{dp_{h}(T)}{dT} - p_{h}(T)$. to obtain at temperatures $T \ll m_{\pi}$

$$\varepsilon(T) \simeq \varepsilon - \frac{3TM_{\pi}^3}{8\pi^2} K_1 \left( \frac{m_{\pi}}{T} \right).$$

$m_{\pi R} = 250$ MeV, $\langle \bar{\psi}\psi \rangle_R = -(100$ MeV$)^3$, $\varepsilon \simeq (250$ MeV$)^4$, $m_{\pi} = 140$ MeV we plot these quantities. Up to the temperatures $\sim 20$ MeV we can disregard hadronic contributions.
Macroscopic properties of a replica-filled domain

![Graph showing exact and approximate ε(T) vs. T [MeV]]
Approximate the internal-energy density of the domain by the constant value $\varepsilon(T) \simeq \varepsilon = (250 \text{ MeV})^4$. 

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Tolman–Oppenheimer–Volkoff defines the maximal possible radius of the domain as

$$R_G = \frac{1}{\sqrt{3\pi\varepsilon G}} \simeq 14 \text{ km},$$
Assume that the matter forming a star is a perfect fluid: This implies the energy-momentum tensor of the form $T^{\mu\nu} = (p + \varepsilon) u^{\mu} u^{\nu} - pg^{\mu\nu}$, with $u^{\mu}(x)$ being the four-velocity of the fluid, such that $g_{\mu\nu} u^{\mu} u^{\nu} = 1$. 
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In the local rest frame of the fluid, where \( u_\mu = (\sqrt{g_{00}}, 0) \), the energy-momentum tensor is diagonal:

\[
T^\mu_{\ \nu} = (p + \varepsilon)u_\mu u_\nu - p \delta^\mu_{\ \nu} = \text{diag} (\varepsilon, -p, -p, -p).
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Gravitational field of a spherical object with constant energy density

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For \( r < R \), we have \( M(r) = 4\pi \int_0^r dr' r'^2 \varepsilon(r') \). The Einstein equation \( R^{0\,0} - \frac{1}{2} R = 8\pi GT^{0\,0} \) can be written as

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\frac{e^{-b}}{r} \left( \frac{db}{dr} - \frac{1}{r} \right) + \frac{1}{r^2} = 8\pi G\varepsilon.
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$$e^{-b} \left( \frac{db}{dr} - \frac{1}{r} \right) + \frac{1}{r^2} = 8\pi G \varepsilon.$$

The function $a(r)$ inside the star can be found from the covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$,

$$-\partial_\mu p \cdot g^{\mu\nu} + \partial_\mu [(p + \varepsilon)u^\mu u^\nu] + (p + \varepsilon) \left( \Gamma^\mu_{\lambda\mu} u^\lambda u^\nu + \Gamma^\nu_{\lambda\mu} u^\mu u^\lambda \right) = 0.$$
Gravitational field of a spherical object with constant energy density

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\[ \partial_\mu [(p + \varepsilon) u^\mu u^\nu] = \partial_0 [(p + \varepsilon) u^0 u^0] = 0. \]
We get
\[ \frac{d \ln g_{00}}{dr} = -2 \cdot \frac{dp/dr}{p + \varepsilon}. \]
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The solution is \[ g_{00}(r) = g_{00}(R) \cdot \exp \left[ 2 \int_r^R dr' \frac{dp/dr'}{p + \varepsilon} \right], \quad g_{00}(R) = 1 - \frac{r g}{R}. \] We have \[ a(r) = \ln g_{00}(r), \quad b(r) \text{ and } a(r) \text{ interpolate smoothly } r < R \text{ and } r > R. \]
Gravitational field of a spherical object with constant energy density

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The equation for \( p(r) \) can be obtained from the Einstein equation \( \mathcal{R}_{1 1} - \frac{1}{2} \mathcal{R} = 8\pi G T_{1 1}, \) that is
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\[ \frac{da}{dr} = e^b \left( \frac{1}{r} + 8\pi Gr p \right) - \frac{1}{r}. \]

That yields the so-called Tolman–Oppenheimer–Volkoff equation for \( p(r): \)
\[ -\frac{dp}{dr} = \frac{G\varepsilon M}{r^2} \left(1 - \frac{2GM}{r} \right)^{-1} \left(1 + \frac{p}{\varepsilon} \right) \left(1 + \frac{4\pi r^3 p}{M} \right). \]

Together with \( \frac{dM}{dr} = 4\pi r^2 \varepsilon \) and the equation of state, they form a set of three equations for the three unknown functions: \( p, \varepsilon, \) and \( M. \)
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In the particular case \( \varepsilon = \text{const} \) with the boundary condition \( p(R) = 0 \) we can define an upper limit for the star radius:
\[ R \leq \frac{1}{\sqrt{3\pi \varepsilon G}}. \]
Conclusions

We put forward a conjecture that the domains of coherently excited pions could have been created in the early Universe.
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- We put forward a conjecture that the domains of coherently excited pions could have been created in the early Universe.

- Such domains are hot (as the estimates made in this paper remain valid up to temperatures of order 20 MeV) and stable against the gravitational collapse up to the maximum radius of about 14 km.
**Conclusions**

- We put forward a conjecture that the domains of coherently excited pions could have been created in the early Universe.

- Such domains are hot (as the estimates made in this paper remain valid up to temperatures of order 20 MeV) and stable against the gravitational collapse up to the maximum radius of about 14 km.

- Moreover, since the decay width of the coherent pionic states into photons is quite small:

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\Gamma(\pi \rightarrow \gamma\gamma) \cdot \frac{\langle \bar{\psi} \psi \rangle_R}{\langle \bar{\psi} \psi \rangle_0} \cdot \frac{m^2_{\pi}}{m^2_{\pi R}} \simeq 0.17 \text{ eV}
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these domains cannot evaporate by means of the electromagnetic radiation.
Conclusions

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  these domains cannot evaporate by means of the electromagnetic radiation.

- Since one can also argue for the stability of the coherent pionic states against the strong and weak decays, such encapsulated domains can have had a chance to survive till the present time, remaining however dark to external observers.