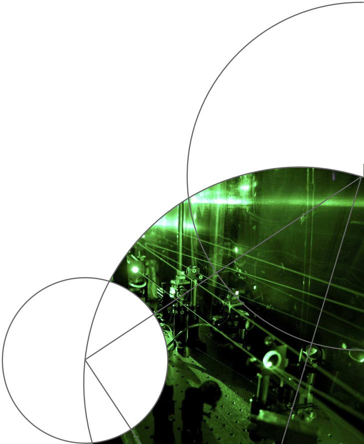
T(R)OPICAL QCD 2010

The Radius of the Proton:
Size Does Matter.

Jonathan D. Carroll
Centre for the Subatomic Structure of Matter

September 27, 2010



Outline

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1. The Puzzle
   - The Publication
   - The Experiment
   - The Original Calculations
2. The Investigation
   - The New Calculations
3. The Results

The Publication

Pohl *et al.* **Nature 466, 213-216 (8 July 2010)**

POHL et al.: $r_p = 0.84184(67)$ fm

CODATA: $r_p = 0.8768(69)$ fm

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Conclusion

" ... obtain a new value of the Rydberg constant,
$R_\infty = 10,973,731.568160(16)\ \mathrm{m}^{-1}$.

This is 2110 kHz/c or 4.9σ away from the CODATA value, but 4.6 times more precise."

"Finally, the origin of the discrepancy with the H data could originate from wrong or missing QED terms or from unexpectedly large contributions of yet uncalculated higher order terms."

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The Experiment

Muonic Hydrogen

The experiment focussed on muonic hydrogen (μp)

$$\frac{m_\mu}{m_e} \sim 200$$

so the wavefunction overlap with the proton is $(m_\mu/m_e)^3 \sim 10^7$ times stronger than that of the electron in the hydrogen atom.

This significantly increases the modification by the proton charge distribution.

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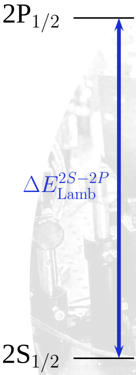

The Lamb shift is the splitting of the degenerate 2S$_{1/2}$ and 2P$_{1/2}$ eigenstates, due to vacuum polarization

$$V_{VP}(r) = -\frac{Z\alpha}{r}\frac{\alpha}{3\pi}\int_4^\infty \frac{d(q^2)}{q^2} e^{-m_e q r}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right)$$

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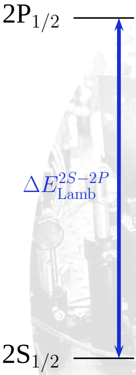

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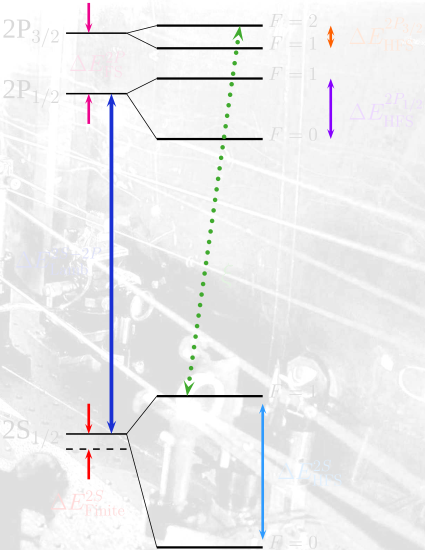
The Experiment

Muonic Hydrogen

The measured transition corresponds to the following sum of effects:

$$
\begin{aligned}
\xi &= E_{2P_{3/2}^{F=2}} - E_{2S_{1/2}^{F=1}} \\
&= \Delta E_{\text{Lamb}}^{2S-2P} + \Delta E_{\text{Finite}}^{2S} \\
&\quad + \Delta E_{FS}^{2P} \\
&\quad + \frac{3}{8}\Delta E_{HFS}^{2P_{3/2}} - \frac{1}{4}\Delta E_{HFS}^{2S}
\end{aligned}
$$



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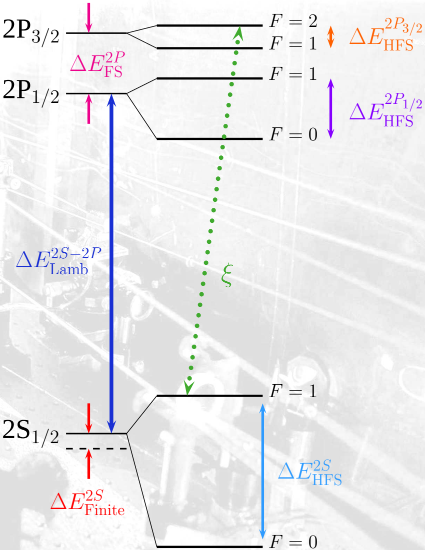

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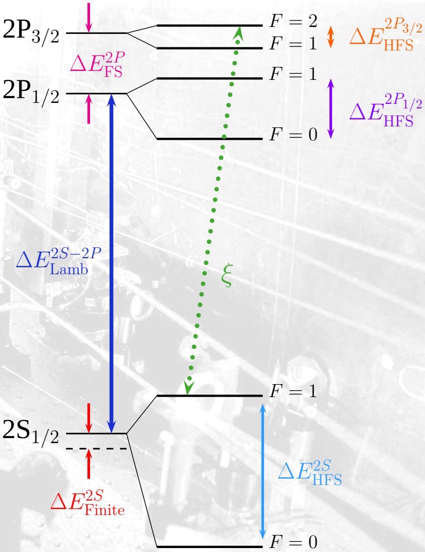
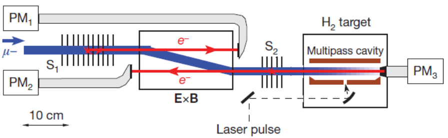
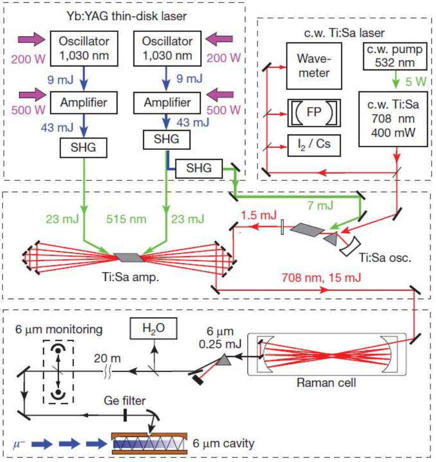


The Experiment

Muonic Hydrogen

The experiment was performed at the πE5 beam-line of the proton accelerator at PSI in Switzerland.

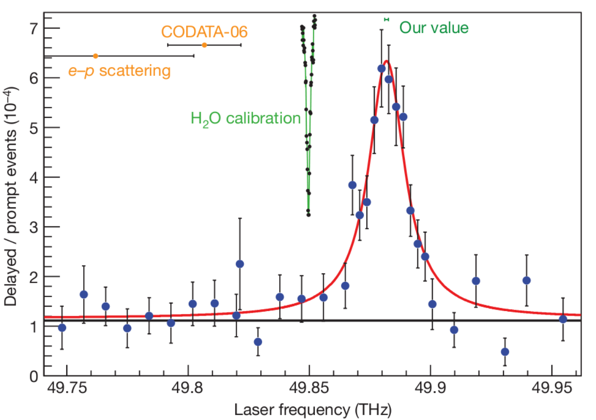




The Experiment

Muonic Hydrogen

The data is clearly in disagreement with previous measurements and the world average



The Experiment

Muonic Hydrogen

"The 1S-2S transition in H has been measured to 34 Hz, that is, 1.4×10^{-14} *relative accuracy. Only an error of about 1,700 times the quoted experimental uncertainty could account for our observed discrepancy."*

The Experiment

Muonic Hydrogen

"This new value of the proton radius

$$r_p = 0.84184(67)\ fm$$

is 10 times more precise, but 5.0σ smaller, than the previous world average, which is mainly inferred from H spectroscopy. It is 26 times more accurate, but 3.1σ smaller, than the accepted hydrogen-independent value extracted from electron-proton scattering."

"The origin of this large discrepancy is not known."

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Outline

The Original Calculations

The Shift

The extraction of the proton radius from the shift involves comparing the measured shift to the following sum of contributions

$$\begin{aligned} \xi &= 206.2949(32) \text{ meV} \\ &= 206.0573(45) - 5.2262 \, \langle r_p \rangle^2 + 0.0347 \, \langle r_p \rangle^3 \text{ meV} \end{aligned}$$

The radius-dependence is entirely contained in the finite-size contributions.

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The Approximations

To calculate the theoretical shift corresponding to the measured transition, `Borie`, `Pachucki`, `Martynenko`, *et al.* have used perturbation theory with non-relativisic wavefunctions to predict the size of the contributing effects, including relativistic effects.

$$V(r) = -\frac{Z\alpha}{r} + V'(r)$$

$$\Delta E^{nlm}_{V'} \sim \int_0^\infty V'(r)\,|\Psi^{nlm}_{\text{Schröd.}}(r)|^2\, d^3r$$

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The Lamb shift value(s) of `Borie` is(are) cited in the Nature article.

$$\Delta E^{nlm}_{\text{Lamb}} \sim \int_0^\infty V_{VP}(r)\,|\Psi^{nlm}_{\text{Schröd.}}(r)|^2 d^3r$$

$$\Delta E^{2S-2P}_{\text{Lamb}} \sim \int_0^\infty V_{VP}(r)\,|\Psi^{2S}_{\text{Schröd.}}(r)|^2 d^3r - \int_0^\infty V_{VP}(r)\,|\Psi^{2P}_{\text{Schröd.}}(r)|^2 d^3r$$

Still need to add recoil, radiative, and higher-order terms.

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Still need to add ~~recoil, radiative, and higher-order~~ 26 other terms.

#	Contribution		Our selection		Pachucki[1–3]		Borie[5]	
		Ref.	Value	Unc.	Value	Unc.	Value	Unc.
1	NR One loop electron VP	1,2			205.0074			
2	Relativistic correction (corrected)	1–3,5			0.0169			
3	Relativistic one loop VP	5	205.0282				205.0282	
4	NR two-loop electron VP	5,14	1.5081		1.5079		1.5081	
5	Polarization insertion in two Coulomb lines	1,2,5	0.1509		0.1509		0.1510	
6	NR three-loop electron VP	11	0.00529					
7	Polarisation insertion in two and three Coulomb lines (corrected)	11,12	0.00223					
8	Three-loop VP (total, uncorrected)				0.0076		0.00761	
9	Wichmann-Kroll	5,15,16	−0.00103				−0.00103	
10	Light by light electron loop contribution (Virtual Delbrück scattering)	6	0.00135	0.00135			0.00135	0.00015
11	Radiative photon and electron polarization in the Coulomb line $\alpha^2(Z\alpha)^4$	1,2	−0.00500	0.0010	−0.006	0.001	−0.005	
12	Electron loop in the radiative photon of order $\alpha^2(Z\alpha)^4$	17–19	−0.00150					
13	Mixed electron and muon loops	20	0.00007				0.00007	
14	Hadronic polarization $\alpha(Z\alpha)^4 m_r$	21–23	0.01077	0.00038	0.0113	0.0003	0.011	0.002
15	Hadronic polarization $\alpha(Z\alpha)^5 m_r$	22,23	0.000047					
16	Hadronic polarization in the radiative photon $\alpha^2(Z\alpha)^4 m_r$	22,23	−0.000015					
17	Recoil contribution	24	0.05750		0.0575		0.0575	
18	Recoil finite size	5	0.01300	0.001			0.013	0.001
19	Recoil correction to VP	5	−0.00410				−0.0041	
20	Radiative corrections of order $\alpha^n(Z\alpha)^k m_r$	2,7	−0.66770		−0.6677		−0.66788	
21	Muon Lamb shift 4th order	5	−0.00169				−0.00169	
22	Recoil corrections of order $\alpha(Z\alpha)^5 \frac{m}{M} m_r$	2,5–7	−0.04497		−0.045		−0.04497	
23	Recoil of order α^6	2	0.00030		0.0003			
24	Radiative recoil corrections of order $\alpha(Z\alpha)^n \frac{m}{M} m_r$	1,2,7	−0.00960		−0.0099		−0.0096	
25	Nuclear structure correction of order $(Z\alpha)^5$ (Proton polarizability contribution)	2,5,22,25	0.015	0.004	0.012	0.002	0.015	0.004
26	Polarization operator induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	23	0.00019					
27	Radiative photon induced correction to nuclear polarizability $\alpha(Z\alpha)^5 m_r$	23	−0.00001					
	Sum		206.0573	0.0045	206.0432	0.0023	206.05856	0.0046

The Original Calculations

Finite-Size Correction

The finite-size correction value(s) of `Borie` is(are) cited in the Nature article.

$$\Delta E_{\text{finite}} = -\frac{2Z\alpha}{3}\left(\frac{Z\alpha\mu}{2}\right)^3\left[\langle r_p^2\rangle - \frac{Z\alpha\mu}{2}\langle r_p^3\rangle\right]$$

The Original Calculations

2P Fine Structure

The 2P fine structure shift value(s) of `Martynenko` is(are) cited in the Nature article.

$$\Delta E_{FS}^{2P} = \frac{\mu^3 (Z\alpha)^4}{32 m_\mu^2} \left(1 + \frac{m_\mu}{2 m_p} \right)$$

The Original Calculations

2S Hyperfine Structure

The 2S hyperfine structure shift value(s) of `Martynenko` is(are) cited in the Nature article.

$$\Delta E_{HFS}^{2S_{1/2}} = \frac{1}{3}(Z\alpha)^4 \frac{\mu^3}{m_\mu m_p}(1+\kappa)$$

The Original Calculations

2P Hyperfine Structure

The 2P hyperfine structure shift value(s) of `Martynenko` is(are) cited in the Nature article.

$$\Delta E^{2P_{3/2}}_{HFS} = (Z\alpha)^4 \frac{\mu^3(1+\kappa)}{3m_\mu m_p}\left[\frac{2}{15} - \frac{a_\mu}{30} + \frac{m_\mu(1+2\kappa)}{2m_p(1+\kappa)}\right]$$

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The New Calculations

Dirac Equation

To calculate the shifts exactly, we can use the Dirac Equation with the appropriate potential to calculate the perturbed wavefunctions.

$$U_\alpha(x) = U_{n\kappa m t}(x) = \begin{bmatrix} iG_{n\kappa t}(r)/r & \Phi_{\kappa m}\eta_t \\ -F_{n\kappa t}(r)/r & \Phi_{-\kappa m}\eta_t \end{bmatrix}$$

$$\int_0^\infty dr \left(|G_\alpha(r)|^2 + |F_\alpha(r)|^2\right) = 1$$

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The New Calculations

Dirac Equation

$$\frac{d}{dr}G_\alpha(r) = -\frac{\kappa}{r}G_\alpha(r) + [\epsilon_\alpha + \mu - V(r)]\,F_\alpha(r)$$

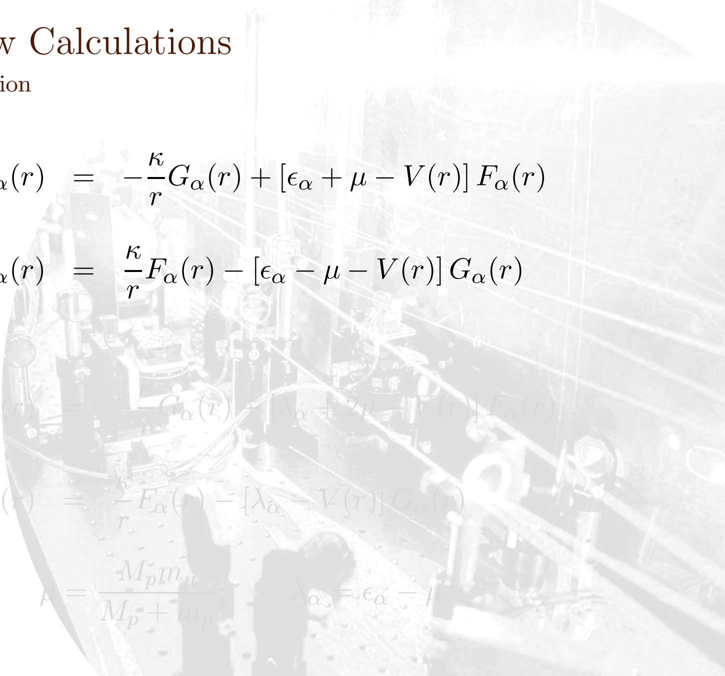
$$\frac{d}{dr}F_\alpha(r) = \frac{\kappa}{r}F_\alpha(r) - [\epsilon_\alpha - \mu - V(r)]\,G_\alpha(r)$$

$\Rightarrow$

$$\frac{d}{dr}G_\alpha(r) = -\frac{\kappa}{r}G_\alpha(r) + [\lambda_\alpha + 2\mu - V(r)]\,F_\alpha(r)$$

$$\frac{d}{dr}F_\alpha(r) = \frac{\kappa}{r}F_\alpha(r) - [\lambda_\alpha - V(r)]\,G_\alpha(r)$$

$$\mu = \frac{M_p m_\mu}{M_p + m_\mu}, \qquad \lambda_\alpha = \epsilon_\alpha - \mu$$



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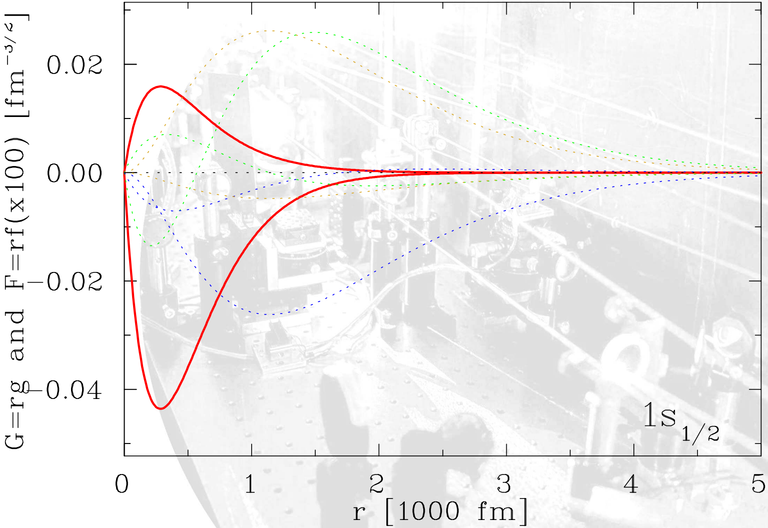

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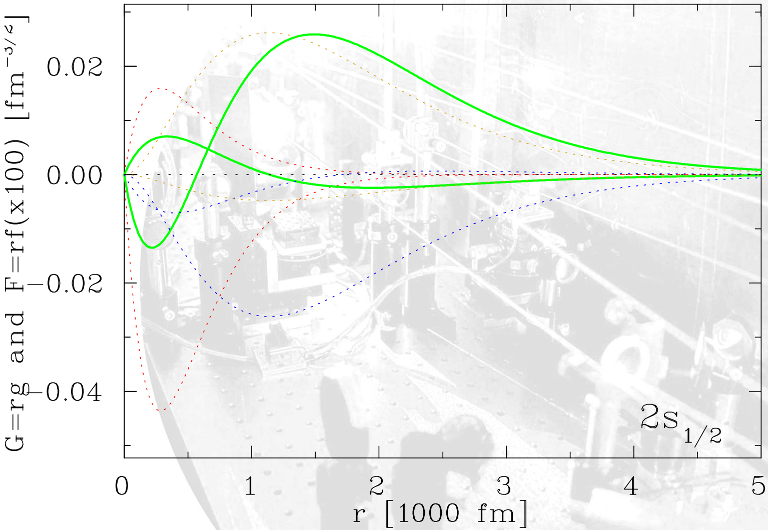

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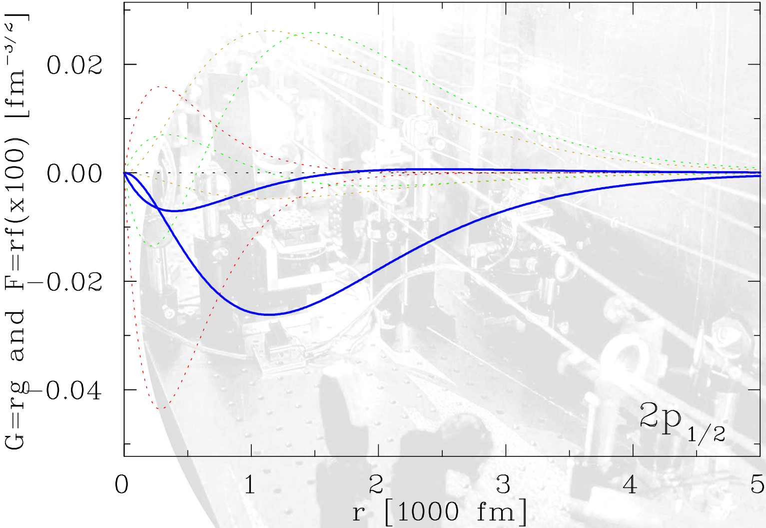

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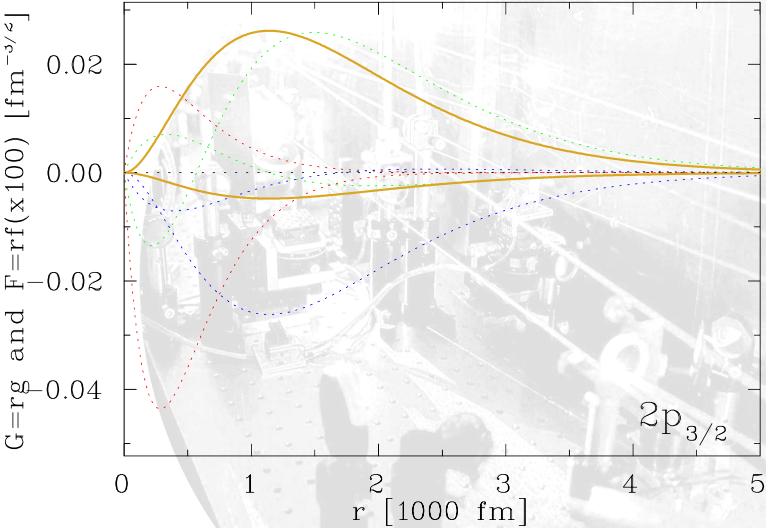

The New Calculations

Dirac Equation

The (shifted) eigenvalues can be reliably reproduced by using the point-Coulomb potential

$$V(r) = -\frac{Z\alpha}{r}$$

and the unperturbed eigenvalues are known exactly

$$\lambda_\alpha = \epsilon_\alpha - \mu = \mu\left(\left[1 + \frac{Z^2\alpha^2}{(n_\alpha - |\kappa_\alpha| + \sqrt{\kappa_\alpha^2 - Z^2\alpha^2})^2}\right]^{-\frac{1}{2}} - 1\right)$$

so we can check the accuracy and precision of our code...

The New Calculations

Dirac Equation

$1S_{1/2}: \quad \lambda = \text{-}2.5285267981975594 \text{ keV}$

$2S_{1/2}: \quad \lambda = \text{-}0.6321338034582128 \text{ keV}$

$2P_{1/2}: \quad \lambda = \text{-}0.6321338034582128 \text{ keV}$

$2P_{3/2}: \quad \lambda = \text{-}0.6321253878229215 \text{ keV}$

The New Calculations

Dirac Equation

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$2P_{3/2}: \quad \lambda = -0.6321253878229213 \text{ keV} \leftarrow \delta = 0.2 \text{ peV !}$

The New Calculations

Lamb Shift

We can calculate the Lamb shift by including the vacuum polarization potential such that

$$V(r) = -\frac{Z\alpha}{r} - \frac{Z\alpha}{r}\frac{\alpha}{3\pi}\int_4^\infty \frac{d(q^2)}{q^2} e^{-m_e q r}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right)$$

and calculating a converged wavefunction, then taking the difference in eigenvalues.

The New Calculations

Lamb Shift

We can calculate the Lamb shift by including the vacuum polarization potential such that

$$V(r) = -\frac{Z\alpha}{r} - \frac{Z\alpha}{r}\frac{\alpha}{3\pi}\int_4^\infty \frac{d(q^2)}{q^2} e^{-m_e q r}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right)$$

and calculating a converged wavefunction, then taking the difference in eigenvalues.

The New Calculations

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The New Calculations

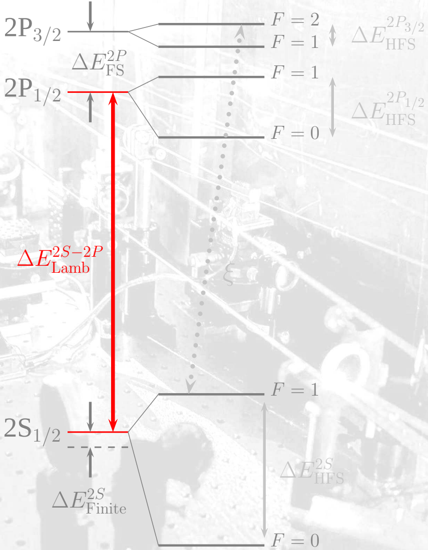
Lamb Shift

This is almost the Lamb shift... still need to add the 24 additional terms.

$$\text{Full Calculation} = \underbrace{\left\{ \begin{array}{c} \text{Relativistic one-loop VP} \\ + \text{ Polarization insertion} \\ \text{in 2 and 3 Coulomb lines} \end{array} \right.}_{\text{Perturbative Calculation}}$$

The New Calculations

Lamb Shift



The New Calculations

More Approximations

$$\Psi_{\text{Schröd.}} \neq \Psi_{\text{Dirac}}$$

But even this is not perfect...

There are still recoil corrections to include to approximate the Bethe-Salpeter equation.

For now we will assume these are unaffected by the vacuum polarization.

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Finite-Size Correction

The point-Coulomb potential assumes that the proton is a point-particle.

This is not an accurate assumption to make, so we use the finite-size Coulomb potential instead

$$V_C(r) = -\frac{Z\alpha}{r} \to -Z\alpha \int \frac{\rho(r')}{|\vec{r} - \vec{r}'|} d\tau'$$

Since this involves the charge-density, this leads to a radius-dependent quantity

$$\rho(r) = \frac{\eta}{8\pi} e^{-\eta r}; \quad \eta = \sqrt{12/{r_p}^2}$$

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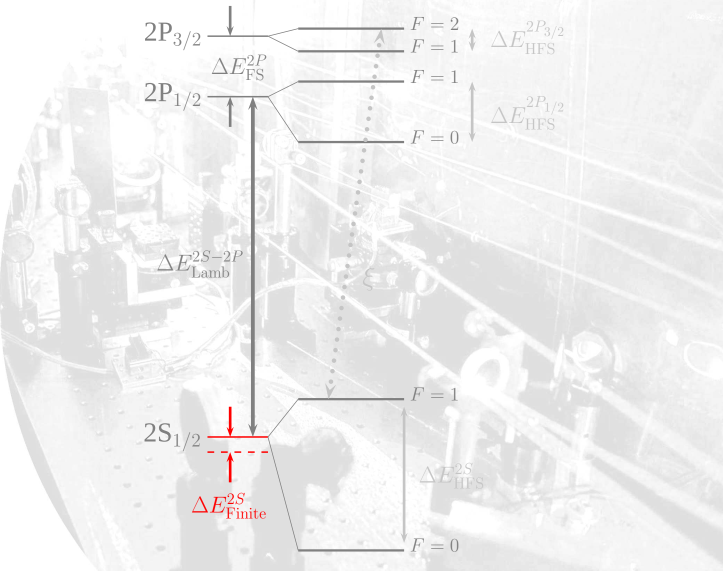

The New Calculations

2p Fine Structure

Subtracting the exact eigenvalues of the $2P_{1/2}$ and $2P_{3/2}$ gives the fine structure splitting precisely.

$$\Delta E_{FS}^{2P} = \epsilon_{2P_{3/2}} - \epsilon_{2P_{1/2}}$$

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$$V(r) = -\frac{Ze}{r} + V_{VP}(r)$$

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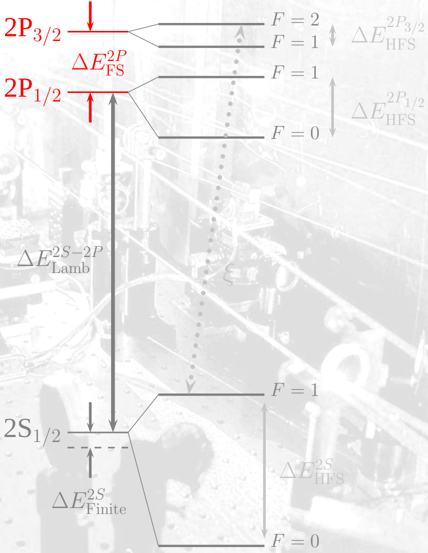

The New Calculations

Hyperfine Structure

The hyperfine structure is a measure of the $\ell \cdot \sigma$ coupling. The Hamiltonian is

$$\mathcal{H} = 2\beta\gamma\hbar \frac{\ell(\ell+1)}{j(j+1)} \left\langle \frac{1}{r^3} \right\rangle \mathbf{I}\cdot\mathbf{J} + A_F \mathbf{I}\cdot\mathbf{S}$$

$$A_F = \frac{16\pi}{3}\beta\gamma\hbar|\psi(0)|^2$$

$$\begin{aligned} \mathbf{I}\cdot\mathbf{J} &= \frac{1}{2}[F(F+1) - I(I+1) - j(j+1)] \\ \mathbf{I}\cdot\mathbf{S} &= \frac{1}{2}[F(F+1) - \frac{3}{2}] \end{aligned}$$

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For $\ell = 0$ the contact term in the Hamiltonian is non-zero, while the dipole term vanishes

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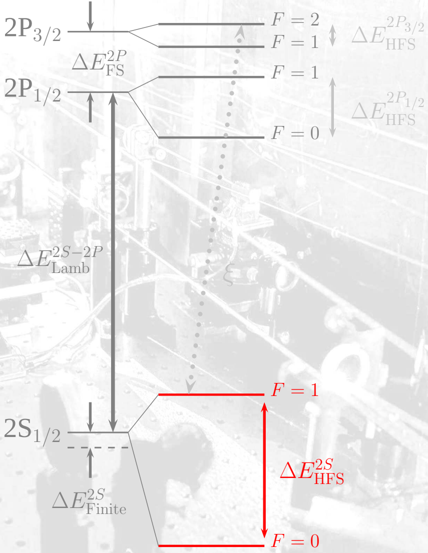


The New Calculations

2p Hyperfine Structure

For $\ell \neq 0$ the dipole term in the Hamiltonian is non-zero, while the contact term vanishes

$$E_{HFS}^{2P} = 2\beta\gamma\hbar \frac{\ell(\ell+1)}{j(j+1)} \left\langle \frac{1}{r^3} \right\rangle \mathbf{I} \cdot \mathbf{J}$$

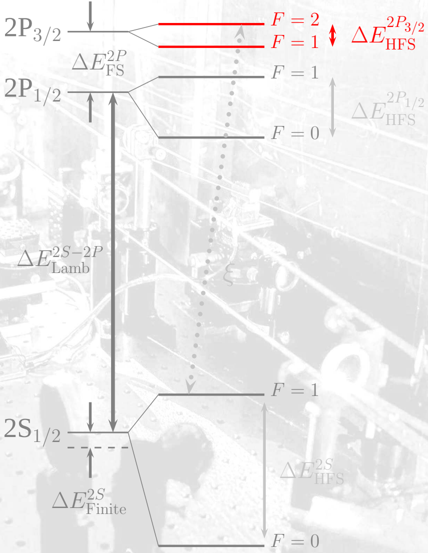
$\Rightarrow$

$$\Delta E_{HFS}^{2P_{3/2}(F=2-F=1)} = 2\beta\gamma\hbar \frac{\ell(\ell+1)}{j(j+1)} \left\langle \frac{1}{r^3} \right\rangle \times [F(F+1) - I(I+1) - j(j+1)]$$

TO BE COMPLETED!

The New Calculations

2p Hyperfine Structure



The New Calculations

Vacuum Polarization Finite-Size Correction

Since we are solving for exact wavefunctions, we can also add additional terms to the potential. For example, we can investigate the finite-size effect of the vacuum polarization

$$V_{VP}(r) \to -\frac{2Z\alpha^2}{3\pi}\int \frac{\rho(r')}{|\vec{r}-\vec{r}'|} Z_0(|\vec{r}-\vec{r}|)\, d\tau'$$

$$Z_n(|\vec{r}|) = \int_1^\infty e^{-\frac{2}{\lambda}|\vec{r}|\xi}\left(1+\frac{1}{2\xi^2}\right)\frac{(1+\xi)^{\frac{1}{2}}}{\xi^n\xi^2}d\xi$$

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Outline

1. The Puzzle
   - The Publication
   - The Experiment
   - The Original Calculations
2. The Investigation
   - The New Calculations
3. The Results

The Results

The Data

In order to reproduce the Nature result, we require a missing contribution of ~ 0.15 meV.

!	Contribution	Carroll	Nature	Other	$\lvert\delta\rvert$	Precision
	$E(1S_{1/2}) : V = V_C$	-2.5285267981975605 keV		-2.5285267981975594 keV [exact]	1.1 peV	QP
	$E(1S_{1/2}) : V = V_C + V_{VP}$	-2.530427808554 keV				DP
	$E(2S_{1/2}) : V = V_C$	-0.6321338034390614 keV		-0.6321338034582128 keV [exact]	19.15 neV	QP
	$E(2S_{1/2}) : V = V_C + V_{VP}$	-0.632353566988 keV				DP
	$E(2P_{1/2}) : V = V_C$	-0.632133803154 keV		-0.632133803457 keV [exact]	0.30 μeV	DP
	$E(2P_{1/2}) : V = V_C + V_{VP}$	-0.632148387529 keV				DP
	$E(2P_{3/2}) : V = V_C$	-0.6321253878229213 keV		-0.6321253878229215 keV [exact]	0.2 peV	QP
	$E(2P_{3/2}) : V = V_C + V_{VP}$	-0.632139967131 keV				DP
	NR One loop electron VP	205.008 meV		205.0074 meV [Pachucki]	0.6 μeV	DP
	Relativistic one loop VP	205.029 meV	205.0282 meV	205.0282 meV [Borie]	0.8 μeV	DP
	+Polarization insertion in two Coulomb lines		+0.1509 meV			DP
	+Polarisation insertion in two and three Coulomb lines		+0.00223 meV			DP
	$= \Delta E(2P_{1/2}) - \Delta E(2S_{1/2}) : V = V_C + V_{VP}$	205.17953 meV	205.18133 meV		1.8 μeV	DP
	$\langle r_p^2 \rangle$ nuclear size correction	5.22047 $\langle r_p^2 \rangle$	5.22619 $\langle r_p^2 \rangle$		5.7e-3 $\langle r_p^2 \rangle$	DP
!	$\langle r_p^3 \rangle$ nuclear size correction	0.0484745 $\langle r_p^3 \rangle$	0.0347 $\langle r_p^3 \rangle$		1.4e-2 $\langle r_p^3 \rangle$	DP
	$\Delta E_{FS}^{2P} : V = V_C$	8.415349 meV	8.352082 meV	8.415643 meV [exact]	0.29 μeV	DP
!	$\Delta E_{FS}^{2P} : V = V_C + V_{VP}$	8.420398 meV				DP
	(NR) ΔE_{HFS}^{2S}	22.8053 meV		22.8054 meV [Martynenko]	0.1 μeV	DP
	+Relativistic correction $\mathcal{O}(\alpha^6)$			+0.0026 meV		DP
	+One-loop e^- VP $\mathcal{O}(\alpha^6)$, $\mathcal{O}(\alpha^6)$			+0.0482 meV		DP
	+VP corrections in second-order PT $\mathcal{O}(\alpha^6)$, $\mathcal{O}(\alpha^6)$			+0.00746 meV		DP
	$= \Delta E_{HFS}^{2S} : V = V_C + V_{VP}$	22.8988 meV		22.9308 meV	32.5 μeV	DP
	$\Delta E_{HFS}^{2S} : V = V_C$	22.8182 meV				DP
	$\Delta E_{HFS}^{2P_{3/2}} : V = V_C$	? meV	3.392588 meV	3.392588 meV [Martynenko]		DP
	$\Delta E_{HFS}^{2P_{3/2}} : V = V_C + V_{VP}$	? meV				DP

16 quantities recomputed.

No significant deviations from approximations?

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$\sim$ 16 quantities re-calculated.

No significant deviations from approximations.

The Results

The Data

Lamb Shift:

	Relativistic one-loop VP
	+ Polarization insertion in 2 and 3 Coulomb lines
	= 205.18133 meV
Full calculation	= 205.17953 meV

The Results

The Data

Finite Size Correction:

	coefficient $\langle r_p \rangle^2$	coefficient $\langle r_p \rangle^3$
	$= 5.22619$	$= 0.0347$
Full calculation	$= 5.22047$	$= 0.0484745$

The Results

The Data

2P Fine Structure:

Dirac + Vacuum Polarization
$= 8.42064$ meV

Full calculation $= 8.420398$

The Results

The Data

$2S_{1/2}$ Hyperfine Structure:

$$\begin{aligned} & \text{Non-relativistic shift} \\ & + \text{Relativistic correction} \\ & + \text{One-loop } e^- \text{ VP } \mathcal{O}(\alpha^5), \mathcal{O}(\alpha^6) \\ & + \text{VP corrections in second-order PT } \mathcal{O}(\alpha^6) \\ & = 22.9308 \text{ meV} \end{aligned}$$

$$\text{Full calculation} \quad = 22.8988 \text{ meV}$$

The Results

The Data

$2P_{3/2}$ Hyperfine Structure:

	Non-relativistic shift
	?
	?
	= 3.392588 meV
Full calculation	= ? meV

TO BE COMPLETED!

The Results

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We have confirmed the [majority of] results of `Borie`, `Pachucki`, `Martynenko`, *et al.* to within sufficient error such as to eliminate these terms as a source of the discrepancy.

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Further Reading I

- R. Pohl et al.
  The size of the proton
  Nature Vol 466 (8 July 2010)
  [doi:10.1038/nature09250]

- E. Borie
  Lamb shift in muonic hydrogen
  Phys.Rev.A71:032508 (2005)

- K. Pachucki
  Theory of the Lamb shift in muonic hydrogen
  Phys.Rev.A53:2092-2100 (1996)

- A. P. Martynenko
  2S Hyperfine Splitting of Muonic Hydrogen
  Phys.Rev.A71:022506 (2005)

Further Reading II

- A. P. Martynenko
  Fine and hyperfine structure of P-wave levels in muonic hydrogen
  Phys.At.Nucl.71:125-135 (2008)

- J. Rafelski, L. P. Fulcher
  Fermions And Bosons Interacting With Arbitrarily Strong External Fields
  Phys.Rep.38:227-361 (1978)

Cheers, Everyone!

The spectrum of the hydrogen atom has proved to be the Rosetta stone of modern physics.

– T.W. Hänsch