Finite Volume Corrections to the Nucleon Axial Charge

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Outline

Lattice vs Experiment

Hedgehog Model

Nucleon-Pion Interaction

Conclusions

The Lattice

Lattice approximations:

- a box of finite volume is used to approximate all space (L the box width is finite)
- finite spacing, a, between lattice points are used instead of the continuum of space
- quark masses m_q used are much larger than the physical masses.

To compare with experiments we need to take limits:

- $L \to \infty$
- $a \to 0$
- $m_q \to$ physical m_q

Lattice has had a number of successes:

- calculated proton and neutron masses to within a few percent[Dürr *et al.* Science, 1163233; Young and Thomas, PRD 014503]
- pion and kaon decay constants also to within a few percent [Davies *et al.* PRL 92, 022001]
- axial charge (within 10%) [Edwards *et al.* PRL 96, 052001]

However, there are still some challenges:

- Axial Form Factor
- Resonances (unstable particles)
- "Disconnected" diagrams

Axial Form Factor

Is...

"a measurable and physical manifestation of the nature of the nucleons constituents and the dynamics that binds them together."

[Arrington et al. nucl-th/0611050]

Matrix element of the axial isospin current in the nucleon:

$$\langle \mathrm{N}|j^{\mu 5a}(q)|\mathrm{N}\rangle =$$
$$\bar{u}\left[\gamma^\mu\gamma^5 G_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\gamma^5 G_2(q^2) + q^\mu\gamma^5 G_3(q^2)\right]\tau^a u$$

Physical axial form factor:

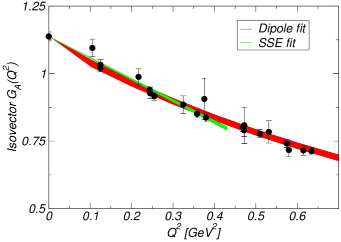
$$G_A(Q^2) = g_A\left(\frac{1}{1+\frac{Q^2}{\Lambda^2}}\right)^2$$

Lattice vs Experiment

Bratt *et al.* [arXiv:1001.3620]

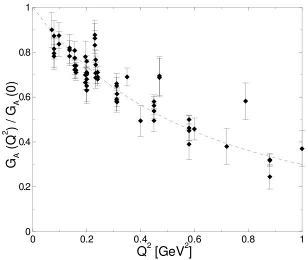
- $m_\pi = 356$ MeV
- $28^3 \times 64$ lattice
- $a = 0.1241(25)$ fm

$\Longrightarrow \Lambda = 1.6$ GeV



Compared with a *normalised* plot of the experimental value [Bernard et al. hep- ph/0107088],

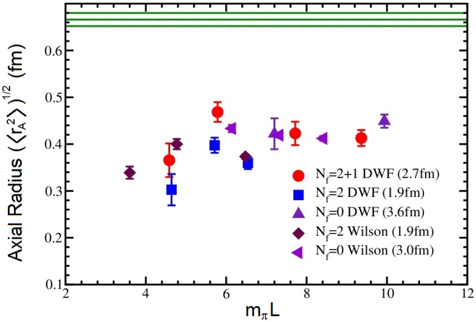
$\Longrightarrow \Lambda = 1.1$ GeV



Axial Radius

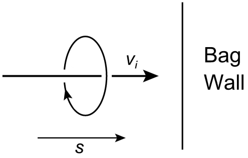
$$\langle r_A^2 \rangle = -\frac{6}{G_A(0)} \frac{d}{dQ^2} G_A(Q^2)|_{Q^2=0}$$

Ohta and Yamazaki [arXiv:0810.0045]:

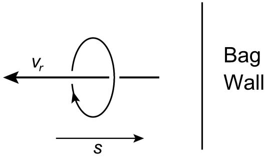


The Hedgehog Model

Used by Chodos and Thorn to solve lack of chiral symmetry in an earlier bag model [Phys. Rev. D12 (1975)].



Incident (Helicity +1)



Reflected (Helicity -1)

[A.W. Thomas Adv.Nucl.Phys 1984]

Important features:

- valence quarks confined to a "bag"
- pion and sigma fields couple only to the surface of the bag
- equations of motion can be solved exactly
- neither an eigenstate of spin or isospin and so is definitely not physical (!)
- respects chiral symmetry of QCD
- its pion field has a radial dependence.

Chodos and Thorn Lagrangian

$$\mathcal{L}_{CT} = [\bar{\psi} i \partial\!\!\!/ \psi - B]\theta_V - \lambda\bar{\psi}\left(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5\right)\psi\delta_S + \frac{1}{2}\left(\partial_\mu\sigma\right)\left(\partial^\mu\sigma\right) + \frac{1}{2}\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial^\mu\vec{\pi}\right)$$

Equations of motion [Phys. Rev. D12 (1975)]:

$$\begin{aligned} i\partial\!\!\!/\psi &= 0, \qquad r < R; \\ i\hat{r}\cdot\vec{\gamma}\psi &= -\xi\left(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5\right)\psi, \qquad r = R; \\ \nabla^2\sigma &= \frac{1}{2}\xi\bar{\psi}\psi\delta(r-R); \\ \nabla^2\vec{\pi} &= \frac{1}{2}\xi\bar{\psi}i\vec{\pi}\gamma_5\psi\delta(r-R); \\ \frac{\partial}{\partial r}[\bar{q}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)q] &= -2(\sigma^2+\vec{\pi}^2)^{1/2}B,\ r = R \end{aligned}$$

where $\xi = \left[\sigma^2(R) + \pi^2(R)\right]^{-1/2}$

Field equations:

$$q(\vec{r}) = \begin{pmatrix} j_0\left(\frac{\omega r}{R}\right) \\ i\vec{\sigma}\cdot\hat{r}j_1\left(\frac{\omega r}{R}\right) \end{pmatrix}\chi_h$$

$$\sigma(\vec{r}) = f(r)$$

$$\vec{\pi}(\vec{r}) = g(r)\hat{r},$$

Where the spin-flavour wave function χ_h is defined as,

$$|\chi_h\rangle = \frac{1}{\sqrt{2}}(|u\downarrow\rangle - |d\uparrow\rangle$$

Finding the Axial Form Factor

Our Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{HH}} &= [\bar{\psi}(i\partial\!\!\!/ - m_q)\psi - B]\theta_V - \lambda\bar{\psi}\left(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5\right)\psi\delta_S \\ &\quad + \frac{1}{2}\left(\partial_\mu\sigma\right)\left(\partial^\mu\sigma\right) + \frac{1}{2}\left(\partial_\mu\vec{\pi}\right)\cdot\left(\partial^\mu\vec{\pi}\right) - \frac{1}{2}m_\pi^2\vec{\pi}\cdot\vec{\pi}\,,\end{aligned}$$

and axial current,

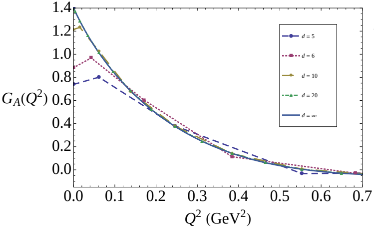
$$\vec{A}^\mu = \frac{1}{2}\bar{\psi}\gamma^\mu\gamma_5\vec{\tau}\psi\theta_{\mathrm{V}} + (\partial^\mu\sigma)\vec{\pi} - \sigma(\partial^\mu\vec{\pi})\,.$$

$$\begin{aligned}\langle\,\mathrm{HH}\,|j^{\mu 5a}(q)|\,\mathrm{HH}\,\rangle &= \langle\,\mathrm{HH}\,|\int \mathrm{d}^3\mathrm{x}\, e^{i\vec{q}.\vec{x}} A^{\mu a}(\underline{x})|\,\mathrm{HH}\,\rangle \\ &= \bar{u}\left[G_{\mathrm{A}}(q^2)\vec{\sigma}\,\underline{\tau} + G_{\mathrm{P}}(q^2)\,\vec{\sigma}.\hat{q}\,\hat{q}\,\underline{\tau}\right] u\end{aligned}$$

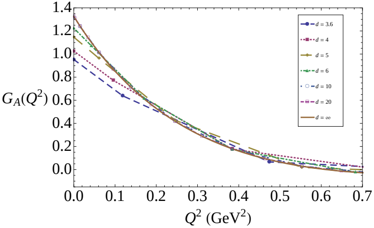
Full expression for axial form factor:

$$
\begin{aligned}
G_{Ahh}\left(q^2\right) = {} & \\
& 2\pi N^2 \int_0^R \mathrm{d}r\, r^2 \left\{ \left[\alpha_+^2 j_0^2\left(\frac{\omega r}{R}\right) - \alpha_-^2 j_1^2\left(\frac{\omega r}{R}\right)\right] j_0(qr) + \alpha_-^2\, 2 j_1^2\left(\frac{\omega r}{R}\right)\frac{j_1(qr)}{qr} \right\} \\
& \quad + 4\pi \int_0^\infty \mathrm{d}r\, r^2 f'(r) g(r) \frac{j_1(qr)}{qr} \\
& \qquad - 4\pi \int_0^\infty \mathrm{d}r\, r^2 f(r) \left[g'(r) \frac{j_1(qr)}{qr} + \frac{g(r)}{3r}\left(2 j_0(qr) - j_2(qr)\right)\right]
\end{aligned}
$$

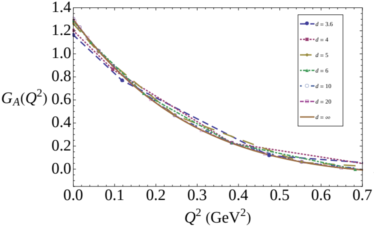
Results



$m_\pi = 140$ MeV



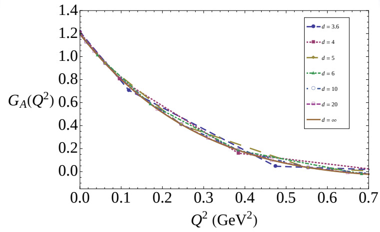
$m_\pi = 300$ MeV

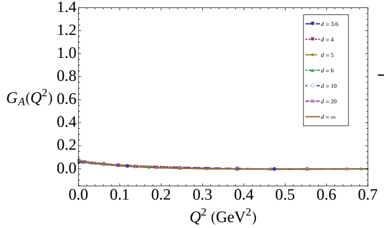


$m_\pi = 500$ MeV

[NH, Thomas and Young AIP Conf.Proc.1354:206-212,2011]

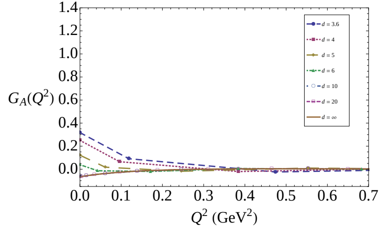
Quark field contribution:





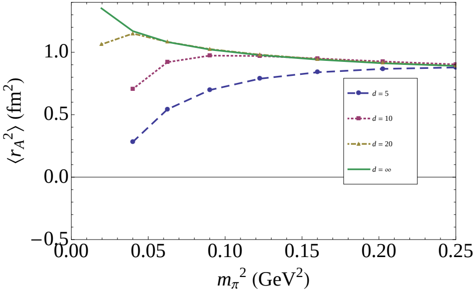
- sigma field contribution.

Pion field contribution:



Axial Radius

$$\langle r_A^2 \rangle = -\frac{6}{G_A(0)} \frac{d}{dQ^2} G_A(Q^2) \Big|_{Q^2=0}$$



[NH, Thomas and Young AIP Conf.Proc.1354:206-212,2011]

Initial Conclusions

For small Q^2 the axial form factor is significantly reduced for finite volumes.

$\implies$ this leads to a small axial radius.

So, if the hedgehog model accurately describes the nucleon, then these results argue that the discrepancy between the lattice calculations and the experimental value is due to finite volume effects. However...

- It is difficult to fully reconcile the situation described here with that on the lattice:

- periodic boundary conditions.

$$\left.\frac{\partial \pi}{\partial r}\right|_{r=L} = 0$$

- Finite volume effects due to the delocalisation of the pion-pole contribution shown by Cohen [T. D. Cohen Phys. Let. 2002] to be invalid.

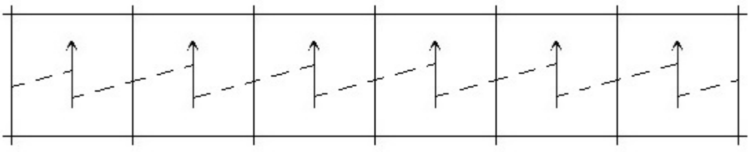
Cloudy Bag Model

Lagrangian:

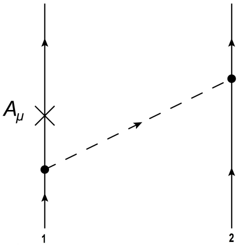
$$\mathcal{L}_{\mathsf{CBM}} = (i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - B)\theta_{\mathsf{V}} - \frac{1}{2}\bar{\psi}\psi\delta_{\mathsf{S}} + \frac{1}{2}(\partial_{\mu}\vec{\phi})^2 - \frac{1}{2}m_{\pi}^2(\vec{\phi})^2 - \frac{i}{2f_{\pi}}\bar{\psi}\gamma_5\vec{\tau}\cdot\vec{\phi}\,\psi\,\delta_{\mathsf{S}}\,,$$

- pion "cloud" surrounding the nucleon
- chirally symmetric
- contains no χ_h function

Pion emission and absorption on a periodic/antiperiodic lattice:



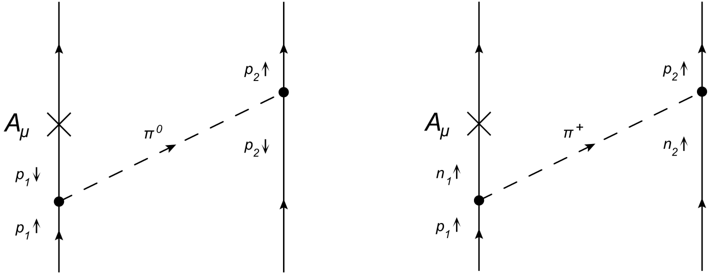
- single dimension
- particular point in time



The general amplitude for these diagrams looks like:

$$\left\langle N_{1\alpha'} N_{2\beta'} \,|\, H_{\text{int}} \,|\, N_{1\alpha}\, \pi^i_{\underline{k}}\, N_{2\beta'} \right\rangle G_0$$
$$\times \left\langle N_{1\alpha}\, \pi^i_{\underline{k}}\, N_{2\beta'} \,|\, A^{(1)}_{3z} \,|\, N_{1\alpha}\, \pi^i_{\underline{k}}\, N_{2\beta'} \right\rangle G_0$$
$$\times \left\langle N_{1\alpha}\, \pi^i_{\underline{k}}\, N_{2\beta'} \,|\, H_{\text{int}} \,|\, N_{1\alpha}\, N_{2\beta} \right\rangle$$

Contributions to the axial charge:



The left-hand diagram gives,

$$\Big\langle p_1 \downarrow p_2 \uparrow \,|H_{\rm int}\,|\, p_1 \downarrow \pi^0_{\underline{k}}\; p_2 \downarrow \Big\rangle\, G_0 \times \Big\langle p_1 \downarrow \pi^0_{\underline{k}}\; p_2 \downarrow \,|\, A^{(1)}_{3z}\,|\, p_1 \downarrow \pi^0_{\underline{k}}\; p_2 \downarrow \Big\rangle\, G_0 \times \Big\langle p_1 \downarrow \pi^0_{\underline{k}}\; p_2 \downarrow \,|\, H_{\rm int}\,|\, p_1 \uparrow p_2 \downarrow \Big\rangle$$

$$= \langle A^{(1)}_{3z} \rangle \Big\langle -\frac{1}{2}\; \frac{1}{2} \,\Big|\, \frac{\tau_{2i}\tau_{1i}}{2(2\pi)^3} \left(\frac{g_A}{2f_\pi}\right)^2 \vec{\sigma}_2 \cdot \frac{\nabla}{i} \vec{\sigma}_1 \cdot \frac{\nabla}{i} \times \int d^3k \frac{e^{i\vec{k}\cdot\vec{L}}}{(k^2+m_\pi^2)^{3/2}} \Big|\, \frac{1}{2}\; -\frac{1}{2} \Big\rangle$$

$$= \langle A^{(1)}_{3z} \rangle \Big\langle -\frac{1}{2}\; \frac{1}{2} \,\Big|\, \left(\frac{g_A}{2f_\pi}\right)^2 \frac{\tau_{2i}\tau_{1i}}{(2\pi)^2}\, \vec{\sigma}_2 \cdot \frac{\nabla}{i} \vec{\sigma}_1 \cdot \frac{\nabla}{i}\, K_0(m_\pi L) \Big|\, \frac{1}{2}\; -\frac{1}{2} \Big\rangle$$

For isospin,

- need $\tau_{23}\tau_{13}$ from $\tau_{2i}\tau_{1i}$.

Where as for spin,

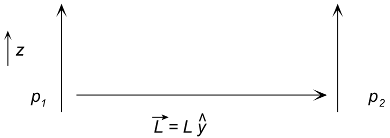
- need σ_{1-} from $\vec{\sigma}_1 \cdot \nabla \Longrightarrow \sigma_{1-}\nabla_+$
- need σ_{2+} from $\vec{\sigma}_2 \cdot \nabla \Longrightarrow \sigma_{2+}\nabla_-$

So therefore,

$$\vec{\sigma}_2 \cdot \frac{\nabla}{i}\vec{\sigma}_1 \cdot \frac{\nabla}{i} = \sigma_{2+}\sigma_{1-}\left(\nabla_x\nabla_x + \nabla_y\nabla_y\right)$$

Maximum when,

$$\nabla_y = \frac{\partial}{\partial L}$$

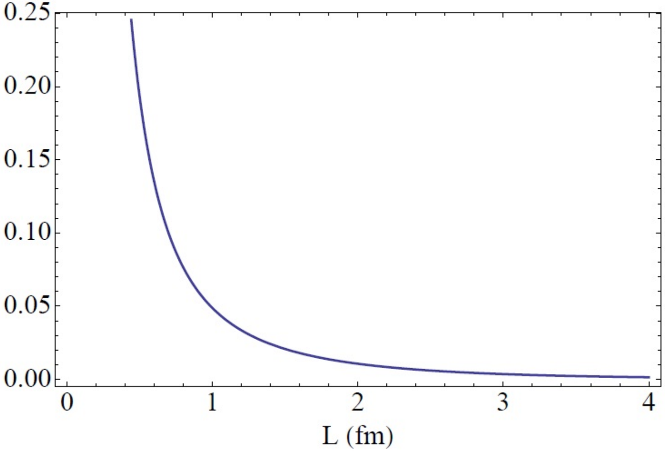


Substituting all this into the amplitude we get:

$$\langle A_{3z}^{(1)}\rangle \left\langle -\frac{1}{2}\ \frac{1}{2}\ \right| \left(\frac{g_A}{2f_\pi}\right)^2 \frac{\tau_{2i}\tau_{1i}}{(2\pi)^2}\ \frac{\sigma_{2+}\sigma_{1-}}{2} \frac{\partial^2}{\partial L^2} K_0(m_\pi L) \left| \frac{1}{2}\ -\frac{1}{2} \right\rangle$$

$$= \langle A_{3z}^{(1)}\rangle \left\langle -\frac{1}{2}\ \frac{1}{2}\ \right| \left(\frac{g_A}{2f_\pi}\right)^2 \frac{\tau_{2i}\tau_{1i}}{(2\pi)^2}\ \frac{\sigma_{2+}\sigma_{1-}}{2} \cdot \frac{1}{2} m_\pi^2 \left(K_0(m_\pi L) + K_0(m_\pi L)\right) \left| \frac{1}{2}\ -\frac{1}{2} \right\rangle$$

Looking at a plot of this function,



[Hall, Thomas and Young in progress.]

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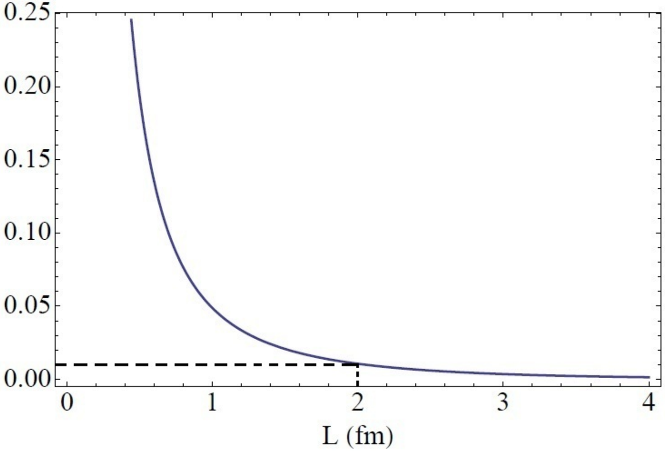
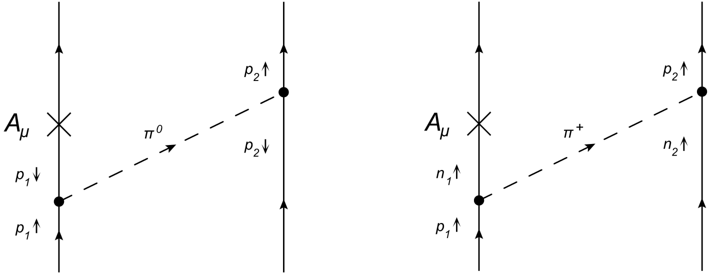


[Hall, Thomas and Young in progress.]

Contributions to the axial charge:



Similarly the right-hand side diagram gives,

$$\left\langle n_1 \uparrow p_2 \uparrow \left| H_{\text{int}} \right| n_1 \uparrow \pi^+_{\underline{k}}\; n_2 \uparrow \right\rangle G_0$$
$$\times \left\langle n_1 \uparrow \pi^+_{\underline{k}}\; n_2 \uparrow \left| A^{(1)}_{3z} \right| n_1 \uparrow \pi^+_{\underline{k}}\; n_2 \uparrow \right\rangle G_0$$
$$\times \left\langle n_1 \uparrow \pi^+_{\underline{k}}\; n_2 \uparrow \left| H_{\text{int}} \right| p_1 \uparrow n_2 \uparrow \right\rangle$$
$$= \langle A^{(1)}_{3z} \rangle \left\langle -\frac{1}{2}_{(is)}\; \frac{1}{2}_{(is)} \right| \frac{\tau_{2i}\tau_{1i}}{2(2\pi)^3} \left(\frac{g_A}{2f_\pi}\right)^2 \vec{\sigma}_2 \cdot \frac{\nabla}{i} \vec{\sigma}_1 \cdot \frac{\nabla}{i}$$
$$\times \int d^3k \frac{e^{i\vec{k}\cdot\vec{L}}}{(k^2 + m_\pi^2)^{3/2}} \left| \frac{1}{2}_{(is)} \; -\frac{1}{2}_{(is)} \right\rangle$$
$$= \langle A^{(1)}_{3z} \rangle \left\langle -\frac{1}{2}_{(is)}\; \frac{1}{2}_{(is)} \right| \left(\frac{g_A}{2f_\pi}\right)^2 \frac{\tau_{2i}\tau_{1i}}{(2\pi)^2} \vec{\sigma}_2 \cdot \frac{\nabla}{i} \vec{\sigma}_1 \cdot \frac{\nabla}{i} K_0(m_\pi L) \left| \frac{1}{2}_{(is)} -\frac{1}{2}_{(is)} \right\rangle$$

However this time,

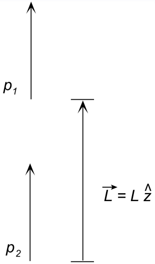
- need $-\tau_{2+}\tau_{1-}$ from $\tau_{2i}\tau_{1i}$.

Where as for spin,

- need σ_{13} from $\vec{\sigma}_1 \cdot \nabla \implies \sigma_{13}\nabla_3$
- need σ_{2+} from $\vec{\sigma}_2 \cdot \nabla \implies \sigma_{23}\nabla_3$

and so therefore the amplitude becomes:

$$\langle A_{3z}^{(1)} \rangle \left\langle -\frac{1}{2}_{(is)} \, \frac{1}{2}_{(is)} \right| \left(\frac{g_A}{2f_\pi} \right)^2 \frac{\tau_{2+}\tau_{1-}}{(2\pi)^2} \, \sigma_{23}\sigma_{13}\nabla_z\nabla_z \, K_0(m_\pi L) \left| \frac{1}{2}_{(is)} -\frac{1}{2}_{(is)} \right\rangle$$



Maximum when,

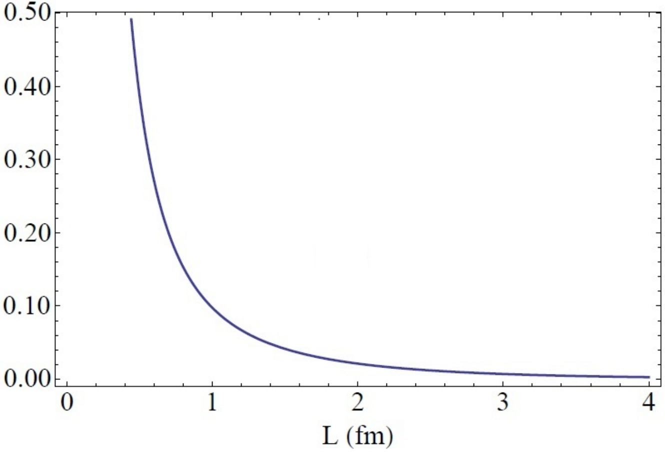
$$\nabla_z = \frac{\partial}{\partial L}$$

Substituting all this into the amplitude we get:

$$\langle A^{(1)}_{3z}\rangle \left\langle -\frac{1}{2}_{(is)}\ \frac{1}{2}_{(is)}\right| \left(\frac{g_A}{2f_\pi}\right)^2 \frac{\tau_{2+}\tau_{1-}}{(2\pi)^2}\ \sigma_{23}\sigma_{13}\frac{\partial^2}{\partial L^2}\, K_0(m_\pi L) \left|\frac{1}{2}_{(is)} - \frac{1}{2}_{(is)}\right\rangle$$

$$= \langle A^{(1)}_{3z}\rangle \left\langle -\frac{1}{2}_{(is)}\ \frac{1}{2}_{(is)}\right| \left(\frac{g_A}{2f_\pi}\right)^2 \frac{\tau_{2+}\tau_{1-}}{(2\pi)^2} \cdot \frac{1}{2} m_\pi^2 \left(K_0(m_\pi L) + K_0(m_\pi L)\right) \left|\frac{1}{2}_{(is)} - \frac{1}{2}_{(is)}\right\rangle$$

Looking at a plot of this function,



[Hall, Thomas and Young in progress.]

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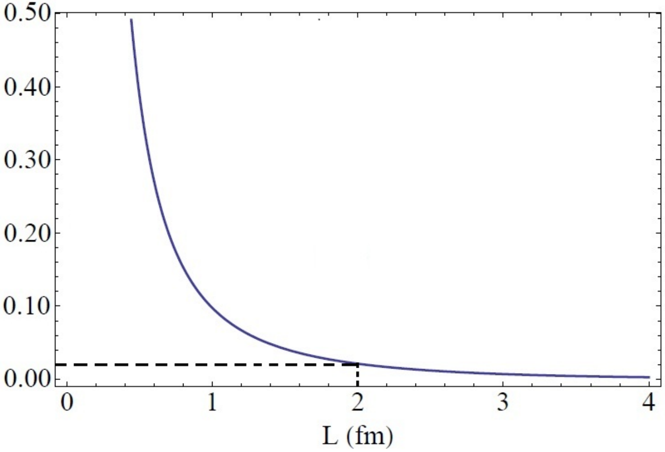
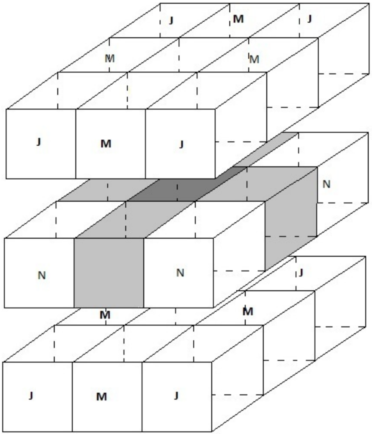


[Hall, Thomas and Young in progress.]

For the spin-flip correction 4 (closest) neighbours in total
$\implies$ 4%

For the isospin-flip correction only 2 (closest) neighbours
$\implies$ 4%

In total, the closest neighbours give 8%.



Conclusion

- Although the hedgehog showed that pion corrections occurred under certain circumstances it was difficult to reconcile this with the situation on the lattice.
- However the pion-mediated tensor-force between nucleons provided significant corrections to the axial charge which were large enough to account for the difference between the two values.
- This interaction may also be involved in hardening the axial form factor.