Feynman gauge on the lattice: new results and perspectives

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Abstract

Recently we have introduced a new implementation of the Feynman gauge on the lattice, based on a minimizing functional that extends in a natural way the Landau-gauge case, while preserving all the properties of the continuum formulation. The only remaining difficulty with our approach is that, using the standard (compact) discretization, the gluon field is bounded while the four-divergence of the gluon field satisfies a Gaussian distribution, i.e. it is unbounded. This can give rise to convergence problems when a numerical implementation is attempted. In order to overcome this problem one can use different discretizations for the gluon field or consider a SU(N_c) group with N_c sufficiently large. Here we discuss these two possible solutions and present results for the transverse and longitudinal gluon propagators.

Why study the linear covariant gauge?

- Study Green's functions in the infrared limit of Yang-Mills theories in order to understand low-energy properties of the theory.
- Since they depend on the gauge, consider different gauges (Landau gauge, Coulomb gauge, λ-gauge, maximally Abelian gauge).
- Linear covariant gauge, very popular in continuum studies, proved quite hostile to the lattice approach.

Some analytic results

What do we know about linear covariant gauge?

- Coupled Dyson-Schwinger equations for gluon and ghost propagators (R. Alkofer et al., 2003): scaling solution (the gluon propagator is infrared suppressed whereas the ghost propagator is infrared enhanced).
- Infrared-finite ghost propagator (A.C. Aguilar and J. Papavassiliou, 2008) using Schwinger-Dyson equation of the ghost propagator.
- Gribov's analysis for small values of the gauge parameter ξ (R.F. Sobreiro and S.P. Sorella, 2005): the transverse gluon propagator is infrared suppressed, the longitudinal part is unchanged and the "ghost propagator" $[-\partial_\mu D_\mu(A_t)]^{-1}$ is infrared enhanced.
- It has been proven (D. Binosi and J. Papavassiliou, 2009) that the background-field Feynman gauge is equivalent (to all orders) to the pinch technique.

Linear covariant gauge on the lattice (I)

We want to impose the condition

$$\partial_\mu A_\mu^b(x) = \Lambda^b(x),$$

for real-valued functions $\Lambda^b(x)$, generated using a Gaussian distribution with width $\sqrt{\xi}$.

Landau gauge [$\Lambda^b(x) = 0$] is obtained on the lattice by minimizing the functional

$$\mathcal{E}_{LG}[U^g] = -\text{Tr} \sum_{\mu,x} g(x) U_\mu(x) g^\dagger(x + e_\mu).$$

From the second variation of $\mathcal{E}_{LG}[U^g]$ we define the (positive-definite) Faddeev-Popov operator $\mathcal{M}$. The set of local minima defines the first Gribov region Ω.

Linear covariant gauge on the lattice (II)

Problem: a no-go theorem (L. Giusti, 1996).
There is no minimizing functional $\mathcal{E}_{LCG}[U^g, \Lambda]$ for the linear covariant gauge!

Proof: Suppose $\mathcal{E}_{LCG}[U^g, \Lambda]$ exists. Then, it should be given by $\mathcal{E}_{LG}[U^g] + \mathcal{F}[U^g]$, for some $\mathcal{F}[U^g]$. The gauge-fixing condition is obtained from the stationarity condition $\frac{\partial \mathcal{E}_{LCG}}{\partial w^b(x)} = 0$, when $g(x) = e^{iw(x)}$. Also, the second variation should satisfy

$$\frac{\partial^2 \mathcal{E}_{LCG}}{\partial w^b(x) \partial w^c(y)} = \frac{\partial^2 \mathcal{E}_{LCG}}{\partial w^c(y) \partial w^b(x)} \, .$$

However, one can show that this equality is not realized since the two terms are, respectively, proportional to f^{acb} and f^{abc}.

First Solution

Consider a different gauge (L. Giusti, 1996):

$$F[\partial_\mu A^a_\mu(x) - \Lambda^a(x)] = 0$$

with $F[s] = 0$ when $s = 0$, for which the minimizing functional $\mathcal{E}[U^{(g)}, \Lambda]$ exists; for example $F[s] = s^2$.

Problems:

- Possible spurious solutions, $F[s] = 0$ when $s \neq 0$; with the above choice, solutions of $D_\nu \partial_\nu [\partial_\mu A^a_\mu(x) - \Lambda^a(x)] = 0$.
- With the considered $F[s]$, the gauge fixing is numerically difficult [non-linear in $g(x)$].
- The Faddeev-Popov matrix is also different from that of the linear covariant gauge.

Second solution

Do not minimize! (A. C., A. Maas, T. Mendes, 2008)

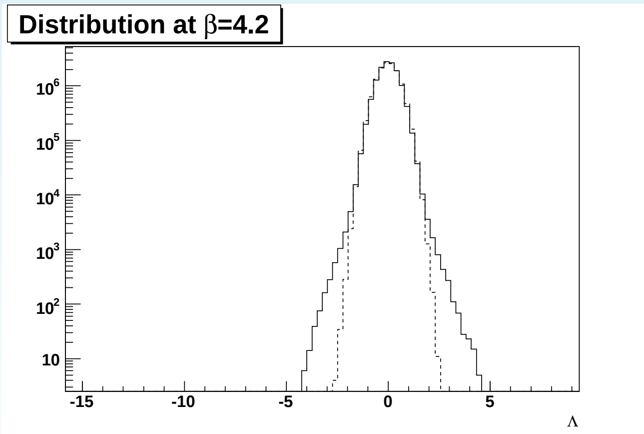
1) Fix $U_\mu(x)$ to Landau gauge.

2) Solve $\left(\partial_\mu D^{ab}_\mu \phi^b\right)(x) = \Lambda^a(x)$. For small $\phi^b(x)$ one has $\partial_\mu {A'}^a_\mu(x) = \partial_\mu \left(A^a_\mu + D^{ab}_\mu \phi^b\right)(x) = \Lambda^a(x)$.

Problem: only correct for infinitesimal gauge transformations.

If one considers functions $\Lambda^a(x)$ with a Gaussian distribution of width $\sqrt{\xi}$, then we should check if $\partial_\mu {A'}^a_\mu(x)$ also has a Gaussian distribution of width $\sqrt{\xi}$ and if $p^2 D_l(p^2) = \xi$.

Numerical tests have shown that the distributions of $\partial_\mu {A'}^b_\mu(x)$ and of $\Lambda^b(x)$ do not agree very well and the relation $p^2 D_l(p^2) = \xi$ is also not well verified by the data at small momenta.



Third solution (2009)

Some details of the no-go theorem: for the minimizing functional $\mathcal{E}_{LCG}[U^g, \Lambda]$ the gauge condition is given by the first variation, i.e.

$$\delta \mathcal{E}_{LCG}[U^g, \Lambda] = \frac{\partial \mathcal{E}_{LCG}}{\partial U} \cdot \frac{\partial U}{\partial g} \, \delta g = 0 \; .$$

Solution: remove an implicit hypothesis of the no-go theorem, i.e. that the gauge transformation appears in the minimizing functional in the "canonical" way $g(x) U_\mu(x) g(x + e_\mu)$. Thus, we can look for a minimizing functional of the type $\mathcal{E}_{LCG}[U^g, g, \Lambda]$!

Simple hint: if you want to solve $B\phi = c$ just minimize $\frac{1}{2}\phi B \phi - \phi c$! In our case

$$\mathcal{E}_{LCG}[U^g, g, \Lambda] \sim \mathcal{E}_{LG}[U^g] - g\Lambda \, .$$

The minimizing functional

The lattice linear covariant gauge condition can be obtained by minimizing the functional

$$\mathcal{E}_{LCG}[U^g, g, \Lambda] \;=\; \mathcal{E}_{LG}[U^g] + \Re\, Tr \sum_x i\, g(x)\, \Lambda(x)$$

where

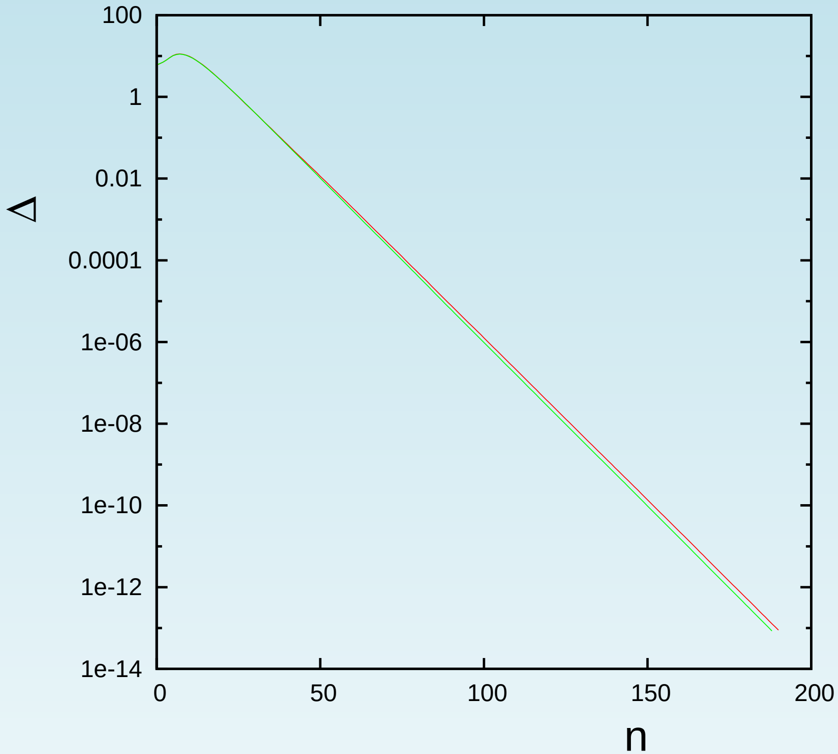
$$\mathcal{E}_{LG}[U^g] = -\mathrm{Tr} \sum_{\mu, x} g(x) U_\mu(x) g^\dagger(x + e_\mu) .$$

One can interpret the Landau-gauge functional $\mathcal{E}_{LG}[U^g]$ as a spin-glass Hamiltonian for the spin variables $g(x)$ with a random interaction given by $U_\mu(x)$. Then, our new functional corresponds to the same spin-glass Hamiltonian when a random external magnetic field $\Lambda(x)$ is applied.

Note: the functional $\mathcal{E}_{LCG}\{U^g, g\}$ is linear in the gauge transformation $\{g(x)\}$.

By considering a one-parameter subgroup $g(x, \tau) \,=\, \exp\left[i\tau\gamma^b(x)\lambda^b\right]$ it is easy to check that the stationarity condition implies the lattice linear covariant gauge condition. Also, the second variation of the term $i\, g(x)\, \Lambda(x)$ is purely imaginary and it does not contribute to the Faddeev-Popov matrix, i.e. $\mathcal{M}$ is a discretized version of the usual Faddeev-Popov operator $-\partial \cdot D$.

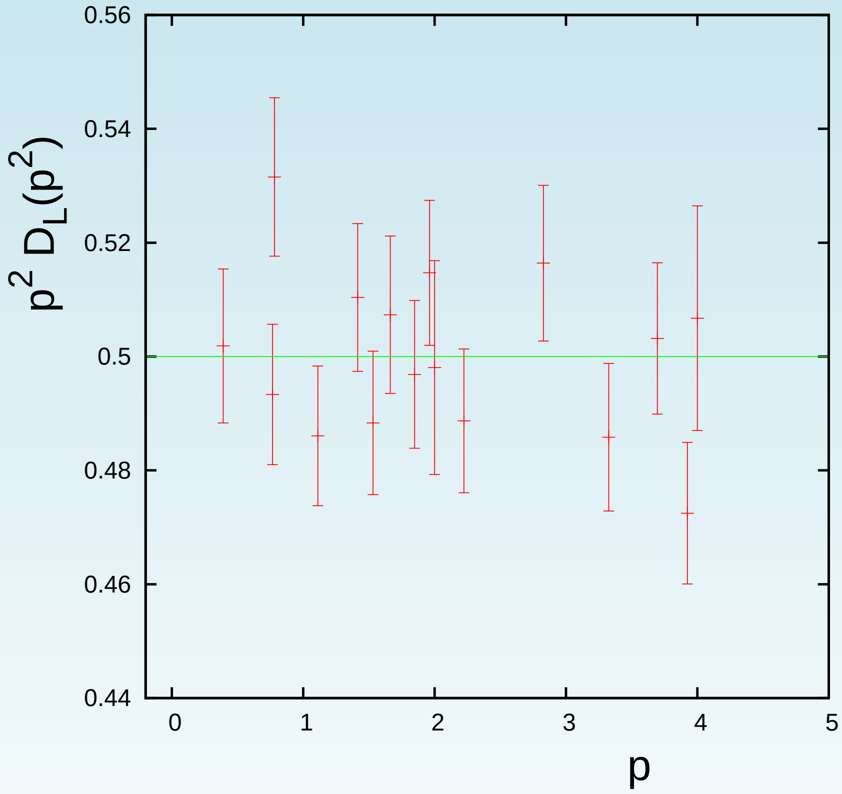
Numerical tests of the gauge fixing



We have performed some numerical tests using the so-called stochastic-overrelaxation algorithm for the 4d SU(2) case at $\beta = 4$, with $V = 8^4$ and 16^4, and $\xi = 0, 0.01, 0.05, 0.1, 0.5$. We show the value of $\Delta = \sum_{x,b} [\nabla \cdot A^b(x) - \Lambda^b(x)]^2$ as a function of the number of iterations n for $\beta = 4$, $V = 16^4$, $\xi = 0$ (red line) and $\xi = 0.05$ (green line).

Note that periodic boundary conditions imply $\sum_x \Lambda^b(x) = 0$. Also, the rate of convergence is essentially the same in the two cases (but the tuning of the stochastic-overrelaxation algorithm is different).

Longitudinal gluon propagator



We also checked that

$$p^2 D_l(p^2) = \xi$$

In the SU(2) case, for $V = 16^4$, $\beta = 4$ and $\xi = 0.5$ a fit a/p^b for $D_l(p^2)$ gives $a = 0.502(5)$ and $b = 2.01(1)$ with a $\chi^2/dof = 1.1$.

Discretization effects (I)

- The gluon field $A_\mu^a(x)$ is (usually) bounded (compact formulation).
- The functions $\Lambda^b(x)$ satisfy a Gaussian distribution, i.e. they are unbounded.
- Convergence problems (the problem gets more severe when ξ is larger).
- The problem is more severe for a larger volume.

Observation: One can try to use different discretizations of the gluon fields.

We did some tests using the angle projection (K. Amemiya and H. Suganuma, 1999) and the stereographic projection (L. von Smekal et al., 2007). In the latter case, the gluon field is in principle unbounded even for a finite lattice spacing.

Discretization effects (II)

We tested the standard discretization, the angle projection and the stereographic projection using $V = 8^4$, $\xi = 0.01, 0.05, 0.1, 0.5, 1.0$ and $\beta = 2.2, 2.3, \ldots, 2.9, 3.0$.

ξ	stand.	angle	stereog.
0.01	2.2	2.2	2.2
0.05	2.2	2.2	2.2
0.1	2.2	2.2	2.2
0.5	2.8	2.6	2.5
1.0	—	3.0	2.5

Smallest value of β for which the numerical gauge-fixing algorithm showed convergence. Results are reported for the three different discretizations and for five different values of the gauge parameter ξ.

Continuum Limit (I)

Note that the continuum relation

$$\partial_\mu A^b_\mu(x) = \Lambda^b(x)$$

can be made dimensionless — working in a generic d-dimensional space — by multiplying both sides by $a^2 g_0$. Since $\beta = 2N_c/(a^{4-d} g_0^2)$ [in the SU(N_c) case] we have that the lattice quantity

$$\frac{\beta/(2N_c)}{2\xi} \sum_{x,b} \left[a^2 g_0 \Lambda^b(x)\right]^2$$

becomes

$$\frac{1}{2\xi} \frac{1}{a^{4-d} g_0^2} \int \frac{d^d x}{a^d} \sum_b \left[a^2 g_0 \Lambda^b(x)\right]^2 \;=\; \frac{1}{2\xi} \int d^d x \sum_b \left[\Lambda^b(x)\right]^2$$

in the formal continuum limit.

Continuum Limit (II)

Thus, if we consider a gauge parameter ξ in the continuum, the lattice quantity $a^2 g_0 \Lambda^b(x)$ is generated from a Gaussian distribution with width

$$\sigma = \sqrt{2 N_c \xi / \beta}$$

instead of a width $\sqrt{\xi}$.

Note that $\sigma = \sqrt{\xi}$ if $\beta = 2N_c$ and that for $\beta < 2N_c$ the lattice width σ is larger than the continuum width $\sqrt{\xi}$.

In the SU(2) case, one has $\sigma = \sqrt{\xi}$ only for $\beta = 4$, corresponding to a lattice spacing $a \approx 0.001$ fm. On the contrary, in the SU3 case, one has $\sigma = \sqrt{\xi}$ for $\beta = 6$, corresponding to $a = 0.102$ fm. Also, for a fixed t'Hooft coupling $g_0^2 N_c =$ constant we have $\beta \propto N_c^2$ and $\sigma \propto \sqrt{1/N_c}$, i.e. simulations for the linear covariant gauge are probably easier in the SU(N_c) case for large N_c.

Continuum Limit (III)

We simulated the SU(2), SU(3) and SU(4) cases for $\xi = 1$, $V = 8^4, 16^4, 24^4, 32^4$ and the following values of β.

N_c	β_1	β_2	β_3	β_4
2	3.0	2.485	2.295	2.44
3	6.75	6.67	6.07	5.99
4	12.0	12.59	11.43	10.97

They correspond, respectively, to a t'Hooft coupling $g_0^2 N_c = 8/3$ (β_1), a plaquette average value of about 0.65 (β_2) and 0.6 (β_3) and a string tension (in lattice units) of about $a^2\sigma = 0.044$ (β_4) giving $a \approx 0.09$ fm.

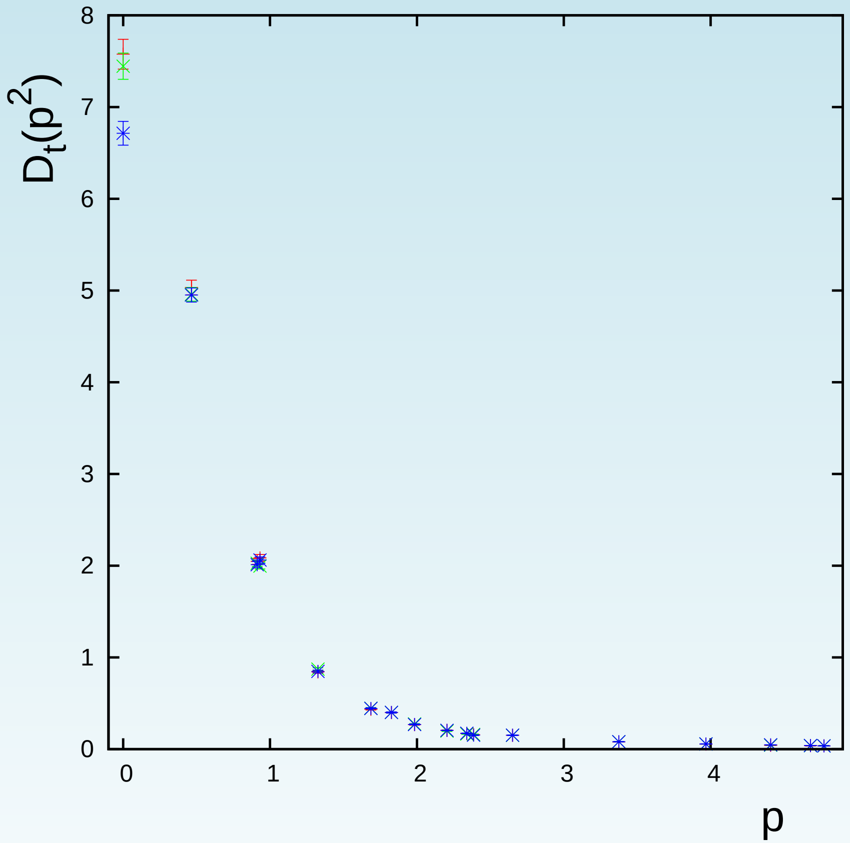
Continuum Limit (IV)

	8^4	16^4	24^4	32^4
2	β_1, β_2	—	—	—
3	all	β_1, β_2	β_1, β_2	—
4	all	all	all	β_1, β_2 *

Values of β for which the numerical gauge-fixing algorithm showed convergence. Results are reported for three different gauge groups and four different lattice volumes. In all cases the gauge parameter ξ was 1 (Feynman gauge).

* We did not test β_3 and β_4 yet.

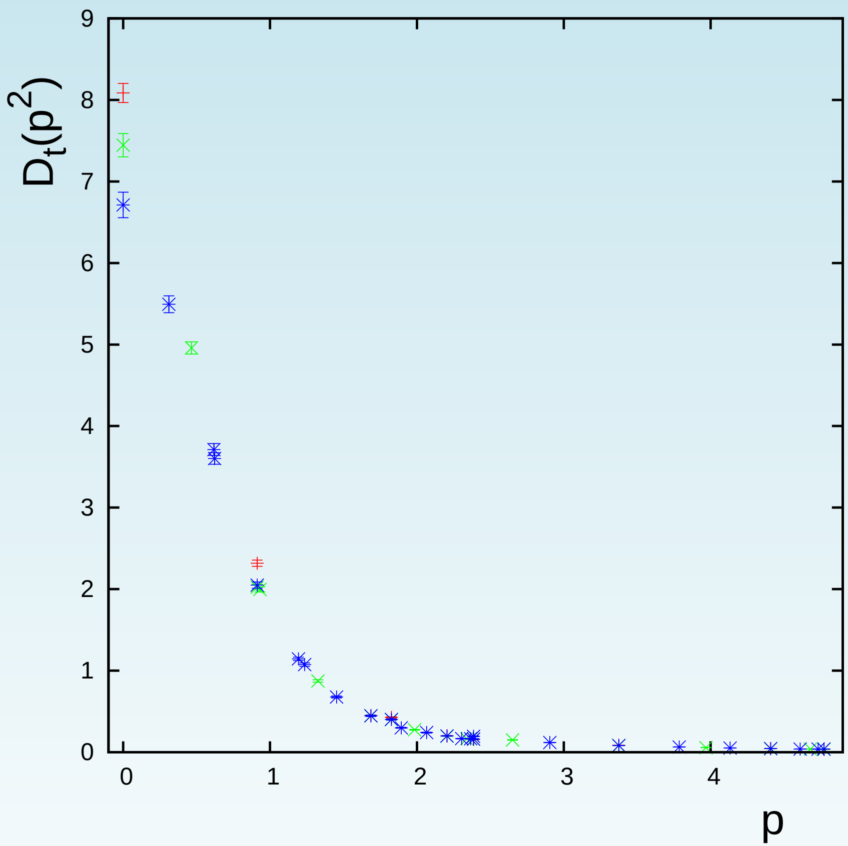
Transverse gluon propagator (I)



Transverse gluon propagator $D_t(p^2)$ [using the stereographic projection in the SU(2) case] as a function of the momentum p (both in physical units) for the lattice volume $V = 16^4$, $\beta = 2.3$ and $\xi = 0\,(+), 0.05\,(\times)$, $0.1\,(*)$.

$D_t(0)$ decreases as ξ increases (in agreement with L. Giusti et al., 2001).

Transverse gluon propagator (II)



Transverse gluon propagator $D_t(p^2)$ [using the stereographic projection in the SU(2) case] as a function of the momentum p (both in physical units) for the gauge coupling $\xi = 0.05$, $\beta = 2.3$, with the lattice volumes $V = 8^4\,(+), 16^4\,(\times)$ and $24^4\,(*)$.

$D_t(0)$ decreases as V increases (as in Landau gauge).

Conclusions

- We have found a minimizing functional for the linear covariant gauge which is a simple generalization of the Landau-gauge functional.
- This approach solves most problems encountered in earlier implementations and ensures a good quality for the gauge fixing with a ratio $D_l(p^2)p^2/\xi \approx 1$ for all cases considered.
- Simulations for large lattice volumes when the gauge parameter ξ is large can probably be done in the SU(N_c) case for larger N_c.