Lattice Hadron Physics

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Motivation for Investigation of Hadron Structure

• We know the nucleon is not a point-like particle but in fact is composed of quarks and gluons.

• But how are these constituents distributed inside the nucleon?
  
  • E.g. The neutron has zero net charge, but does it have a +/- core?

• How do they combine to produce its experimentally observed properties?

• For example
  
  • “Spin crisis”: quarks carry on ~30% of the proton’s spin
  • gluons? orbital angular momentum?

• Understanding how the nucleon is built from its quark and gluon constituents remains one the most important and challenging questions in modern nuclear physics.
Elastic Scattering

- Elastic scattering cross-section in the lab frame

\[
\frac{d\sigma}{d\Omega} = \sigma_{\text{Mott}} \left[ \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2\tau G_M^2(Q^2) \tan^2 \frac{\theta}{2} \right]
\]

- where

\[
G_E(Q^2) = F_1(Q^2) - \tau F_2(Q^2)
\]

\[
G_M(Q^2) = F_1(Q^2) + F_2(Q^2)
\]

\[
\tau = \frac{Q^2}{4M^2}
\]

- are the Sachs electric and magnetic form factors

- Rewriting in terms of the virtual photon’s longitudinal polarisation

\[
\frac{d\sigma}{d\Omega} = \sigma_{\text{Mott}} \left[ \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} \right]
\]

\[
\epsilon^{-1} = 1 + (1 + \tau)2 \tan \frac{\theta}{2}
\]

- Need cross sections at fixed $Q^2$ but different scattering angle: Rosenbluth separation
• Precise results now available up to 8-9 GeV$^2$

• Does $G_E^p$ change sign?

• What is the origin of the linear fall-off?
FIG. 17. (color online) Comparison of selected theoretical predictions to data for all four nucleon FFs at space-like $Q^2$. The theory curves are [15] (Diehl05), [18] (Eichmann11), [72] (Lomon06), [91] (Gross08) and [94] (Santopinto10). Data are from [5, 80, 81, 100–105] (cross section data, empty circles) and [1, 2, 25, 49–53, 106] (polarization data, filled circles), where the results of [2] have been replaced by the results of the present work (Table IV). Data are from [5, 80, 81, 100–102, 104, 105, 107–109]. Data are from [20, 110–121]. Data are from [21, 122–132].

$D_D = (1 + Q^2/Λ^2)^{-2}$, with $Λ^2 = 0.71 \text{ GeV}^2$, is the standard dipole form factor.

World nucleon form factor data compared to theory

Figure 17 summarizes the theoretical interpretation of the nucleon electromagnetic form factors, with representative examples from each of the classes of models discussed compared to the world data for all four nucleon electromagnetic form factors. Published results for $R = Gp_E/Gp_M$ were converted to $Gp_E$ values using the global fit of $Gp_E$ and $Gp_M$ from [43], updated to use the $R$ values of the present work, a change that does not noticeably affect $Gp_M$. Except at very low $Q^2$, the contribution of the uncertainty in $Gp_M$ to the resulting uncertainty in $Gp_E$ is negligible. At this juncture, it is worth recalling that the $Gp_E$ results extracted from cross section data are believed to be unreliable at high $Q^2$ due to incompletely understood TPEX corrections, which have not been applied to the data shown in Figures 14-17. Except for the DSE calculation of [18], all of the models shown describe existing data very well, which is to be expected given that the parameters of the models are fitted to reproduce the data. However, their predictions tend to diverge when extrapolated outside the $Q^2$ range of the data. That the DSE-based calculation of [18] fails to describe the data as well as the other calculations is not surprising, since it represents a more fundamental ab initio approach with virtually no adjustable parameters, but requires approximations that are not yet well-controlled. Significant progress in the quality of the predictions is nonetheless evident, as the data expose the weaknesses of different approximation schemes. Since the hard scattering mechanism leading to the asymptotic pQCD scaling relations is not expected to dominate the form factor behavior at presently accessible $Q^2$ values, phenomenological models and the ambitious ongoing efforts in lattice QCD and DSE calculations are of paramount importance to understanding the internal structure and dynamics of the nu-
Size of the Proton

- $>5\sigma$ discrepancy between muonic hydrogen and $e-p$ scattering
  
  - $r_p=0.84184(67)$ fm  
    [Nature 466, 213 (2010)]
  
  - $r_p=0.875(8)(6)$ fm  
    [arXiv:1102.0318]

\[
\langle r^2 \rangle = -6 \frac{dG_E(Q^2)}{dQ^2} \bigg|_{Q^2=0}
\]
**Transverse Spatial Distributions**

- Model independent relation between form factors and transverse spatial distributions occurs in the \textit{infinite momentum frame}.

- Quark (charge) distribution in the transverse plane:
  \[
  q(b_\perp^2) = \int d^2q_\perp e^{-i \vec{b}_\perp \cdot \vec{q}_\perp} F_1(q^2)
  \]

\textit{Distance of (active) quark to the centre of momentum in a fast moving nucleon}

\textit{Provide information on the size and internal charge densities}
Electromagnetic Form Factors

- Can some of these questions be answered by a calculation from QCD?
- Form factors are nonperturbative quantities

\[ \langle p', s' | J^\mu(q) | p, s \rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2m} F_2(q^2) \right] u(p, s) \]
Calculating Matrix Elements

\[ \langle H' | \mathcal{O} | H \rangle \]

\( H, H' : \pi, K p, n, \ldots \)
\( \mathcal{O} : V_\mu, A_\mu, \ldots \)
Lattice 3pt Functions

• Recall that we want to compute a three-point function

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}}(t-\tau)} e^{-E_{\vec{p}'}\Gamma_{\beta\alpha}} \langle \Omega | \chi_{\alpha}(0) | N(p', s') \rangle \langle N(p', s') | \mathcal{O}(\bar{q}) | N(p, s) \rangle \langle N(p, s) | \bar{\chi}_{\beta}(0) | \Omega \rangle \]

• Use the following interpolating operator to create a proton

\[ \chi_{\alpha}(x) = \epsilon^{abc} \left( u^{T}a(x) \ C \gamma_{5} \ d^{b}(x) \right) u^{c}_{\alpha}(x) \]

• And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \)

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

• Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^{T}a(x_{2}) \ C \gamma_{5} \ d^{b}(x_{2}) \right) u^{c}_{\alpha}(x_{2}) \bar{u}(x_{1}) \mathcal{O} u(x_{1}) \bar{u}^{c'}(0) \left( \bar{d}^{b'}(0) C \gamma_{5} \bar{u}^{T}a'(0) \right) \]

• Similar for d-quark, but only 2 terms
Lattice 3pt Functions

- Recall that we want to compute a three-point function

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}}\tau} \Gamma_{\beta \alpha} \langle \Omega | \chi_{\alpha}(0) | N(p', s') \rangle \langle N(p', s') | \mathcal{O}(\bar{q}) | N(p, s) \rangle \langle N(p, s) | \bar{\chi}_\beta(0) | \Omega \rangle \]

- Use the following interpolating operator to create a proton

\[ \chi_{\alpha}(x) = \epsilon^{abc} \left( u^T_{Ta}(x) \, C \gamma_5 \, d^b(x) \right) u^c(x) \]

- And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \)

\[ \mathcal{O} : \text{Combination of } \gamma \text{ matrices and derivatives} \]

- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^T_{Ta}(x_2) \, C \gamma_5 \, d^b(x_2) \right) u^c_\alpha(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^{T{a'}}(0) \right) \]

- Similar for d-quark, but only 2 terms
Recall that we want to compute a three-point function $G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}, (t-\tau)}} e^{-E_{\vec{p}'}} \Gamma_{\beta\alpha} \langle \Omega \mid \chi_{\alpha}(0) \mid N(p', s') \rangle \langle N(p', s') \mid \mathcal{O}(\bar{q}) \mid N(p, s) \rangle \langle N(p, s) \mid \bar{\chi}_\beta(0) \mid \Omega \rangle$

- Use the following interpolating operator to create a proton
  \[
  \chi_{\alpha}(x) = \epsilon^{abc} \left( u^T a(x) \ C \gamma_5 \ d^b(x) \right) u^c(x)
  \]

- And insert the local operator (quark bi-linear) $\bar{q}(x) \mathcal{O} q(x)$

- Perform all possible (connected) Wick contractions

- Similar for d-quark, but only 2 terms

Lattice 3pt Functions

\[
\mathcal{O}(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}, (t-\tau)}} e^{-E_{\vec{p}'}} \Gamma_{\beta\alpha} \langle \Omega \mid \chi_{\alpha}(0) \mid N(p', s') \rangle \langle N(p', s') \mid \mathcal{O}(\bar{q}) \mid N(p, s) \rangle \langle N(p, s) \mid \bar{\chi}_\beta(0) \mid \Omega \rangle
\]
Lattice 3pt Functions

- Recall that we want to compute a three-point function

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{p'}(t-\tau)} e^{-E_{p'} \Gamma_{\beta\alpha}} \langle \Omega | \chi_{\alpha}(0) | N(p', s') \rangle \langle N(p', s') | \mathcal{O}(\bar{q}) | N(p, s) \rangle \langle N(p, s) | \bar{\chi}_{\beta}(0) | \Omega \rangle \]

- Use the following interpolating operator to create a proton

\[ \chi_{\alpha}(x) = \epsilon^{abc} \left( u^{T_a}(x) C \gamma_5 d^b(x) \right) u^c_{\alpha}(x) \]

- And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \)

\[ \mathcal{O} \text{: Combination of } \gamma \text{ matrices and derivatives} \]

- Perform all possible (connected) Wick contractions

\[
\begin{align*}
\epsilon^{abc} \epsilon^{a'b'c'} \left( u^{T_a}(x_2) C \gamma_5 d^b(x_2) \right) u^c_{\alpha}(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^{T_{a'}}(0) \right)
\end{align*}
\]

- Similar for d-quark, but only 2 terms
Lattice 3pt Functions \textit{at the quark level}

\[
C_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i\vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle_{\{U\}}
\]
Lattice 3pt Functions at the quark level

\[ C_\Gamma(t, \tau; \vec{p'}, \vec{p}) = \sum_{\vec{x}_1} e^{i \vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_\Gamma(\vec{0}, 0; \vec{p'}, t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle_{\{U\}} \]

\[ \Sigma_\Gamma(\vec{0}, 0; \vec{x}_1; \vec{p'}, t) = \sum_{\vec{x}_2} S_\Gamma(\vec{x}_2, t; \vec{0}, 0; \vec{p'}) G(\vec{x}_2, t; \vec{x}_1) \]
Lattice 3pt Functions at the quark level

\[ C_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i\vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) O(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle \{U\} \]

\[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{x}_1; \vec{p}', t) = \sum_{\vec{x}_2} S_{\Gamma}(\vec{x}_2, t; \vec{0}, 0; \vec{p}') G(\vec{x}_2, t; \vec{x}_1) \]

\[ S_{\Gamma}^{u;a'a}(\vec{x}_2, t; \vec{0}, 0; \vec{p}') = e^{-i\vec{p}' \cdot \vec{x}_2} \epsilon^{abc} \epsilon^{a'b'c'} \times \]

\[ \left[ \tilde{G}^{d;bb'}(\vec{x}_2, t; \vec{0}, 0) G^{u;cc'}(\vec{x}_2, t; \vec{0}, 0) \Gamma + \text{Tr}_D [\tilde{G}^{d;bb'}(\vec{x}_2, t; \vec{0}, 0) G^{u;cc'}(\vec{x}_2, t; \vec{0}, 0)] \Gamma \right. \]

\[ + \Gamma G^{u;bb'}(\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d;cc'}(\vec{x}, t; \vec{0}, 0) + \text{Tr}_D [\Gamma G^{u;bb'}(\vec{x}_2, t; \vec{0}, 0)] \tilde{G}^{d;cc'}(\vec{x}_2, t; \vec{0}, 0) \]

\[ S_{\Gamma}^{d;a'a}(\vec{x}_2, t; \vec{0}, 0; \vec{p}') = e^{-i\vec{p}' \cdot \vec{x}_2} \epsilon^{abc} \epsilon^{a'b'c'} \times \]

\[ \left[ \tilde{G}^{u;bb'}(\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{r;cc'}(\vec{x}_2, t; \vec{0}, 0) + \text{Tr}_D [\Gamma G^{u;bb'}(\vec{x}_2, t; \vec{0}, 0)] \tilde{G}^{r;cc'}(\vec{x}_2, t; \vec{0}, 0) \right] \]

\[ \tilde{G} = C\gamma_5 G^T \gamma_5 C \]
Lattice 3pt Functions

- \( \Sigma_\Gamma(\vec{0}, 0; x_1; p', t) = \sum_{x_2} S_\Gamma(x_2, t; \vec{0}, 0; p')G(x_2, t; x_1) \) can be computed from the linear system of equations

\[
\sum_{v} M(v', v) \gamma_5 S^\dagger_\Gamma(\vec{0}, 0; v; p', t) = \gamma_5 S^\dagger_\Gamma(\vec{v}, t; \vec{0}, 0; p') \delta_{v_0, t}
\]

Fermion matrix

- so \( \Sigma_\Gamma(\vec{0}, 0; x_1; p', t) \) is a sequential propagator based on a source \( S_\Gamma(x_2, t; \vec{0}, 0; p') \) constructed from two ordinary propagators at time \( t \)

...
Sequential Source Technique

- First compute ordinary propagators $G(x, 0)$
Sequential Source Technique

- Construct sources

\[ S_{\Gamma}^{u; a' a} (\vec{x}_2, t; \vec{0}, 0; \vec{p}') \quad \text{or} \quad S_{\Gamma}^{d; a' a} (\vec{x}_2, t; \vec{0}, 0; \vec{p}') \]
Sequential Source Technique

- Compute sequential propagators

\[ \Sigma_\Gamma(\vec{0}, 0; \vec{x}_1; \vec{p}', t) = \sum_{\vec{x}_2} S_\Gamma(\vec{x}_2, t; \vec{0}, 0; \vec{p}') S(\vec{x}_2, t; \vec{x}_1) \]

- via the second inversion

\[ \sum_{\nu} M(\nu', \nu) \gamma_5 \Sigma^\dagger_\Gamma(\vec{0}, 0; \nu; \vec{p}', t) = \gamma_5 S^\dagger_\Gamma(\vec{v}, t; \vec{0}, 0; \vec{p}') \delta_{\nu', t} \]
Sequential Source Technique

- Insert operator

\[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) \]
Sequential Source Technique

- Tie everything together with an ordinary propagator

\[
C_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_1} e^{i\vec{q} \cdot \vec{x}_1} \left\langle \text{Tr} \left[ \Sigma_{\Gamma}(\vec{0}, 0; \vec{p}', t) \mathcal{O}(\vec{x}_1, \tau) G(\vec{x}_1, 0) \right] \right\rangle_{\{U\}}
\]

Diagram:

- \( \mathcal{O}(\tau, \vec{q}) \)
- \( \tilde{N}(0, \vec{p}) \)
- \( N(\tau, \vec{p}') \)
Sequential Source Technique

- **Advantages:** Free choice of
  - Momentum transfer
  - Operator (vector/axial/tensor)
  - Ideal for Form Factors, Structure Functions, GPDs

- **Disadvantages:** Separate 3-pt inversion for each
  - Quark flavour $\Sigma, \Delta, \pi, N \rightarrow \gamma \Delta$
  - Hadron eg. p,
  - Polarisation
  - Sink momentum
Sequential Source Technique

- Alternative method involves computing a sequential propagator “through the operator”
Sequential Source Technique

Advantages:

- Free choice of
  - Quark flavour
  - Hadron e.g. $p$, $\Sigma$, $\Delta$, $\pi$, $N \rightarrow \gamma\Delta$
  - Polarisation
  - Sink momentum

Ideal for studying flavour dependence in a hadron multiplet

Disadvantages:

- Separate 3-pt inversion for each
  - Momentum transfer
  - Operator (vector/axial/tensor)
Lattice 3pt Functions in Chroma

\[ S_{\Gamma}^{u;a'a}(\vec{x}_2, t; \vec{0}, 0; \vec{p}', \vec{p}'') = e^{-i\vec{p}' \cdot \vec{x}} \varepsilon^{abc} \epsilon^{a'b'c'} \times \]
\[ \left[ \tilde{G}^{d;bb'}(\vec{x}_2, t; \vec{0}, 0) G^{u;cc'}(\vec{x}_2, t; \vec{0}, 0) \Gamma + \text{Tr}_D[\tilde{G}^{d;bb'}(\vec{x}_2, t; \vec{0}, 0) G^{u;cc'}(\vec{x}_2, t; \vec{0}, 0)] \Gamma \right] \\
+ \Gamma G^{u;bb'}(\vec{x}_2, t; \vec{0}, 0) \tilde{G}^{d;cc'}(\vec{x}, t; \vec{0}, 0) + \text{Tr}_D[\Gamma G^{u;bb'}(\vec{x}_2, t; \vec{0}, 0)] \tilde{G}^{d;cc'}(\vec{x}_2, t; \vec{0}, 0) \]

/* "\bar u 0 u" insertion in NR proton, ie. */
/* "(u Cg5 d) u" */
/* Some generic T */

// Use precomputed Cg5
q1_tmp = quark_propagators[0] * Cg5;
q2_tmp = Cg5 * quark_propagators[1];
di_quark = quarkContract24(q1_tmp, q2_tmp);

// First term
src_prop_tmp = T * di_quark;

// Now the second term
src_prop_tmp += traceSpin(di_quark) * T;

// The third term...
qu1_tmp = q2_tmp * Cg5;
qu2_tmp = quark_propagators[0] * T;

src_prop_tmp -= quarkContract13(q1_tmp, q2_tmp) + transposeSpin(quarkContract12(q2_tmp, q1_tmp));

END_CODE();

return projectBaryon(src_prop_tmp, forward_headers);
Chroma xml for Sequential Source

```
<elem>
  <annotation>; NUCL_U_UNPOL seqsource</annotation>
  <Name>SEQSOURCE</Name>
  <Frequency>1</Frequency>
  <Param>
    <version>1</version>
    <seq_src>NUCL_U_UNPOL</seq_src>
    <t_sink>13</t_sink>
    <sink_mom>0 0 0</sink_mom>
  </Param>
  <PropSink>
    <version>5</version>
    <Sink>
      <version>2</version>
      <SinkType>SHELL_SINK</SinkType>
      <j_decay>3</j_decay>
      <SmearingParam>
        <wvf_kind>GAUGE_INV_GAUSSIAN</wvf_kind>
        <wvf_param>2.0</wvf_param>
        <wfIntPar>5</wfIntPar>
        <no_smear_dir>3</no_smear_dir>
      </SmearingParam>
    </Sink>
  </PropSink>
  <NamedObject>
    <gauge_id>gauge</gauge_id>
    <prop_ids>
      <elem>sh_prop_1</elem>
      <elem>sh_prop_1</elem>
    </prop_ids>
    <seqsource_id>seqsource_NUCL_U_UNPOL</seqsource_id>
  </NamedObject>
</elem>
```

- **u-quark in proton, unpolarised**
- **sink timeslice**
- **sink momentum**
- **sink smearing**
- **ordinary quark props**
  - required for construction of seq source (2 for u, 1 for d)
  - tag (to be used as source for prop calculation)
Extracting matrix elements

- Recall hadronic form of the nucleon 3pt function

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}', (t-\tau)}} e^{-E_{\vec{p}} \tau} \Gamma_{\beta \alpha} \langle \Omega | \chi_\alpha(0) | N(p', s') \rangle \langle N(p', s') | \mathcal{O}(\vec{q}) | N(p, s) \rangle \langle N(p, s) | \bar{\chi}_\beta(0) | \Omega \rangle \]

- Need to remove time dependence and wave function amplitudes

Form a ratios with the nucleon 2pt function

\[ G_2(t, \vec{p}) = \sum_s e^{-E_{\vec{p}} t} \Gamma_{\beta \alpha} \langle \Omega | \chi_\alpha | N(p, s) \rangle \langle N(p, s) | \bar{\chi}_\beta | \Omega \rangle \]

- E.g.

\[ R(t, \tau; \vec{p}', \vec{p}; \mathcal{O}) = \frac{G_1(t, \tau; \vec{p}', \vec{p}; \mathcal{O})}{G_2(t, \vec{p}')} \left[ \frac{G_2(t, \vec{p}') G_2(t, \vec{p}) G_2(t - \tau, \vec{p})}{G_2(t, \vec{p}) G_2(t, \vec{p}) G_2(t - \tau, \vec{p}')} \right]^{1/2} \]
Extracting Matrix Elements

\[ \langle \Omega | \chi | N(p, s) \rangle = \sqrt{\frac{Z(p)}{2E_p}} u(p, s) \] + ν-spinor terms with opposite parity

- We can write the two point function in terms of nucleon spinors as

\[ G_2(t, \vec{p}) = \sum_s \frac{\sqrt{Z^{\text{snk}}(\vec{p})} \sqrt{Z^{\text{src}}(\vec{p})}}{2E_{\vec{p}}} \text{Tr} \Gamma u(\vec{p}, s) \bar{u}(\vec{p}, s) [e^{-E_{\vec{p}} t} + e^{-E'_{\vec{p}} (T-t)}] \]

- Using the relation for spinors (in Euclidean space)

\[ u(\vec{p}, \sigma) \bar{u}(\vec{p}, \sigma') = \frac{(-i \vec{p} + m)}{2} = \frac{(E\gamma_4 - i\vec{p} \cdot \vec{\gamma} + m)}{2} \]

- Use \( \Gamma_4 = \frac{1}{2} (1 + \gamma_4) \) to maximise overlap with positive parity forward propagating state

\[ G_2(t, \vec{p}) = \sqrt{Z^{\text{snk}}(\vec{p}) Z^{\text{src}}(\vec{p})} \left[ \left( \frac{E_{\vec{p}} + m}{E_{\vec{p}}} \right) e^{-E_{\vec{p}} t} + \left( \frac{E'_{\vec{p}} + m'}{E'_{\vec{p}}} \right) e^{-E'_{\vec{p}} (T-t)} \right] \]
Extracting Matrix Elements

• Similarly for the three-point function, if we express the nucleon matrix element under study as

\[ \langle N(p', s')|O(\vec{q})|N(p, s)\rangle = \bar{u}(p', s')\mathcal{J}u(p, s) \]

• E.g., for the EM current \( O = J^\mu \)

\[ \mathcal{J} = \gamma^\mu F_1(Q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2M} F_2(Q^2) \]

• Then we have

\[ G_3(t, \tau; \vec{p}'\vec{p}; \Gamma, O) = \sqrt{Z^{\text{snk}}(\vec{p}')Z^{\text{src}}(\vec{p})} F(\Gamma, \mathcal{J}) e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}\tau}} \]

• where

\[ F(\Gamma, \mathcal{J}) = \frac{1}{4} \text{Tr} \left\{ \Gamma \left( \gamma_4 - i\frac{\vec{p}' \cdot \vec{\gamma}}{E_{\vec{p}'}} + \frac{m}{E_{\vec{p}'}} \right) \mathcal{J} \left( \gamma_4 - i\frac{\vec{p} \cdot \vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}} \right) \right\} \]
Example

• If we consider the particular case

\[ \Gamma = \Gamma_{\text{unpol}} \equiv \frac{1}{2} (1 + \gamma_4), \quad \mathcal{O} = J^\mu, \quad \vec{p}' = \vec{p} \Rightarrow q = 0 \]

• then the contribution from \( F_2 \) to the matrix element drops out (proportional to \( q \))

\[
\langle N(p', s')| J^\mu(0) | N(p, s) \rangle = \bar{u}(p', s') \gamma^\mu u(p, s) F_1(Q^2 = 0) + \bar{u}(p', s') \frac{i \sigma^{\mu\nu} q_\nu}{2M} u(p, s) F_2(Q^2 = 0)
\]

• Euclideanisation

\[ \gamma^M_0 = \gamma^E_4, \quad \gamma^M_i = -i \gamma^E_i \quad \quad p^E_4 = i p^M_0 \equiv i E(\vec{p}), \quad p^E_i = -p^M_i \]

\[
\langle N(p', s')| \bar{q} \gamma^\mu_\mu q | N(p, s) \rangle = \bar{u}(p', s') \gamma^E_\mu u(p, s) F_1(Q^2 = 0) + \bar{u}(p', s') \frac{\sigma^\mu_\nu q^E_\nu}{2M} u(p, s) F_2(Q^2 = 0)
\]

Using the local vector current \( J^\mu = \bar{q} \gamma^\mu q \)
Then the three-point function is now

\[ G_3(t, \tau; \vec{p}', \vec{p}; \Gamma, \mathcal{O}) = \sqrt{Z^{\text{snk}}(\vec{p}')Z^{\text{src}}(\vec{p})} F(\Gamma, \mathcal{J}) e^{-E_{\vec{p}}'(t-\tau)} e^{-E_{\vec{p}}\tau} \]

with

\[ \mathcal{J} = \gamma^\mu F_1(Q^2) \]

and

\[ F(\Gamma_{\text{unpol}}, \gamma_4) = \frac{1}{2E_{\vec{p}}E_{\vec{p}'}} [(E_{\vec{p}} + m)(E_{\vec{p}'} + m) + \vec{p}' \cdot \vec{p}] = 2 \]

\[ F(\Gamma_{\text{unpol}}, \gamma_i) = \frac{-i}{2E_{\vec{p}}E_{\vec{p}'}} [(E_{\vec{p}} + m)\vec{p}' + (E_{\vec{p}'} + m)\vec{p}] = 0 \]
Example

- So our ratio determines

\[
R(t, \tau; \vec{p}', \vec{p}; \mathcal{O}) = \frac{G_1(t, \tau; \vec{p}', \vec{p}, \mathcal{O})}{G_2(t, \vec{p}')} \left[ \frac{G_2(\tau, \vec{p}')G_2(t, \vec{p}')G_2(t - \tau, \vec{p})}{G_2(\tau, \vec{p})G_2(t, \vec{p})G_2(t - \tau, \vec{p}')} \right]^{\frac{1}{2}}
\]

\[
= \sqrt{\frac{E_{\vec{p}'}E_{\vec{p}}}{(E_{\vec{p}} + m)(E_{\vec{p}} + m)}} F(\Gamma, \mathcal{J}_\mathcal{O}(\vec{q})) \quad 0 \ll \tau \ll t \ll \frac{1}{2} T
\]

\[
= F_1(q^2 = 0) \quad \Gamma_{\text{unpol}} = \frac{1}{2} (1 + \gamma_4), \quad \mathcal{O} = V_4 \equiv \gamma_4, \quad \vec{p}' = \vec{p} = 0
\]
Other Useful Combinations

\[ R(t, \tau; \vec{0}, \vec{p}; V_4, \Gamma_4) = F_1(q^2) - \frac{E\vec{p} - M}{2M} F_2(q^2) = G_E(q^2) \]

\[ R(t, \tau; \vec{0}, \vec{p}; V_i, \Gamma_4) = -i \frac{q_i}{E + M} G_E(q^2) \]

\[ R(t, \tau; \vec{0}, \vec{p}; V_i, \Gamma_j) = -i \epsilon_{ijk} \frac{q_k}{E + M} G_M(q^2) \]

\[ \Gamma_j = \frac{1}{2} (1 + \gamma_4) i \gamma_5 \gamma_j \]

- Certain combinations of parameters and kinematics give access to the form factors.

- It is possible to have several choices giving access to the form factors at a fixed \( Q^2 \).

Exercise: Prove them!

Overdetermined set of simultaneous equations that can be solved for \( F_1, F_2 \) or \( G_E, G_M \)
Comparison With Experiment

$F_{1}^{u-d}(Q^{2})$

- Isovector Dirac form factor
- Darker colours ➞ lighter masses
- Grey band ➞ parameterisation of experimental data
- Lattice results lie above experiment with smaller slope
Comparison With Experiment

\[ F_{2}^{u-d}(Q^{2}) \]

- Isovector Pauli form factor
- Darker colours → lighter masses
- Grey band → parameterisation of experimental data
- Lattice results lie above experiment with smaller slope

\[ m_{\pi} \geq 0.8 \text{ GeV} \]

\[ 0.8 \text{ GeV} \geq m_{\pi} \geq 0.4 \text{ GeV} \]
• Isovector Dirac radius (squared)

• Isovector Pauli radius (squared)

• Isovector anomalous magnetic moment

• Dirac radius: different experimental values
Light Quark Mass Dependence

- Radii suppressed at large masses and small volumes
- Hint of sharp rise at small masses
- $r_2$ approaching experimental result
- $\kappa_{u-d}$ shows clear curvature at small masses
- Can the remaining discrepancy be due to the (still) unphysically large quark masses?
- Contact with ChPT?
Light Quark Mass Dependence

J.Hall, et al. [arXiv:1201.6114]

FIG. 8. (color online). The renormalization flow of $c_0$ for $\mu v N$ obtained using a dipole regulator on QCDSF lattice QCD results. Only the lightest seven lattice results are used. For each curve, two arbitrary values of $\Lambda$ are chosen to indicate the general size of the error bars.

FIG. 9. (color online). $\chi^2$ dof for the renormalization flow of $c_0$ for $\mu v N$ obtained using a dipole regulator on QCDSF lattice QCD results, up to and including $m_\pi^2$, max = 0.44 GeV$^2$ only. Statistical uncertainty, and an outer error bar, which also includes the systematic uncertainty due to the regulator in quadrature.

In all extrapolations, the strange quark loops have been unquenched, and the effects of kaons loops that would occur in an $SU(3)$ lattice calculation are estimated. The result is a change of only $\approx 0.7\%$ larger at the physical point when kaons loops are included.

The finite-volume extrapolations of Fig. 10 are generally useful for estimating the result of a lattice QCD calculation at certain box sizes. This can also provide a benchmark for estimating the outcome of a lattice QCD simulation at larger and untested box sizes.

FIG. 10. (color online). Extrapolations of $\mu v N$ at different finite volumes and infinite volume. The curves are based on lattice QCD results from QCDSF, lattice sizes: $1.7 - 2.9$ fm. The experimental value is marked [21, 22]. In all finite-volume extrapolations, the provisional constraint $m_\pi L > 3$ is used.

FIG. 11. (color online). Extrapolations of $\mu v N$ at different finite volumes and infinite volume. The finite-volume lattice QCD results from Ref. [10] are plotted for comparison, with box sizes in the range $1.7 - 2.9$ fm. In all finite-volume extrapolations, the provisional constraint $m_\pi L > 3$ is used.

FIG. 12. (color online). Extrapolations of $\mu v N$ at different finite volumes and infinite volume. The lattice QCD results displayed have been corrected to infinite volume. Only the lightest seven points are used in the fit, corresponding to a value of $m_\pi^2$, max = 0.44 GeV$^2$. Light Quark Mass Dependence
Flavour Distribution

- Individual flavour contributions not accessible directly in experiment

- Must be derived from a combination of proton and neutron form factors

  - (assuming charge symmetry $u^p = d^n$)

\[
F^p = \frac{2}{3} F^p_u - \frac{1}{3} F^p_d \\
F^n = -\frac{1}{3} F^p_u + \frac{2}{3} F^p_d
\]

- On the lattice we compute the individual quark contributions directly
Flavour Distribution

- $d$-quark contribution to $F_1(Q^2)$ falls off faster than the $u$-quark contribution
- Effect is enhanced at lighter quark masses
In terms of charge radii, the d-quark in the proton has a larger charge radius than the u-quark.
Implications for Transverse Densities

Recall: \[ q(b_{\perp}^2) = \int d^2q_\perp e^{-i\vec{b}_\perp \cdot q_\perp} F_1(q^2) \]

\[ r_{1,2}^d > r_{1,2}^u \]

Ph. Hägler (QCDSF) [PRL 98, 222001 (2007)]
Deformed Spin Densities
Nucleon

Ph. Hägler (QCDSF) [PRL 98, 222001 (2007)]

$\mathbf{S}_x \rightarrow S_x$

$r_{1,2}^d > r_{1,2}^u$

$f_{1T}^\perp(x, k_{\perp}^2)$

$h_{1}^\perp(x, k_{\perp}^2)$
Sivers Effect

Expect sizeable effect with opposite sign for up and down quarks (Sivers effect)
Pion Form Factor

$$\langle \pi(p')|J^\mu(\vec{q})|\pi(p)\rangle = P^\mu F_\pi(q^2)$$

$$q^2 = -Q^2 = (p' - p)^2$$

$$P^\mu = p'^\mu + p^\mu$$
• Asymptotic normalisation known from $\pi \rightarrow \mu + \nu$ decay

$$F_\pi(Q^2 \rightarrow \infty) = \frac{16\pi\alpha_s(Q^2)f_\pi^2}{Q^2}$$

• Allows to study the transition from the soft to hard regimes

• Low $Q^2$: measured directly by scattering high energy pions from atomic electrons [CERN]

• High $Q^2$: quasi-elastic scattering off virtual pions [DESY & JLab]
Pion Form Factor

\[ Q^2 = -q^2 \]

**JLQCD**: arXiv:0810.2590 [hep-lat]

\[ m_{ud} = 0.050 \]

- fit: \( \rho \) pole + cubic
- VMD

\[ F_\pi(Q^2) \]

\[ Q^2 \text{ [GeV}^2\text{]} \]
Pion Form Factor

**JLQCD**: arXiv:0810.2590 [hep-lat]

$m_{ud} = 0.050$

- fit: $\rho$ pole + cubic
- VMD

Minimum lattice momentum:

$$\vec{p} = \frac{2\pi}{L} \vec{n}$$

**QCDSF, hep-lat/0608021**

affects determination of:

$$\langle r^2_\pi \rangle$$
Discretised Momentum

• On a periodic lattice with spatial volume $L^3$, quark fields satisfy

$$\psi(x + \vec{e}_i L) = \psi(x), \quad i = 1, 2, 3$$

$$\int d^4 p \ e^{-i p(x + \vec{e}_i L)} \tilde{\psi}(p) = \int d^4 p \ e^{-i p x} \tilde{\psi}(p), \quad i = 1, 2, 3$$

• so we see that momenta are discretised in units of $p_i = \frac{2\pi}{L} n_i, \quad i = 1, 2, 3$

• For typical lattices, smallest non-zero momentum $\sim 400$-$500$ MeV

• Poor momentum resolution

• Can affect phenomenological observables e.g. form factors
Accessing small momenta:
(partially) twisted boundary conditions

• On a periodic lattice with spatial volume $L^3$, quark fields satisfy

$$
\psi(x + \vec{e}_i L) = \psi(x), \quad i = 1, 2, 3
$$

$$
\int d^4p \ e^{-ip(x+\vec{e}_i L)} \tilde{\psi}(p) = \int d^4p \ e^{-ipx} \tilde{\psi}(p), \quad i = 1, 2, 3
$$

• so we see that momenta are discretised in units of

$$
p_i = \frac{2\pi}{L} n_i, \quad i = 1, 2, 3
$$

• Modify boundary conditions on the valence quarks

$$
\psi(x + \vec{e}_i L) = e^{i\theta_i} \psi(x), \quad i = 1, 2, 3
$$

• allows to tune the momenta continuously

$$
p_i = \frac{2\pi}{L} n_i + \frac{\theta_i}{L}, \quad i = 1, 2, 3
$$

• For a meson with quark flavours (1,2)

$$
\vec{p} = \frac{2\pi}{L} \vec{n} + \left( \frac{\theta_1 - \theta_2}{L} \right)
$$
Implementation

- Make a unitary Abelian transformation on the fields
  \[
  \psi(x) \longrightarrow U(\theta, x)\tilde{\psi}(x) = e^{i\theta \cdot \bar{x}} \tilde{\psi}(x)
  \]
- Phase factor cancels in all terms of the lattice fermion action except the spatial hopping term
  \[
  \tilde{\psi}(x) \left[ e^{i \frac{a \theta_i}{L}} U_i(x) (1 - \gamma_i)\tilde{\psi}(x + \hat{i}) + e^{-i \frac{a \theta_i}{L}} U_i^\dagger(x - \hat{i}) (1 + \gamma_i)\tilde{\psi}(x - \hat{i}) \right]
  \]
- In practice, compute quark propagator with gauge links
  \[
  \{U_i(x)\} \longrightarrow \{e^{i \frac{a \theta_i}{L}} U_i(x)\}
  \]
- Twisted boundary conditions for sea quarks requires generating new set of gauge fields for each twist
  - only twist valence quarks \(\rightsquigarrow\) partially twisted boundary conditions
- Introduces an additional finite size effect that is, however, exponentially suppressed
Additional Finite Volume Effects

\( E_{\pi, \rho}^2 = m_{\pi, \rho}^2 + \left( \vec{p}_{\text{lat}} - \frac{\theta_1 - \theta_2}{L} \right)^2 \)

\( m_{\pi}/m_{\rho} = 0.70 \)

Figure 1: The plots in the first line illustrate the results for the second line we show the corresponding relative error as a function of the momentum.

\( aE_{\pi}/\rho \)

\( m_{\pi}/m_{\rho} = 0.70 \)

Figure 2: Magnified view of the dispersion relation of fig. 1 in [hep-lat/0506016].

\( \Delta \pi \)

\( m_{\pi}/m_{\rho} = 0.70 \)

Table 2: \( \Delta \pi \) as a parameter of the fit (third row).

\( \chi^2/\text{d.o.f.} \) for the lattice data with respect the expectations equal to zero (empty and full symbols respectively) for the two choices of \( m_\pi \) (first two rows) and the results obtained from a fit to (20).

\( P_L \)

\( \kappa \)

\( \theta \)

\( Z_P \)
Pion charge radius

[RBC/UKQCD, arXiv:0804.3971]

\[m_\pi \approx 330\text{MeV}\]

\[24^3 \times 64\]

\[\alpha \approx 0.114\text{ fm}\]

<table>
<thead>
<tr>
<th>(\text{maximum } Q^2)</th>
<th>linear</th>
<th>quadratic</th>
<th>cubic</th>
<th>pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.013 GeV(^2)</td>
<td>0.354(28)(11)</td>
<td>–</td>
<td>–</td>
<td>0.361(29)(12)</td>
</tr>
<tr>
<td>0.022 GeV(^2)</td>
<td>0.354(26)(11)</td>
<td>0.353(35)(11)</td>
<td>–</td>
<td>0.364(27)(12)</td>
</tr>
<tr>
<td>0.035 GeV(^2)</td>
<td>0.353(25)(11)</td>
<td>0.355(32)(11)</td>
<td>0.351(41)(11)</td>
<td>0.366(27)(12)</td>
</tr>
<tr>
<td>0.150 GeV(^2)</td>
<td>0.332(28)(11)</td>
<td>0.387(44)(13)</td>
<td>0.406(56)(13)</td>
<td>0.382(37)(12)</td>
</tr>
</tbody>
</table>
Pion form factor
(compared to experiment)
Pion Form Factor

- Many choices of twist angles giving access to extremely small $Q^2$
- Radii results increasing towards the experimental point at smaller quark masses
Other Form Factors

• What about transition form factors? Excited states?

• Can we use the variational method described yesterday to improve our extraction of form factors?

• Recall:

\[ [(G(t_0))^{-1} G(t_0 + \Delta t)]_{ij} u_j^\alpha = c^\alpha u_i^\alpha \]
\[ v_i^\alpha [G(t_0 + \Delta t) (G(t_0))^{-1}]_{ij} = c^\alpha v_j^\alpha \]

\[ v_i^\alpha G_{ij}(t) u_j^\beta \propto \delta^{\alpha \beta} \]

• Apply to a 3pt function to isolate matrix elements of single states
Other Form Factors

Figure 2: Comparison of the standard, single correlation function approach (red-square symbols) with our improved variational approach (blue-circle symbols). Top left is the pion $G_C$, top right is the rho meson $G_C$, bottom left is the rho meson $G_M$ and bottom right is the rho meson $G_Q$. In all cases we find an early onset of ground state dominance and in the case of the more difficult magnetic and quadrupole form factors, the quality of the plateau has dramatically improved. Contrast to this, our variational approach provides a rapid onset of the plateau. A distinct plateau is observed within 2-3 time slices after the point-split current insertion centred at $t_s=21$. This is clear evidence that through the use of optimised operators obtained from a variational procedure one is able to systematically reduce the impact of excited state effects. Similar behaviour is observed for $G_Q$.

5. Excited State Form Factors

The most intriguing aspect of using a variational approach is that it opens up a new realm of interesting physics. As the excited states exhibit the same quantum numbers as their ground-state counterparts, the same expressions can be considered using the relevant projected correlation functions $G_{\alpha}(p, p')$ in the analysis. In Fig. 3 we present the worlds first results for the form factors of an excited state hadron, in this case the first rho meson excitation. We note that similar results are obtained for the first excitation of the pion. As outlined in reference \[6\] one can infer rms-charge radii and magnetic moments for these states from $G_C(Q^2)$ and $G_M(Q^2)$ respectively. In Table 1 this approach adopts a monopole ansatz for the $Q^2$ dependence of the charge form factor. While the multi-particle Fock-space component of the excited state certainly give rise to a more interesting $Q^2$ dependence, the ansatz does provide some insight into the nature of this excited state.\[6\]
Figure 3: The charge and magnetic form factors for the first excitation of the rho meson (green-circle symbols). The shaded region is the best fit to the data in the region where there is a clear signal. The blue-square symbols illustrate the ground-state result for comparison.

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References

Other Form Factors

\[ \Lambda^* \quad (\Lambda(1405)) \]

[B. Menadue: PoS (Latt’12) 178]
Neutron beta decay

\[ n \rightarrow pe^- \bar{\nu}_e \]
Neutron beta decay

- Free neutrons are unstable $\tau \approx 15$ mins.

- The most common way to study the weak interaction.

- The decay rate is proportional the matrix element of the weak V-A current

$$\langle p(p', s')|(V_\mu - A_\mu)|n(p, s)\rangle = \bar{u}_p(p', s')\{\gamma_\mu f_1(q^2) + i\frac{\sigma_{\mu\nu}q^\nu}{2M} f_2(q^2) + \frac{q_\mu}{2M} f_3(q^2)$$

$$- [\gamma_\mu \gamma_5 g_1(q^2) + i\frac{\sigma_{\mu\nu}q^\nu}{2M} \gamma_5 g_2(q^2) + \frac{q_\mu}{2M} \gamma_5 g_3(q^2)] \} u_n(p, s)$$

- Here the momentum transfer is so small that we only need to consider the $f_1$ and $g_1$ terms

$$M_n - M_p \simeq 1.3 \text{ MeV}$$

- By convention, we call $g_V = f_1(0)$ $g_A = g_1(0)$

- with $g_V = 1$ according to the conserved vector current (CVC) hypothesis

- Adler-Weisberger relation predicts $g_A = 1.26$
**Neutron beta decay**

- The decay rate for
  - a neutron at rest and with spin in the $\vec{s}_n$ direction
  - final $e^-$ and $\bar{\nu}_e$ with velocities $\vec{v}_e$, $\vec{v}_\bar{\nu}$

\[
\frac{dR}{dp_e d\Omega_e d\Omega_{\bar{\nu}}} = \frac{G_F^2 |V_{ud}|^2}{(2\pi)^5} \left[ \alpha + \beta \vec{v}_e \cdot \vec{v}_\bar{\nu} + \gamma \vec{s}_n \cdot \vec{v}_e + \delta \vec{s}_n \cdot \vec{v}_\bar{\nu} \right] p_e^2 (E_{\text{max}} - E_e)^2
\]

- with $E_{\text{max}} = M_n - M_p \simeq 1.3$ MeV
  
  \[
  \begin{align*}
  \alpha &= g_V^2 + 3g_A^2 \\
  \beta &= g_V^2 - g_A^2 \\
  \gamma &= 2(g_A g_V - g_A^2) \\
  \delta &= 2(g_A g_V + g_V^2)
  \end{align*}
  \]

- so even without neutron polarisation, we can determine $|g_A/g_V|$ through an accurate determination of the angular correlation between outgoing $e^-$ and $\bar{\nu}_e$

- To determine the sign of $g_A$ spin-dependent measurement

- Current best determination (PDG 2012) $g_A/g_V = 1.2701(25)$
Axial Charge, $g_A$

- The axial charge is defined as the value of the axial form factor at $q^2=0$
  \[ g_A = G_A(q^2 = 0) \]
- Ideal quantity for a benchmark lattice calculation of nucleon structure
  - Zero momentum
  - Statistically clean
  - Isovector
  - Disconnected contributions cancel

\[
\langle p|\bar{u}\gamma^\mu\gamma^5d|n\rangle = \langle p|\bar{u}\gamma^\mu\gamma^5u - \bar{d}\gamma^\mu\gamma^5d|p\rangle
\]
Determination of $g_A$ on the Lattice

• Need access to the matrix element
  \[ \langle n | \bar{u} \gamma^\mu \gamma^5 d | p \rangle = \bar{u}(p', s') \gamma^\mu \gamma^5 u(p, s) g_A \]

• from our three-point functions
  \[
  G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}}} \Gamma_{\beta\alpha} \langle \Omega | \chi_\alpha(0) | N(p', s') \rangle \langle N(p', s') | O(\vec{q}) | N(p, s) \rangle \langle N p, s | \bar{\chi}_\beta(0) | \Omega \rangle
  \]

• From yesterday, we know that after the spin-trace, our 3pt will be proportional to
  \[
  F(\Gamma, \mathcal{J}) = \frac{1}{4} \text{Tr} \left\{ \Gamma \left( \gamma_4 - i \frac{\vec{p}' \cdot \vec{\gamma}}{E_{\vec{p}'}} + \frac{m}{E_{\vec{p}'}} \right) \mathcal{J} \left( \gamma_4 - i \frac{\vec{p} \cdot \vec{\gamma}}{E_{\vec{p}}} + \frac{m}{E_{\vec{p}}} \right) \right\}
  \]

• For $\mathcal{J} = \gamma^\mu \gamma^5$ we have
  \[
  F(\Gamma_{\text{unpol}}, \gamma^4 \gamma^5) = 0 \quad F(\Gamma_{\text{unpol}}, \gamma^i \gamma^5) = 0
  \]

• For $\mathcal{J} = \gamma^i \gamma^5$ we have
  \[
  F(\Gamma_{\text{pol}}, \gamma^4 \gamma^5) = \frac{-1}{2E_{\vec{p}}E_{\vec{p}'}} \left[ (E_{\vec{p}} + m) \vec{p}' \cdot \vec{s} + (E_{\vec{p}'} + m) \vec{p} \cdot \vec{s} \right]
  \]

• When using the projector $\Gamma_{\text{pol}} = \frac{1}{2} (1 + \gamma_4) i \gamma_5 \vec{\gamma} \cdot \vec{s}$ when computing the 3pt function

• Requires nucleon state to be polarised in, e.g. +z direction
Determination of $g_A$ on the Lattice

- At zero momentum, $F(\Gamma_{pol}, \gamma_i\gamma_5) = 2is_i$

- So $g_A$ can be determined by choosing the direction of the axial current to be the same as the direction of the nucleon polarisation. E.g. use a 3pt function with

  $\Gamma_{pol} = \Gamma_3 = \frac{1}{2}(1 + \gamma_4)i\gamma_5\gamma_3$  \hspace{1em} $O = \gamma_3\gamma_5$

- Our ratio from yesterday

  $$R(t, \tau; \vec{p}', \vec{p}; O, \Gamma) = \frac{G_\Gamma(t, \tau; \vec{p}', \vec{p}, O)}{G_2(t, \vec{p}')} \left[ \frac{G_2(\tau, \vec{p}')G_2(t, \vec{p}')G_2(t - \tau, \vec{p})}{G_2(\tau, \vec{p})G_2(t, \vec{p})G_2(t - \tau, \vec{p}')} \right]^{\frac{1}{2}}$$

- now becomes

  $$R(t, \tau; \vec{0}, \vec{0}; \gamma_3\gamma_5, \Gamma_3) = \frac{G_{\Gamma_3}(t, \tau; \vec{0}, \vec{0}, \gamma_3\gamma_5)}{G_2(t, \vec{0})} = ig_A$$
Determination of $g_A$ on the Lattice

- Example of
  \[ R(t, \tau; \bar{0}, \bar{0}; \gamma_3 \gamma_5, \Gamma_3) = \frac{G_{\Gamma_3}(t, \tau; \bar{0}, \bar{0}, \gamma_3 \gamma_5)}{G_2(t, \bar{0})} = ig_A \]

- from [RBC/UKQCD:0801.4016] at 4 different pion masses
Determination of $g_A$ on the Lattice

Results appear to undershoot by ~10%
Determination of $g_A$ on the Lattice

- What about lattice systematic errors?
  - Finite lattice spacing
  - Large quark masses
  - Finite volume
  - Contamination from excited states
Determination of $g_A$ on the Lattice

Lattice spacing dependence

- Different colours correspond to different fermion actions at multiple lattice spacings

No obvious dependence on $a$
Determination of $g_A$ on the Lattice

Lattice volume dependence

QCDSF: 1101.2326

RBC/UKQCD: 0801.4016

• Substantial finite size effects

• $g_A$ suppressed on a finite volume

• See [CSSM: 1205.1608] for attempts to understand the source of this behaviour
Determination of $g_A$ on the Lattice

Excited state contamination

- Test for contamination from excited states by varying the location of the sink
- Evidence that excited state contamination suppresses $g_A$
• Can we apply variational method to remove excited state contamination?
Determination of $g_A$ on the Lattice

Quark mass dependence

- HBChPT form suggests that an enhancement is expected in the infinite volume at light quark masses
Determination of $g_A$ on the Lattice

$g_A = m_{\pi} = \{5.2, 5.25, 5.29, 5.4\}$

$m_{\pi}^2 [GeV^2]$

$g_A$

$0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2$

$0.8 \quad 0.9 \quad 1.0 \quad 1.1 \quad 1.2 \quad 1.3$

$m_{\pi} L$

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$

$g_A$

$0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1.0 \quad 1.1 \quad 1.2 \quad 1.3$

$QCDSF: 1302.2233$
• First moment of the (isovector) nucleon parton distribution function

\[ \langle x \rangle_{u-d}^{\mu} = \int_0^1 dx x(u(x, \mu) - d(x, \mu)) + \int_0^1 dx x(\bar{u}(x, \mu) - \bar{d}(x, \mu)) \]

• Notorious for producing lattice results \( \approx 2x \) too large for isovector nucleon

• What are the possible systematic errors that could account for this

• Quenching? Chiral physics? Finite volume effects?
QCDSF ($N_f=2$)
Excited State Contamination?

QCDSF

\[ a \approx 0.75 \text{ fm}, \quad m_\pi \approx 650 \text{ MeV} \]

Evidence for severe excited state contamination!

Excellent candidate for the correlation matrix method approach!