QCD Relics of Astrophysical Relevance

J. E. F. T. Ribeiro

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Acknowledgments

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- We briefly discuss the phenomenological implication of the Mass-Gap equation in the low energy domain of hadronic physics and show how it unifies, in the same vision, so diverse phenomena like N-N repulsive cores, pion masses, hadronic scattering, notably $\pi - \pi$ scattering lengths.
- Finally we discuss the possibility of a gravity stabilized domains–extra solutions of mass gap equation. A Tolman-Oppenheimer-Volkoff calculation of gravitationally stable such domains will be presented. They are shown to be dark

Effective Action-A simple example: Small loops

- An NJL-type model (P. Bicudo, N. Brambilla, J.E.F.T.R., A. Vairo, '98):

$$S = \int_x \bar{\psi}(\gamma_\mu \partial_\mu + m)\psi + \frac{1}{2}\int_{x,y} j^a_\mu(x) \left\langle g^2 A^a_\mu(x) A^b_\nu(y) \right\rangle j^b_\nu(y),$$

where $j^a_\mu \equiv \bar{\psi}\gamma_\mu T^a \psi$ and $\left\langle g^2 A^a_\mu(x) A^b_\nu(y) \right\rangle \simeq \frac{1}{4} x_\lambda y_\rho \left\langle g^2 F^a_{\mu\lambda} F^b_{\nu\rho} \right\rangle$.

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- Bosonization of the four-quark interaction

$$\left\langle g^2 A^a_\mu(x) A^b_\nu(y)\right\rangle = x_\lambda y_\rho \cdot \frac{\left\langle G^2\right\rangle}{48(N_c^2-1)} \cdot \delta^{ab}(\delta_{\mu\nu}\delta_{\lambda\rho} - \delta_{\mu\rho}\delta_{\lambda\nu})$$

goes via an auxiliary Abelian-like field $\mathcal{A}^a_\mu(x) = \frac{1}{2} x_\nu n^a \mathcal{F}_{\nu\mu}$, where $\mathcal{F}_{\nu\mu}$ and n^a are constant, and $n^a n^b = \delta^{ab}$.

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- The resulting one-loop Euler–Heisenberg–Schwinger Lagrangian in the $\mathcal{A}^a_\mu$-field,

$$\mathrm{tr}\, \ln \frac{\gamma_\mu D_\mu[\mathcal{A}] + M}{\gamma_\mu \partial_\mu + M} = -2N_\mathrm{f} \cdot \mathrm{tr}\,(T^a T^a) \cdot \int_0^\infty ds\, \frac{\mathrm{e}^{-M^2 s}}{(4\pi s)^2} \left[abs^2 \cot(as)\coth(bs) - 1\right],$$

  where $a^2 - b^2 = \mathbf{E}^2 - \mathbf{H}^2$, $ab = |\mathbf{EH}|$, can be expanded at large M in the number of external $\mathcal{A}^a_\mu$-lines $\Rightarrow$ an NJL-based derivation of $\left\langle \bar{\psi}\psi\right\rangle_{\mathrm{heavy};\ N_c=3} = -N_\mathrm{f} \cdot \frac{\alpha_s \left\langle (F^a_{\mu\nu})^2\right\rangle}{12\pi M}$.

Going beyond the Gaussian approximation

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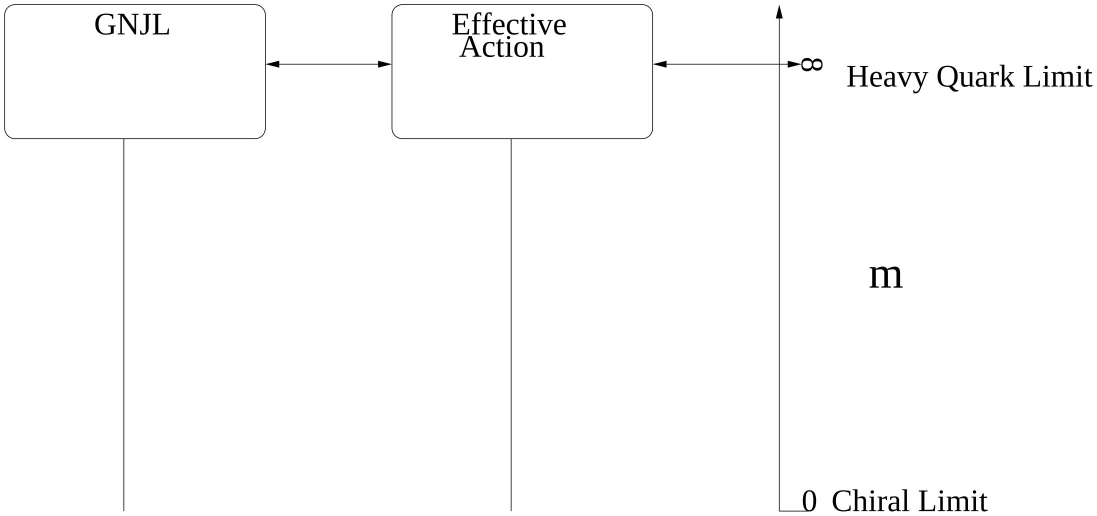

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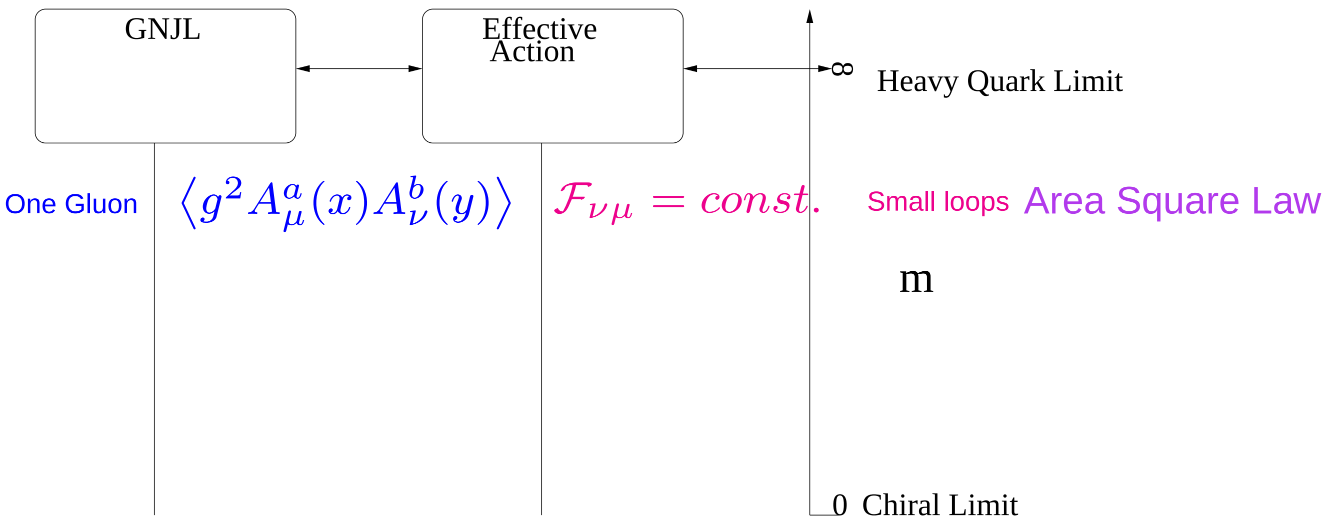

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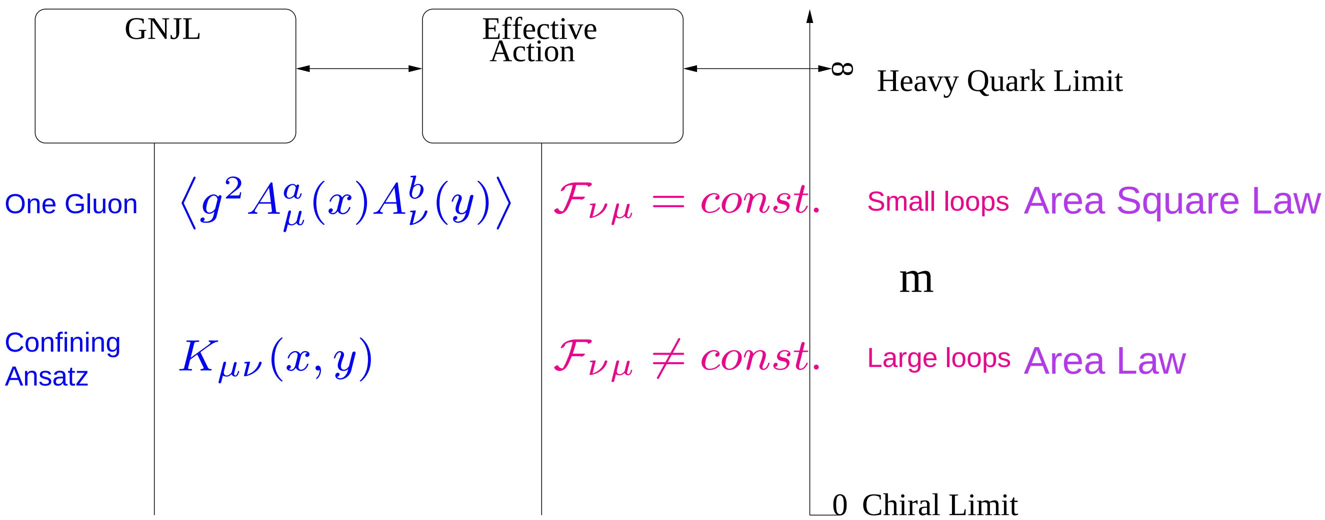

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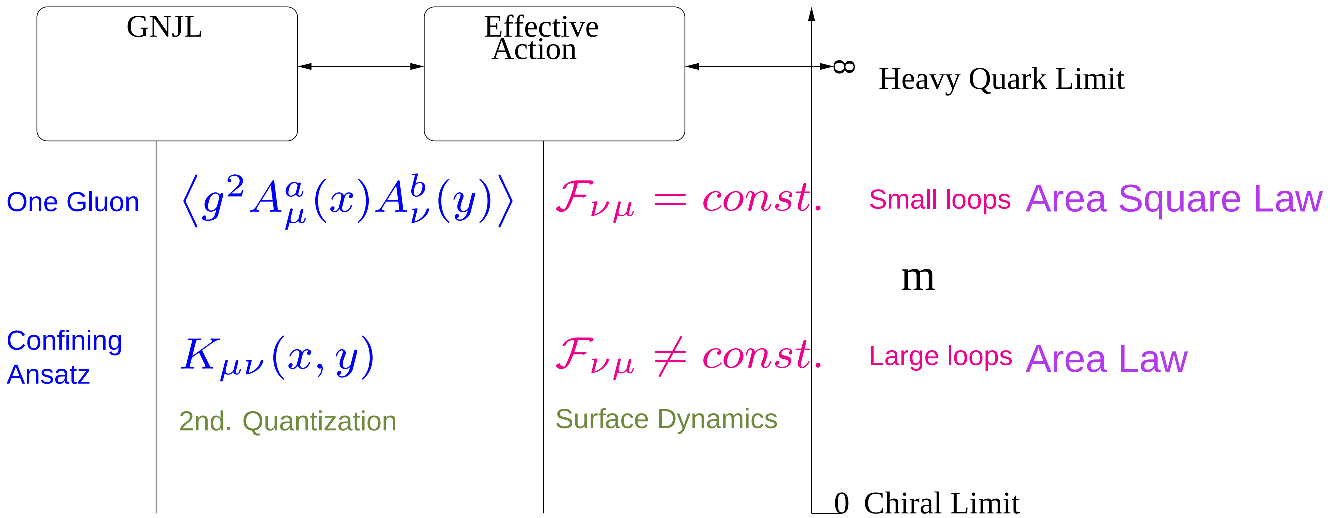

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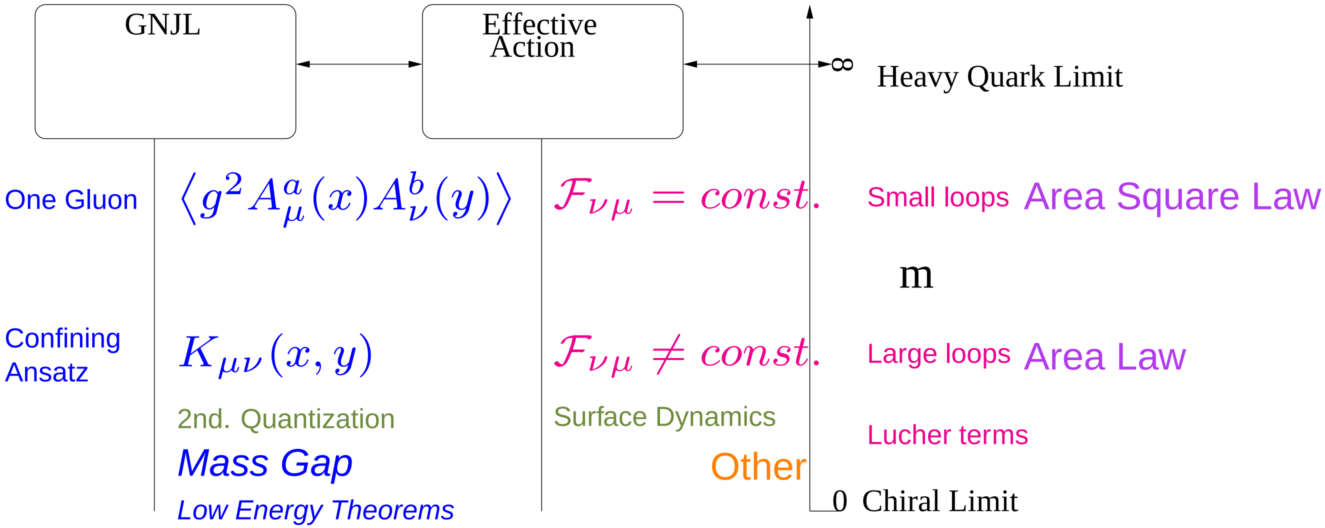

NJL-Linking SCSB with confinement: A general strategy.

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- The starting idea is to get $\langle\bar{\psi}\psi\rangle$ from the one-loop quark effective action:

$$\langle\bar{\psi}\psi\rangle = -\frac{\partial}{\partial m}\langle\Gamma[A^a_\mu]\rangle,$$

assuming for the Wilson loop entering $\langle\Gamma[A^a_\mu]\rangle$ an area law.

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$$\langle \Gamma[A_\mu^a] \rangle = -(2S+1)N_{\rm f} \int_0^\infty \frac{ds}{s} {\rm e}^{-m^2 s} \int_P \mathcal{D}z_\mu \int_A \mathcal{D}\psi_\mu {\rm e}^{-\int_0^s d\tau \left(\frac{1}{4}\dot{z}_\mu^2 + \frac{1}{2}\psi_\mu \dot{\psi}_\mu\right)} \times$$

$$\times \left\{ \left\langle {\rm tr}\, \mathcal{P} \, \exp\left[ig \int_0^s d\tau T^a \left(A_\mu^a \dot{z}_\mu - \psi_\mu \psi_\nu F_{\mu\nu}^a \right) \right] \right\rangle - N_c \right\}.$$

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- Only when $\int_P \mathcal{D}z_\mu \int_A \mathcal{D}\psi_\mu [\cdots] \to \frac{\rm const}{\sqrt{s}}$ at $s \to \infty$, we have a finite quark condensate in the chiral limit:

$$\langle \bar{\psi}\psi \rangle \propto \frac{\partial}{\partial m} \int_0^\infty \frac{ds}{s} {\rm e}^{-m^2 s} \cdot \frac{\rm const}{\sqrt{s}} = -2\sqrt{\pi} \cdot {\rm const}$$

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(T. Banks and A. Casher, ’80).
- The second idea: Parametrize via $z_\mu(\tau)$ the minimal area $S_{\rm min}$, entering the area law:
$$\langle W[z_\mu] \rangle = \left\langle {\rm tr}\, \mathcal{P} \, \exp \left(ig \int_0^s d\tau T^a A_\mu^a \dot{z}_\mu \right) \right\rangle \to N_c \cdot {\rm e}^{-\sigma(s) \cdot S_{\rm min}}.$$

Linking SCSB with confinement: A general strategy.

- Find an ansatz for $S_{\min}[z_\mu]$ so to enable the analytic calculation of $\left\langle \Gamma[A_\mu^a] \right\rangle$, and impose the $\int_P \mathcal{D}z_\mu \int_A \mathcal{D}\psi_\mu[\cdots] \to 1/\sqrt{s}$ asymptotic behavior $\Rightarrow \sigma(s)$.

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$$S_{3\mathrm{d}} = \frac{1}{2}\int_0^s d\tau |\mathbf{z}\times\dot{\mathbf{z}}| \rightarrow S_{4\mathrm{d}} = \frac{1}{2\sqrt{2}}\int_0^s d\tau |\varepsilon_{\mu\nu\lambda\rho} z_\lambda \dot{z}_\rho| \geq \frac{1}{4\sqrt{3}}|\Sigma_{\mu\nu}| := S_{\min}[z_\mu],$$

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where $\Sigma_{\mu\nu}(s)=\varepsilon_{\mu\nu\lambda\rho}\int_0^s d\tau z_\lambda\dot{z}_\rho$.

- The simple exponential ansatz for the Wilson loop at all distances,

$$\left\langle W[z_\mu]\right\rangle=N_c\cdot{\rm e}^{-\tilde{\sigma}|\Sigma_{\mu\nu}|},\ \ {\rm where}\ \ \tilde{\sigma}(s)=\frac{\sigma(s)}{4\sqrt{3}},$$

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- yields for $\left\langle\Gamma[A_\mu^a]\right\rangle$ the Euler–Heisenberg–Schwinger Lagrangian in an auxiliary constant Abelian field $B_{\mu\nu}$, to be averaged with the weight $1/\left(1+\frac{B_{\mu\nu}^2}{4\tilde{\sigma}^2}\right)^{7/2}$.

Linking SCSB with confinement: A general strategy.

- The quark condensate becomes:

$$\langle\bar{\psi}\psi\rangle = -\frac{3N_{\rm f}}{4\pi^2}\cdot m\int_0^\infty ds\,{\rm e}^{-m^2 s}\cdot\tilde{\sigma}^2\cdot f[A(s)],$$

where

$$A(s)\equiv\frac{1}{2\tilde{\sigma}^2 s^2}\ \ \text{and}\ \ f[A]=\frac{\left(\sqrt{1+A}-1\right)^4\cdot\left(5A+4\sqrt{1+A}+6\right)}{(1+A)^{5/2}}.$$

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- We obtain $\tilde{\sigma}(s)$ from the condition

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- Then, we obtain the chiral condensate:

$$\langle\bar{\psi}\psi\rangle\simeq-\frac{3N_{\rm f}}{4\pi^{3/2}}\cdot\sigma_0^{3/2}=-\mathcal{N},\ \mathcal{N}=(250\,\text{MeV})^3\Rightarrow\ \ m\gtrsim 2\sqrt{\pi}\left(\frac{\mathcal{N}}{3N_{\rm f}G_{\rm max}}\right)^{1/3},$$

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- It is possible to obtain reasonable values for the constituent quark mass $m=460\,{\rm MeV}$ while reproducing, at the same time, the heavy-quark limit of the squared area law.

NJL-A simple example: Vacuum Structure in Strong Magnetic fields

The Hamiltonian of a relativistic fermion in an external field A_μ has the following form in 2+1 dimensions:

$$H = \int d^2x\, \bar{\psi}(x) \left[-i\gamma^j D_j + m\right] \psi(x) \ ,$$

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- This constitutes a demonstration model for the more complicated 3+1 dimensions
- Let us use a Bogoliubov-Valatin transformation to obtain the known results

$$_B\langle 0|\psi^\dagger(\boldsymbol{x})\psi(\boldsymbol{x})|0\rangle_B = -\frac{|eB|}{2\pi},$$

$$E_n = \sqrt{m^2 + 2n|eB|}$$

with n standing for the Landau levels

A simple case for Valatin-Bogoliubov Transformations

We need just three steps to construct the wave-function of a particle in a magnetic field. From,

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}} \frac{\mathbf{1}}{\sqrt{\mathbf{L_x L_y}}} \left\{ \mathbf{u}(\mathbf{p})\, \mathbf{a_p} + \mathbf{v}(\mathbf{p})\, \mathbf{b}^{\dagger}_{-\mathbf{p}} \right\} \mathbf{e}^{\mathbf{i p \cdot x}}$$

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The u and v spinors are the solutions of the Dirac equation for positive and negative energy respectively. (with $\cos\phi = \sqrt{\frac{E_{\mathbf{p}}+m}{2E_{\mathbf{p}}}},\ \sin\phi = \sqrt{\frac{E_{\mathbf{p}}-m}{2E_{\mathbf{p}}}}$.)

A simple case for Valatin-Bogoliubov Transformations

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- Step 1: perform the following canonical transformation,

$$\begin{bmatrix} \tilde{a}_\mathbf{p} \\ \tilde{b}^\dagger_{-\mathbf{p}} \end{bmatrix} = R_\phi(\mathbf{p}) \begin{bmatrix} a_\mathbf{p} \\ b^\dagger_{-\mathbf{p}} \end{bmatrix} \qquad \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \mathbf{R}^*_\phi(\mathbf{p}) \begin{bmatrix} u(\mathbf{p}) \\ v(\mathbf{p}) \end{bmatrix}$$

- $R_\phi(\mathbf{p}) = \begin{bmatrix} \cos\phi & -\sin\phi\,(\hat{p}_y + i\hat{p}_x) \\ \sin\phi\,(\hat{p}_y - i\hat{p}_x) & \cos\phi \end{bmatrix},\ \hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$

A simple example of a non-trivial vacuum

- The vacuum associated to the new operators $\tilde{a}$ and $\tilde{b}$ is given by $|\tilde{0}\rangle = S|0\rangle = \prod_{\boldsymbol{p}}(\cos\phi + \sin\phi\, a^{\dagger}_{\boldsymbol{p}} b^{\dagger}_{-\boldsymbol{p}})|0\rangle$
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- Choose ϕ as to ensure that the new spinors $\tilde{u}$ and $\tilde{v}$ are momentum independent: $\tilde{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so that all the momentum dependence of ψ is stored in $\{\tilde{a}_{\mathbf{p}}, \tilde{b}_{\mathbf{p}}\} = S\{\hat{a}, \hat{b}\}S$

Landau Levels

- Use the Landau level representation
- Use $e^{ip_y y} = e^{-i\ell^2 p_x p_y}\sqrt{2\pi}\sum_{n=0}^{\infty} i^n \omega_n(\xi)\,\omega_n(\ell p_y)$
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- The new operators satisfy the anticommutation relations:
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- The vacuum is invariant under this change of basis, i.e., $\hat{a}_{np_x}|\tilde{0}\rangle = 0 \quad , \quad \hat{b}_{np_x}|\tilde{0}\rangle = 0$

An example of Mass Gap Equation

There are several approaches one can use:

- 1-consider the Ward identity or;
- 2-get rid of anomalous Bogoliubov terms or;
- 3-Derive it as the condition for the vacuum energy to be a minimum or;
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- Setting the the anomalous terms in the Hamiltonian to zero finds θ_n. We can obtain the following mass gap equations, $\begin{cases} (\ell\, m\cos\theta_0 + \sin\theta_2/\sqrt{2})\sin\theta_0 = 0\,, \quad n=0\,, \\ \ell\, m\sin 2\theta_n - \sqrt{2n}\cos 2\theta_n = 0\,, \quad n>0\,, \end{cases}$
- For any n have the following solution: $\tan 2\theta_n = \frac{\sqrt{2n|eB|}}{m}$, $E_n = \sqrt{m^2 + 2n|eB|}$

Vacuum Condensates and 3+1

Let us construct the vacuum state in a magnetic field $|0\rangle_B$, annihilated by the operators a_{np_x} and b_{np_x} :

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- In 3+1 Dimensions with quartic interactions we can perform these very same 3-steps.

A class of Hamiltonians

Consider now the simplest Hamiltonian containing the ladder-Dyson-Schwinger machinery for chiral symmetry.

In any case most of the results presented here do not depend on the kernel choice

$$H = \int d^3x \, q^+(x) \left(-i\vec{\alpha}.\vec{\nabla} \right) q(x) + \int \frac{d^3x \; d^3y}{2} J^a_\mu(x) K^{ab}_{\mu\nu}(x-y) J^b_\nu(y)$$

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This class of Hamiltonians has rich phenomenological consequences enabling us to study a variety of hadronic phenomena controlled by global symmetries

- Reproduces in a non-trivial manner the low energy properties of pion physics like, for instance, $\pi - \pi$ Weinberg results for the scattering lengths together with Oakes-Renner, Goldberger-Treiman....
- Possesses the mechanism of pole-doubling in what concerns scalar decays (Unitarization).

Bogoliubov Transformations

We can rotate the creation and annihilation Fock space operators. It is canonical !

- $$|\widetilde{0}>= Exp\left\{\widehat{Q}_0^+ - \widehat{Q}_0\right\}|0>$$

- $$\widehat{Q}_0^+(\Phi) = \sum_{cf}\int d^3p\,\Phi(p)\,M_{ss'}(\theta,\phi)\widehat{b}_{fcs}^+(\overrightarrow{p})\,\widehat{d}_{fcs'}^+(-\overrightarrow{p})$$

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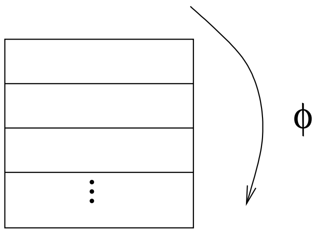
The functions $\Phi(p)$ classify the infinite set of possible Fock spaces:



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- The {u,v}, contain now the information on the angle $\phi(p)$.

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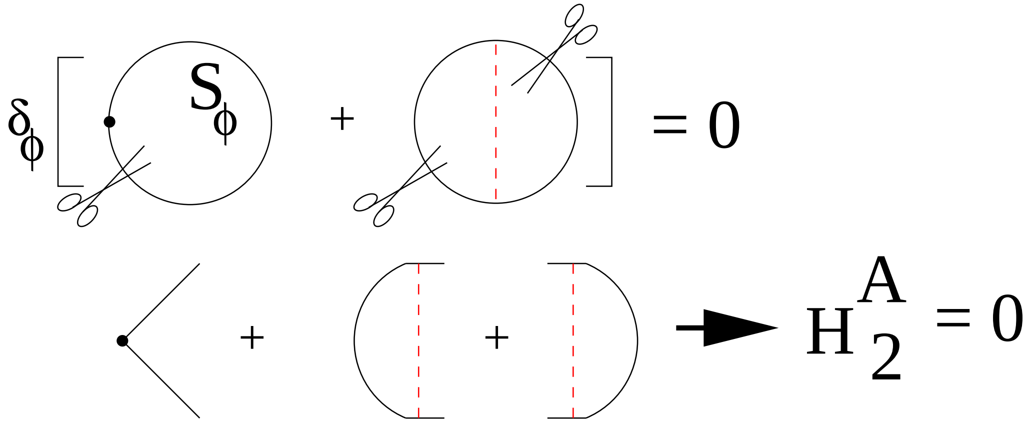

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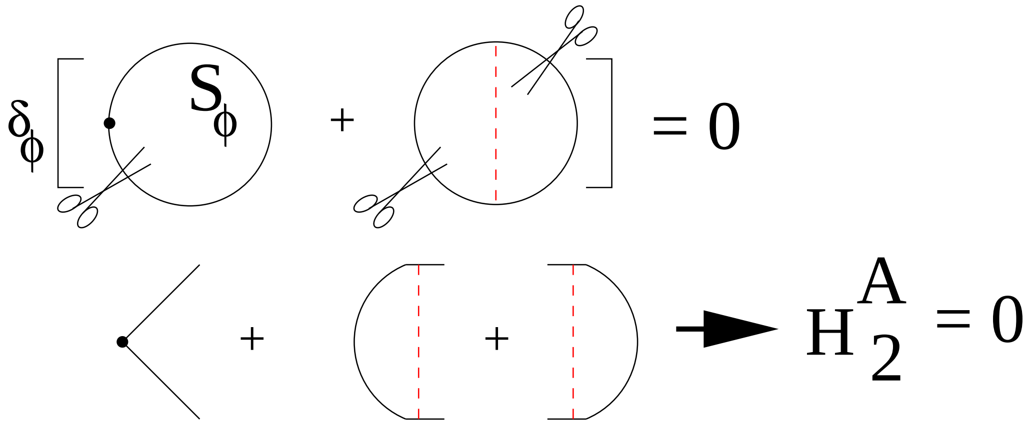

- $\Gamma_\mu(p,p^{'}) = \gamma_\mu + i\int \frac{d^4q}{(2\pi)^4} K(q)\Omega\, S(p^{'}+q)\Gamma_\mu(p^{'}+q,p+q)\Omega S(p^{'}+q)$
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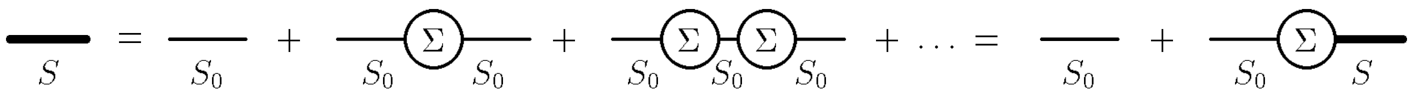

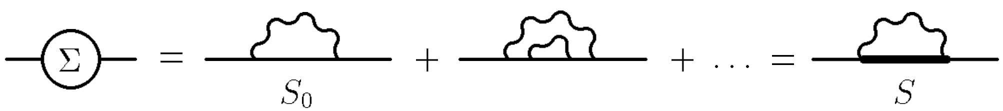

$S = S_0 + S_0\Sigma S, \quad S^{-1}(p_0,\vec{p}) = S_0^{-1}(p_0,\vec{p}) - \Sigma(\vec{p})$

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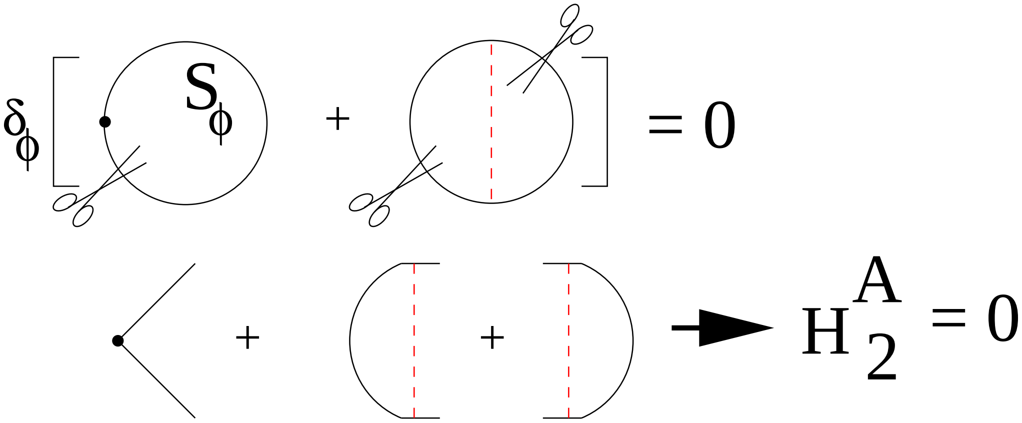

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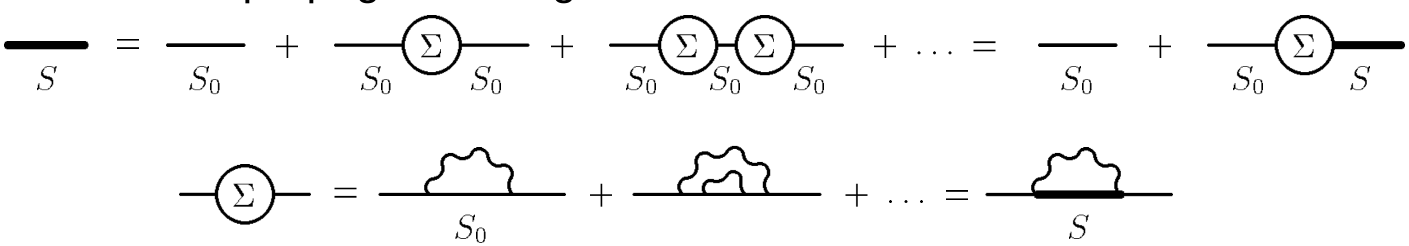

$$S = S_0 + S_0\Sigma S, \quad S^{-1}(p_0,\vec{p}) = S_0^{-1}(p_0,\vec{p}) - \Sigma(\vec{p})$$

Quasi-Classicalregime and $\hbar$ expansions

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- The functions A_p and B_p represent the scalar part and the space–vectorial part of the effective Dirac operator.

- Finally $\tan\varphi_p = \frac{A_p}{B_p}$
  $\varphi_{p\to\infty} \to 0$: only the vectorial part survives
  $\varphi_{p\to 0} \to \pi/2$: only the scalar part survives

Breakdown of the expansion for φ_p in powers of $\hbar$

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- To sum it up: we have two different regimes according to the parameter $m/\sqrt{\sigma}$: The spontaneous breaking of chiral symmetry is relevant for $m \ll \sqrt{\sigma}$, with heavy quark physics relevant for the opposite

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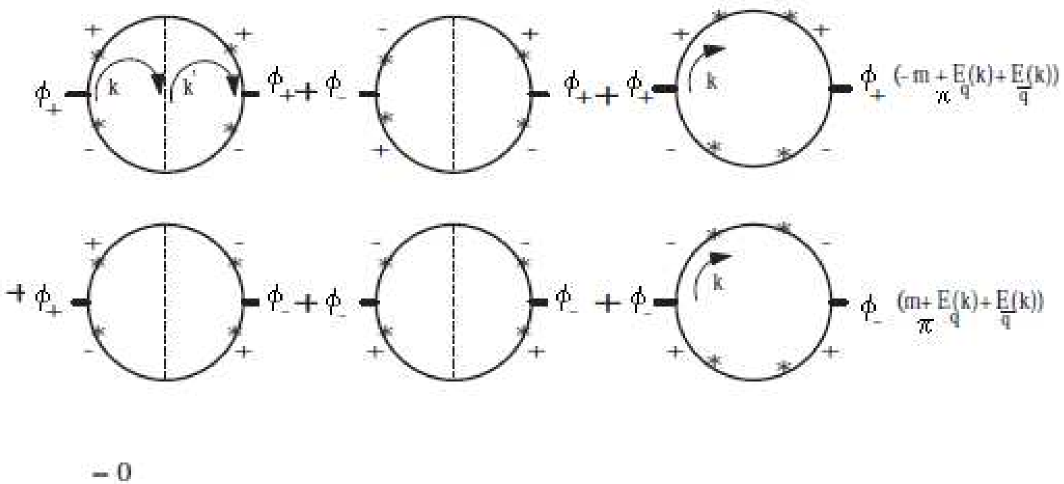
Regardless of the particular form of the $K_{\mu\nu}(x, y)$ these class of models have phenomenologically nice features:

1. For massless quarks it possesses a massless pion. (As an instance of the Mass Gap)
2. It is at least qualitatively successful in describing hadronic scattering, namely the issue of $\pi - \pi$ scattering: (The Adler zeros)

The pion: An example of Mass Gap

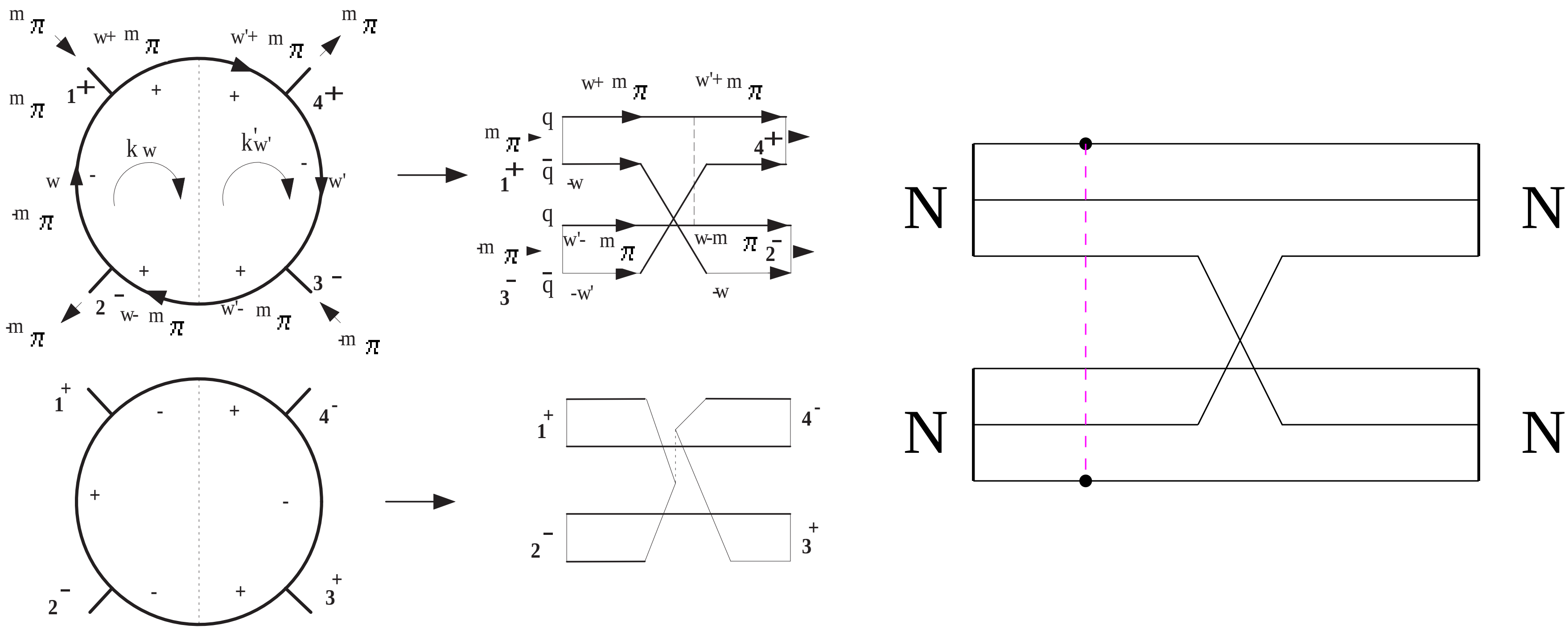


Phys.Rev.D65:076008,2002.
P. Bicudo, S. Cotanch, F. Llanes-Estrada, P. Maris, JEFTR, A. Szczepaniak

FIG. 1. Pion Salpeter equation. In terms of the Dirac matrices β and $\vec{\alpha}$, the projection operators for the quark propagator, with momentum $\vec{k}$, are $\Lambda^{\pm} = (1 \pm \sin(\phi)\beta \pm \cos(\phi)\alpha \cdot \widehat{\mathbf{k}})/2$, and denoted in the figure by $\{+,-\}$. Note that $\Phi^{\pm}$ is consistent with the normalization condition, Eq. (3), and should contain the cluster propagators obtained after integrating the quark propagator energy, E_q. This is the reason the propagator cuts are displayed in the figure. Two such cluster propagators are needed for the two Φ' s but only one is generated per integration loop. This necessitates multiplying and dividing the diagrams by the missing cluster propagator leading to the factors $\pm m_\pi + E_q + E_{\bar{q}}$ appearing in the diagram.

Exotic versus non-exotic: π-π scattering

A pictorial description of the Weinberg formula for the scattering lengths of pions. We have,



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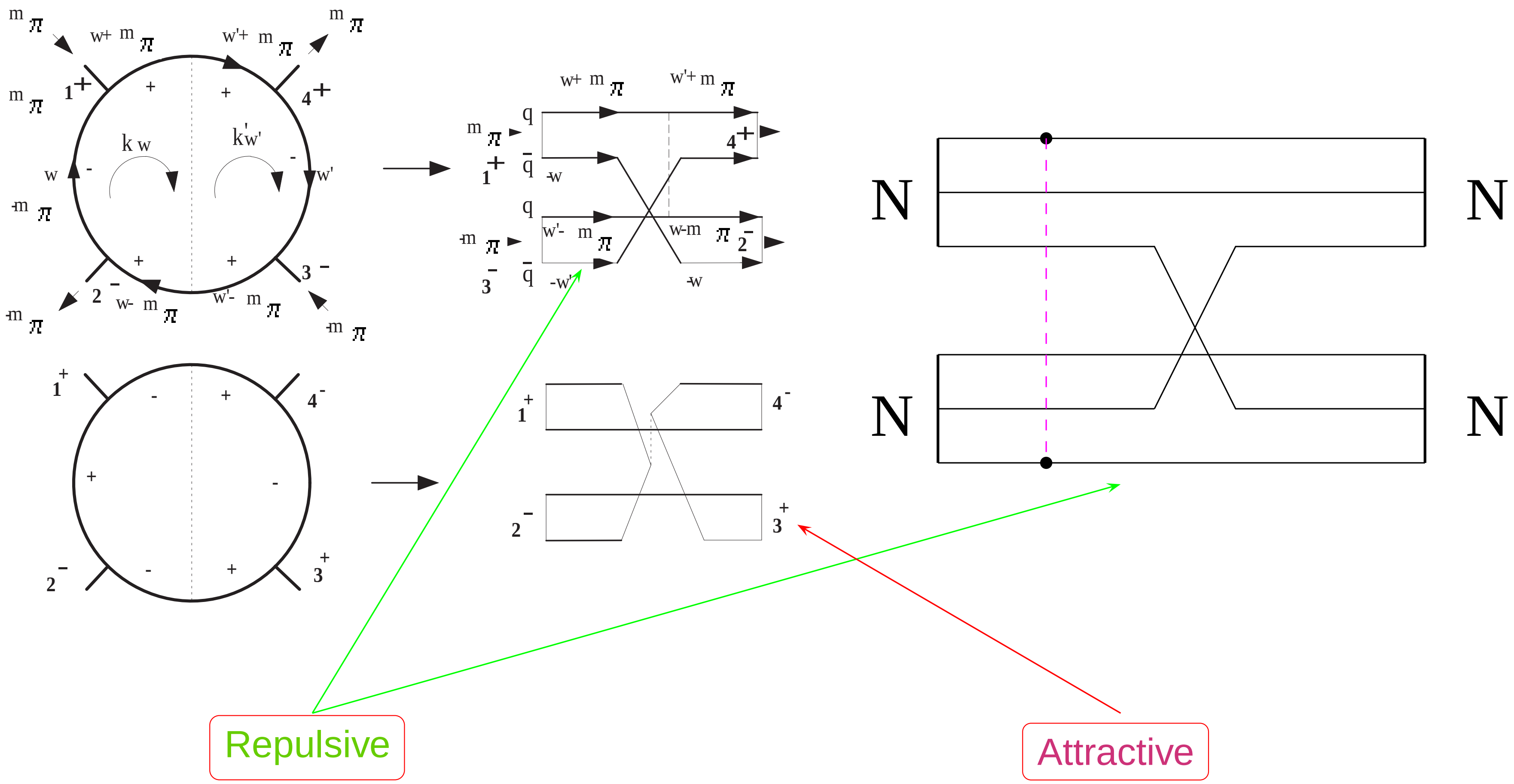

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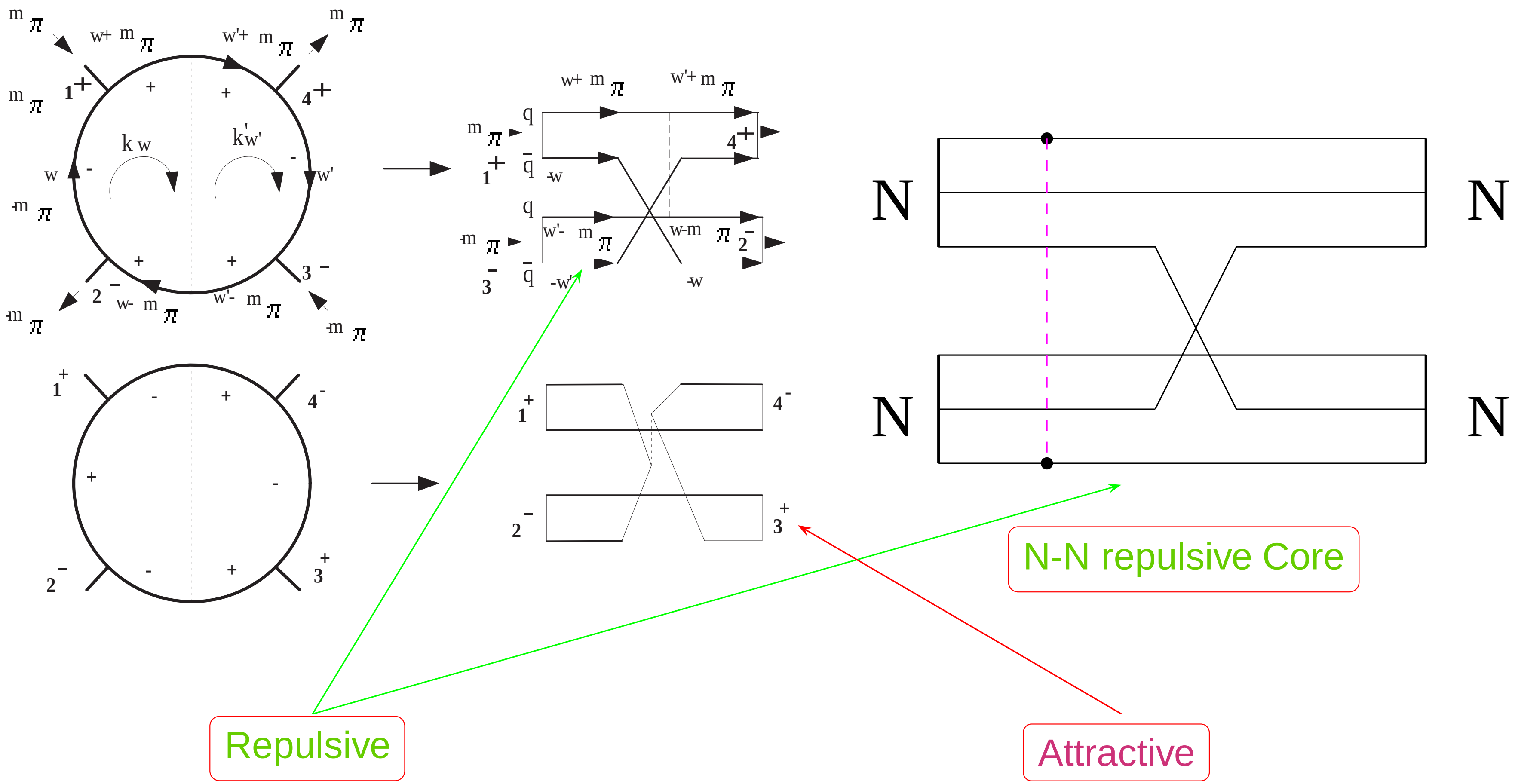

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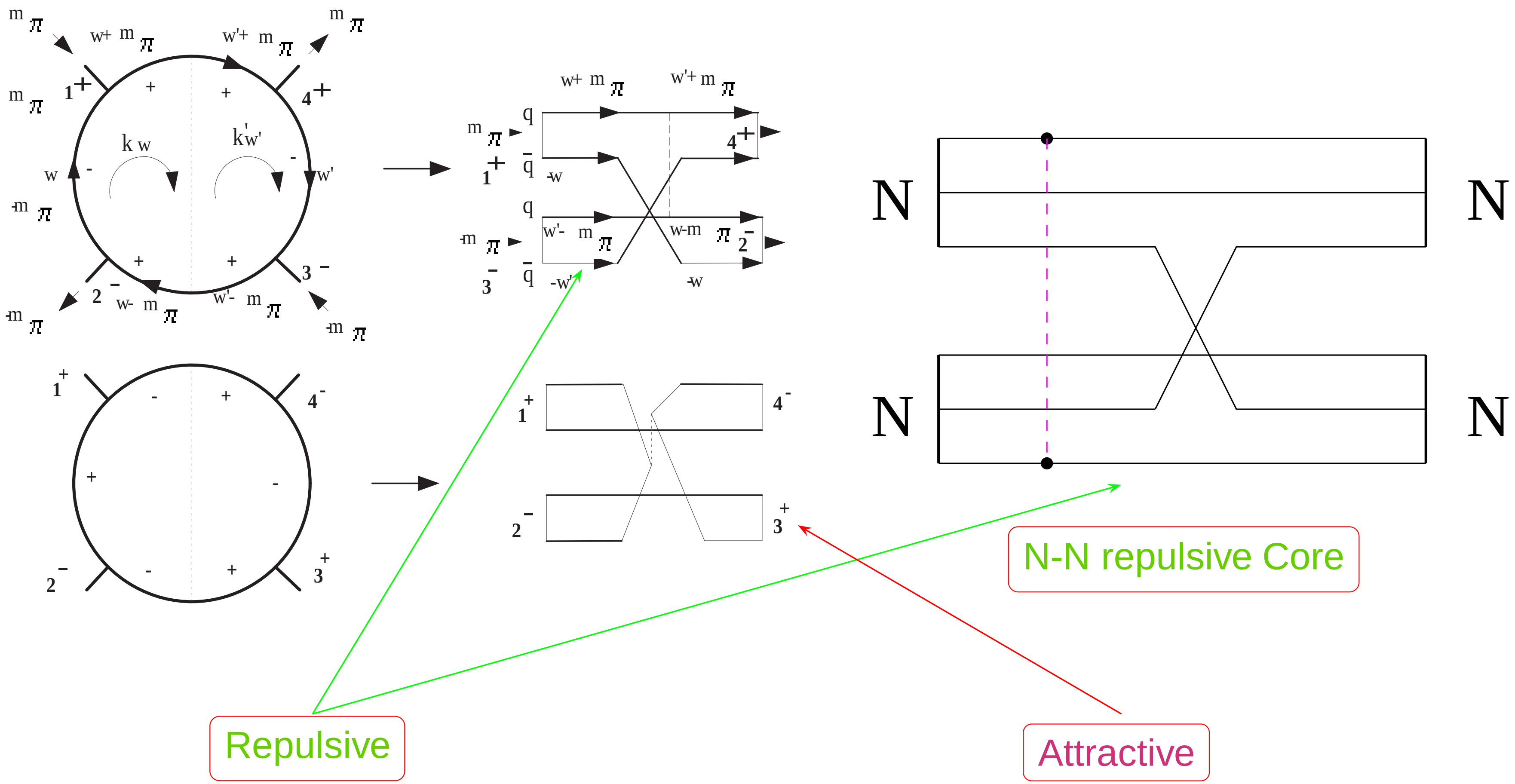

Adler Zero=Mass Gap equation+(Exotic+Non-Exotic=0)

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- Use the known temperature-dependent gluonic and chiral condensates, which at temperatures $T \ll m_\pi$ of interest read

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- Use the known temperature-dependent gluonic and chiral condensates, which at temperatures $T \ll m_\pi$ of interest read

$$\left\langle G^2\right\rangle_T = \left\langle G^2\right\rangle - \frac{24 m_\pi^3 T}{b} S_1\left(\frac{m_\pi}{T}\right), \quad \left\langle\bar{\psi}\psi\right\rangle_T = \left\langle\bar{\psi}\psi\right\rangle\left[1 - \frac{3 m_\pi T}{4\pi^2 f_\pi^2} S_1\left(\frac{m_\pi}{T}\right)\right].$$

- Using the Gell-Mann–Oakes–Renner relation we have

$$\varepsilon_0(T) = \varepsilon_0(0) + \frac{3 m_\pi^3 T}{8\pi^2} S_1\left(\frac{m_\pi}{T}\right),$$

where we used the expansion $S_\nu(x) \equiv \sum_{n=1}^{\infty} \frac{K_\nu(nx)}{n^\nu}$.

Macroscopic properties of a replica-filled domain

- For the internal-energy density of the excited vacuum (GMOR rules), we must have an expression similar:

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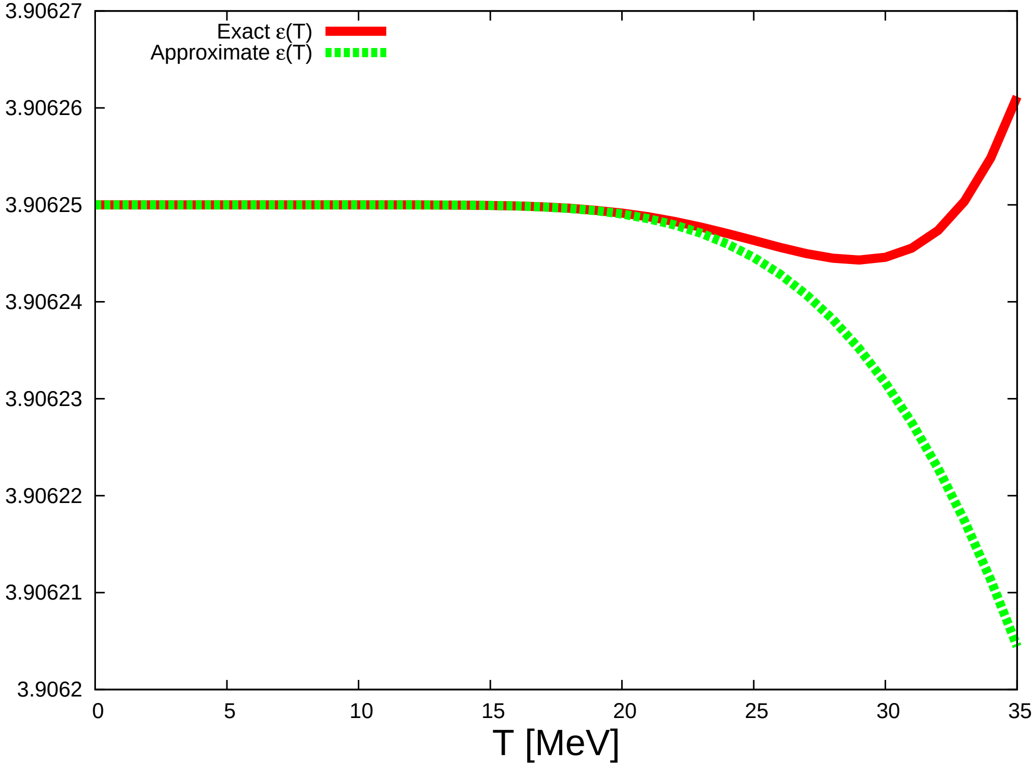
$m_{\pi_R} = 250$ MeV, $\langle\bar\psi\psi\rangle_R = -(100$ MeV$)^3$, $\varepsilon \simeq (250\,\mathrm{MeV})^4$, $m_\pi = 140\,\mathrm{MeV}$ we plot these quantities. Up to the temperatures $\sim 20\,\mathrm{MeV}$ we can disregard hadronic contributions

Macroscopic properties of a replica-filled domain



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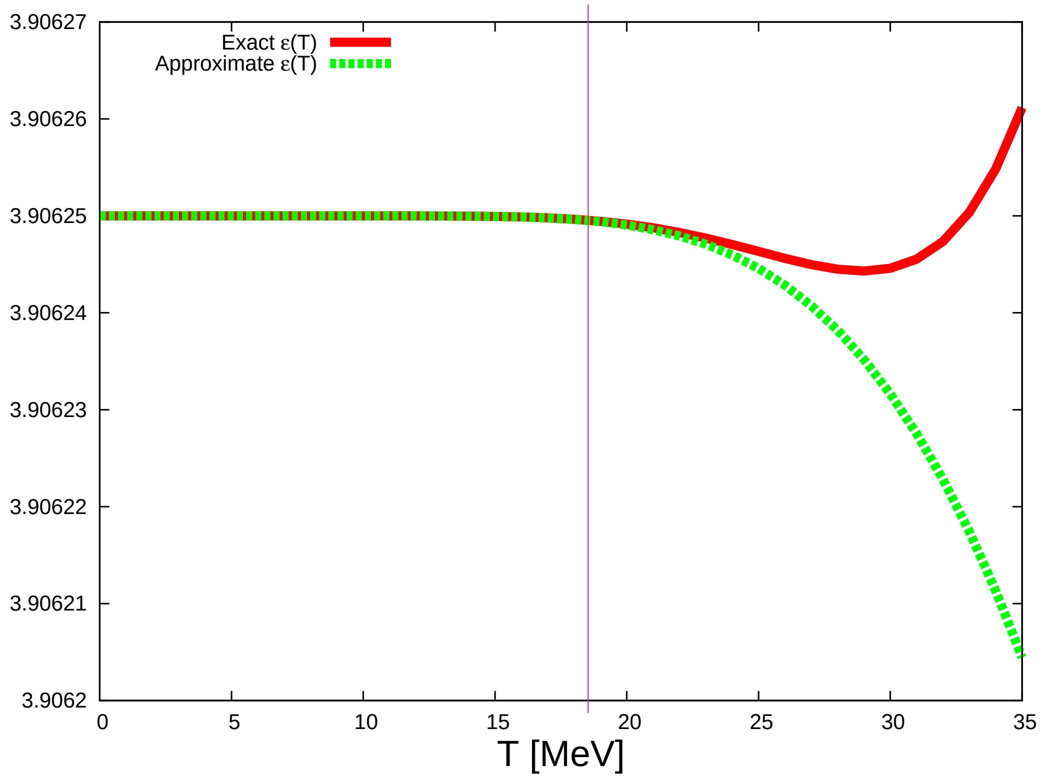

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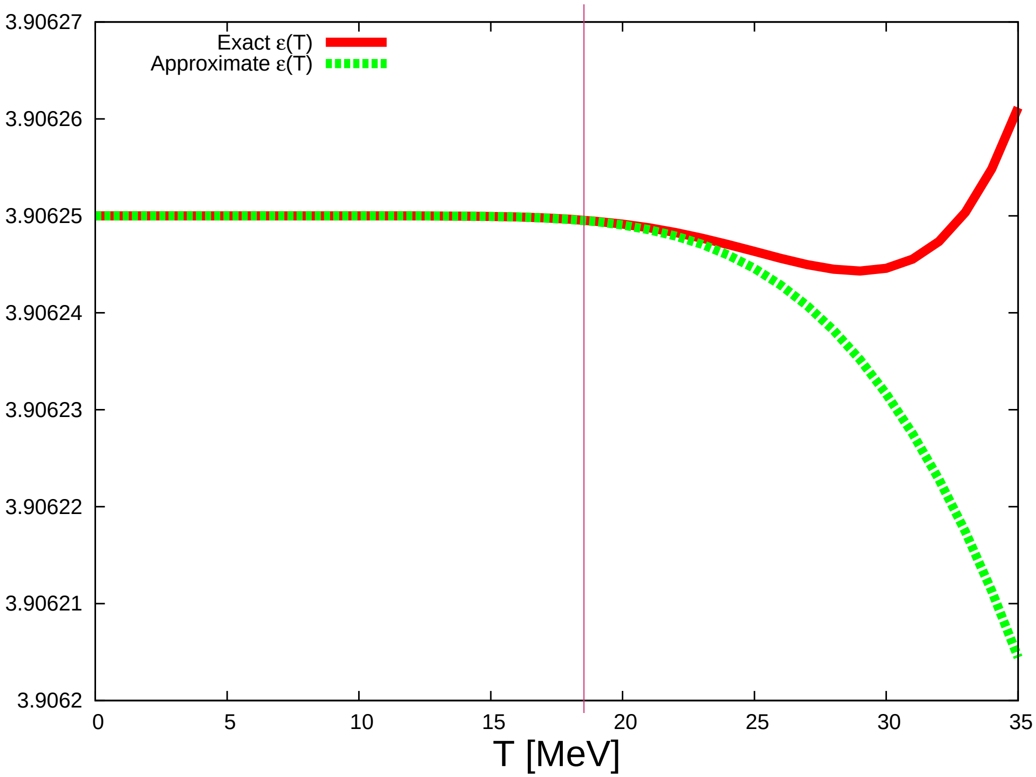

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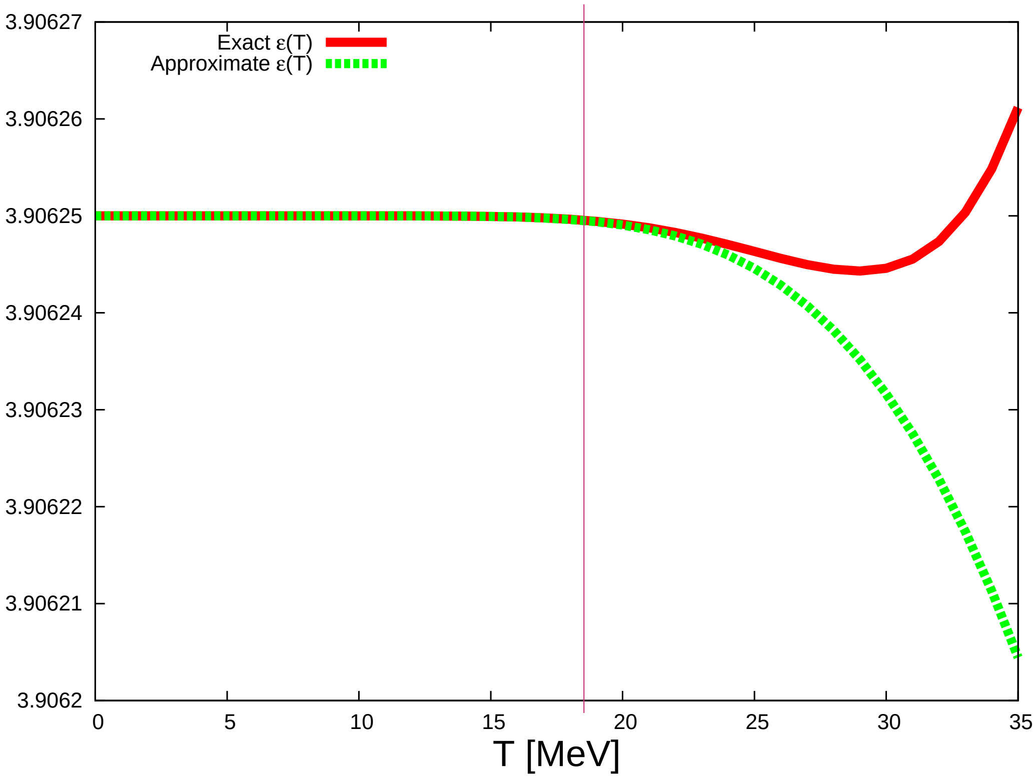

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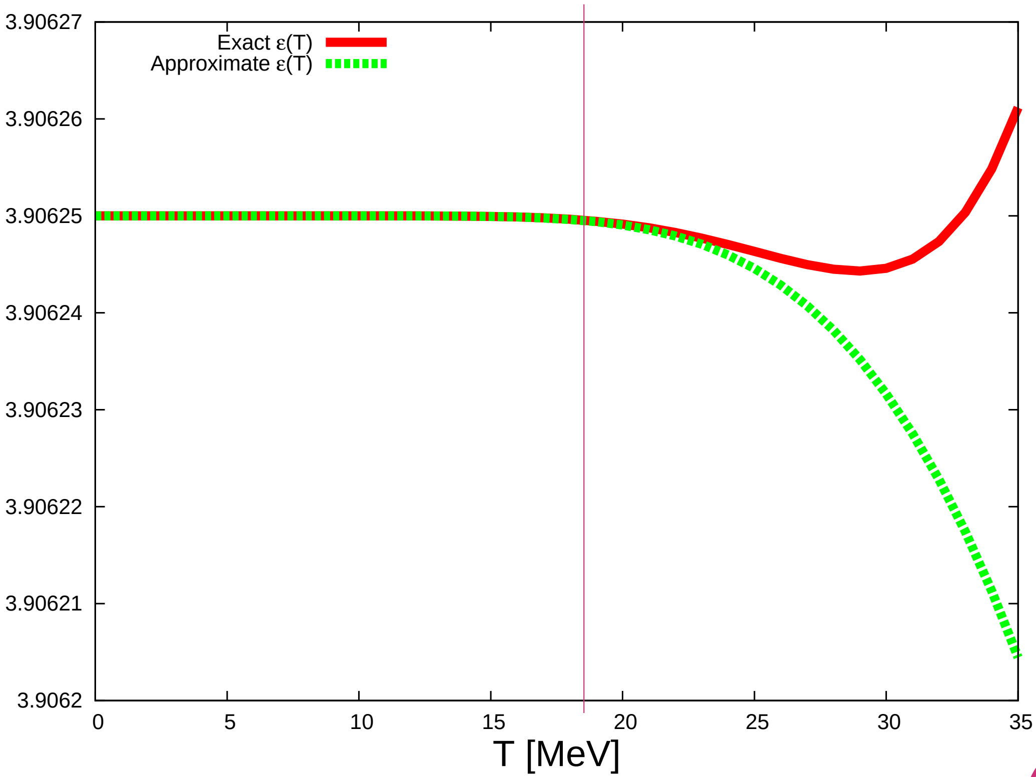

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Tolman–Oppenheimer–Volkoff defines the maximal possible radius of the domain as

$$R_G = \frac{1}{\sqrt{3\pi\varepsilon G}} \simeq 14\,\mathrm{km},$$

Gravitational field of a spherical object with constant energy density

- Assume that the matter forming a star is a perfect fluid: This implies the energy-momentum tensor of the form $T^{\mu\nu} = (p + \varepsilon)u^\mu u^\nu - pg^{\mu\nu}$, with $u^\mu(x)$ being the four-velocity of the fluid, such that $g_{\mu\nu}u^\mu u^\nu = 1$.

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- The function $a(r)$ inside the star can be found from the covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$,

$$-\partial_\mu p \cdot g^{\mu\nu} + \partial_\mu\left[(p+\varepsilon)u^\mu u^\nu\right] + (p+\varepsilon)\left(\Gamma^\mu_{\lambda\mu}u^\lambda u^\nu + \Gamma^\nu_{\lambda\mu}u^\mu u^\lambda\right) = 0.$$

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$$-\frac{dp}{dr} = \frac{G\varepsilon\mathcal{M}}{r^2}\left(1-\frac{2G\mathcal{M}}{r}\right)^{-1}\left(1+\frac{p}{\varepsilon}\right)\left(1+\frac{4\pi r^3 p}{\mathcal{M}}\right).$$

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- In the particular case $\varepsilon = \text{const}$ with the boundary condition $p(R) = 0$ we can define an upper limit for the star radius: $R \leq \frac{1}{\sqrt{3\pi\varepsilon G}}$

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- Since one can also argue for the stability of the coherent pionic states against the strong and weak decays, such encapsulated domains can have had a chance to survive till the present time, remaining however dark to external observers.