Lattice Hadron Physics

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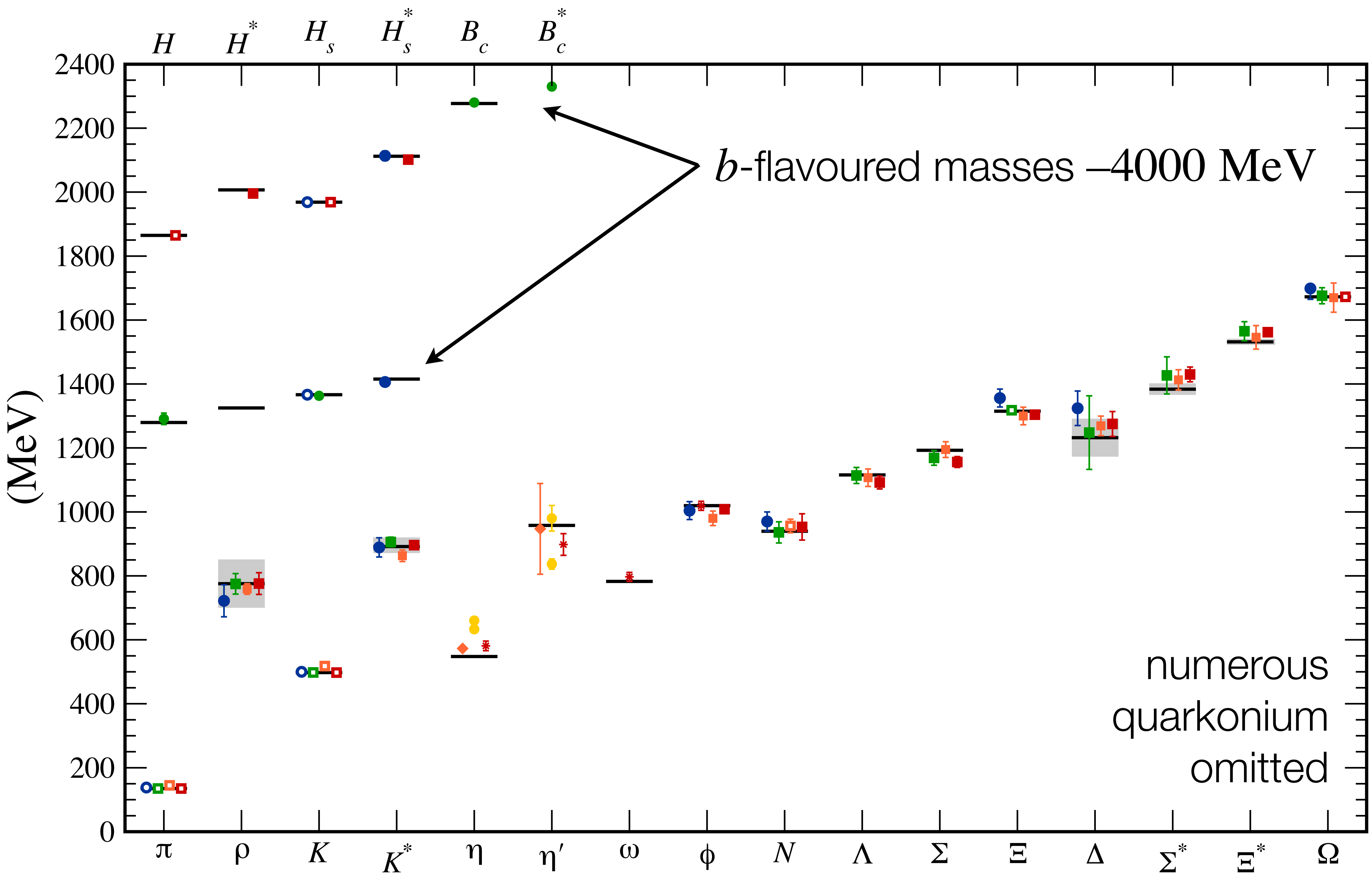
CSSM Summer School, February 11 - 15, 2013, CSSM, Adelaide, Australia

QCD Hadron Spectrum

Plot from A. Kronfeld [1203.1204]

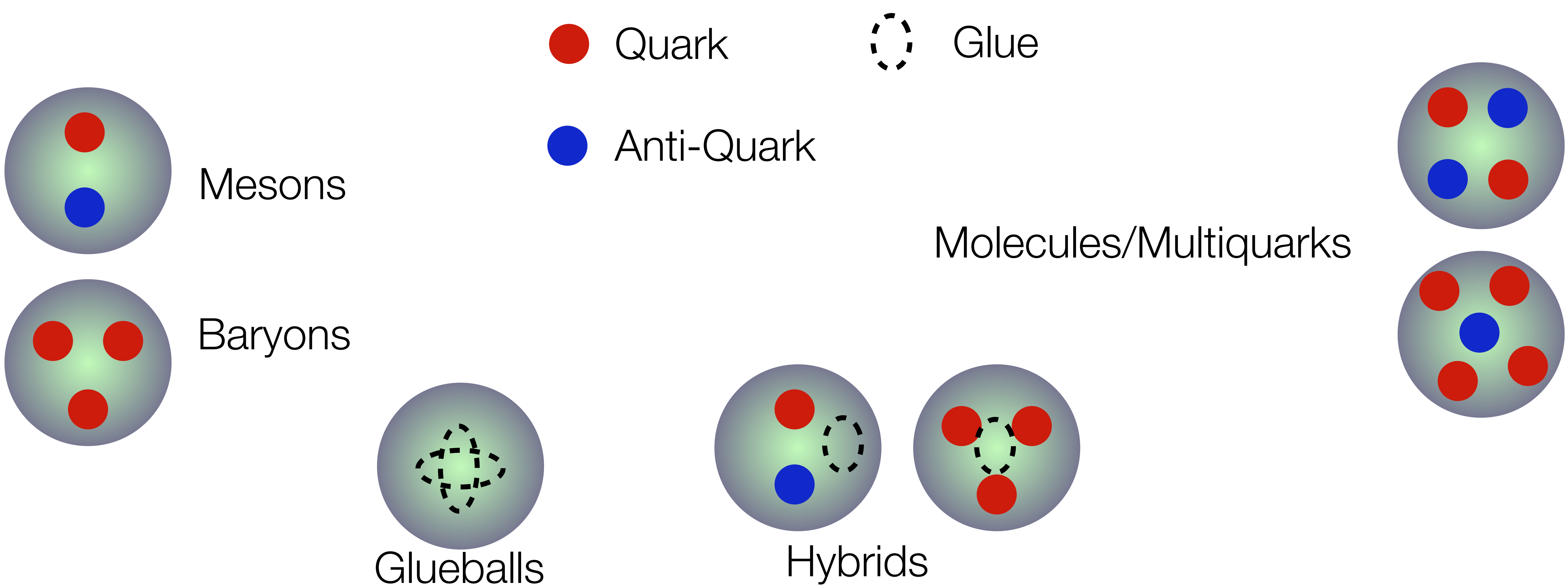
$\pi\ldots\Omega$: BMW, MILC, PACS-CS, QCDSF;
η-η': RBC, UKQCD, Hadron Spectrum (ω);
D, B: Fermilab, HPQCD, Mohler-Woloshyn



Excellent agreement between different collaborations/lattice formulations

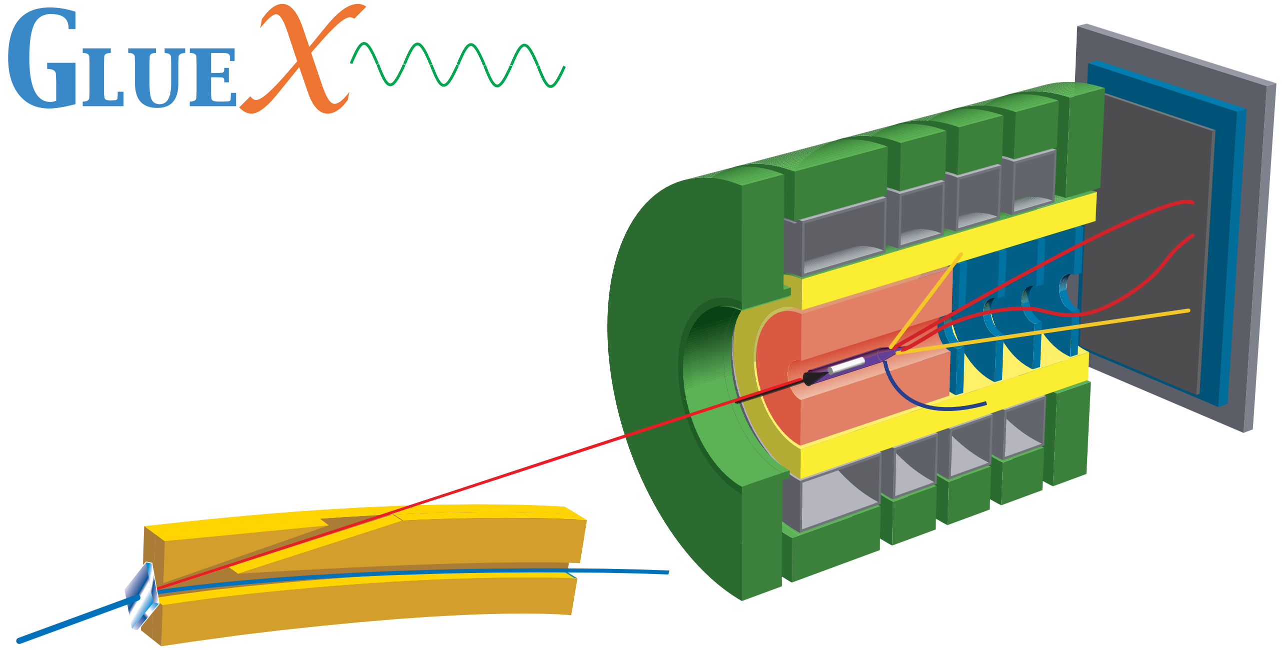
Spectroscopy

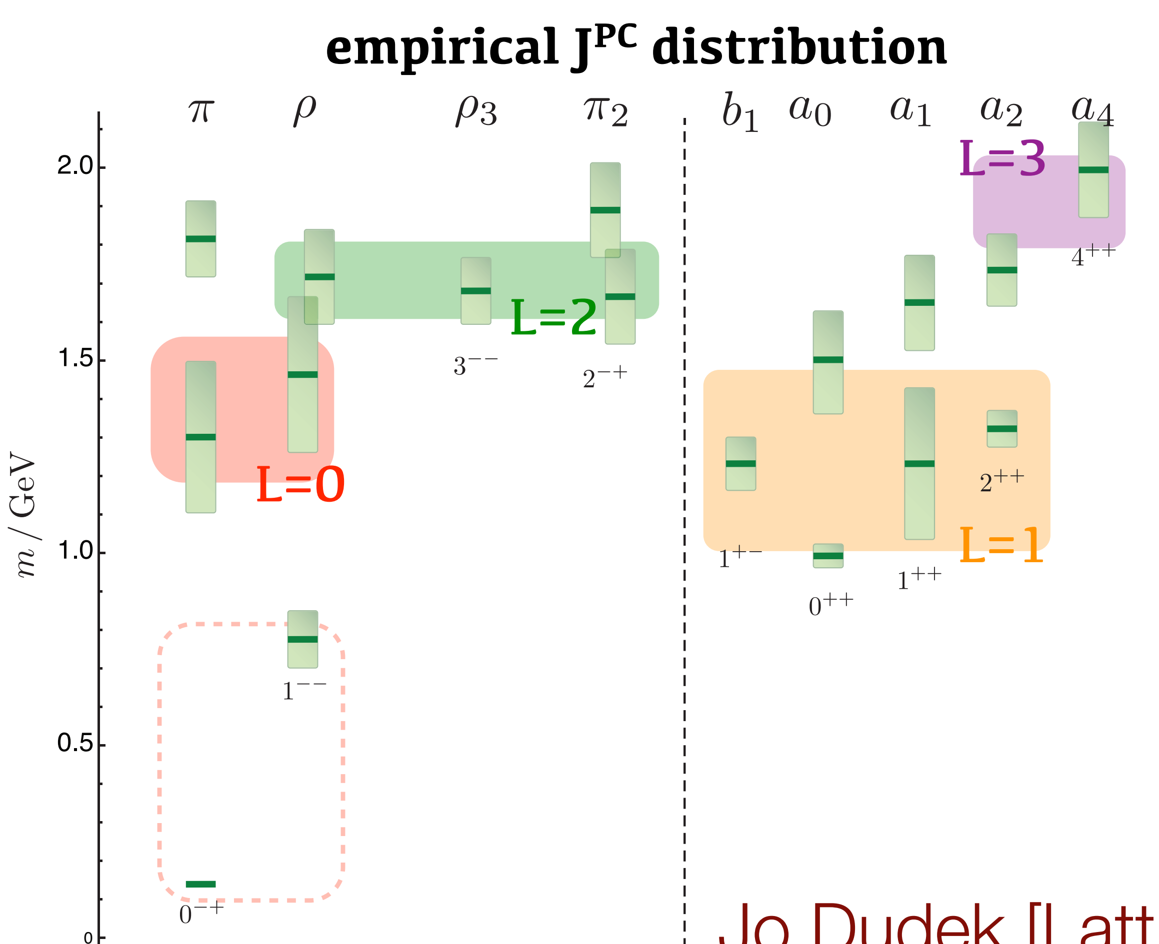
- Rich spectrum of particles observed in experiment
- Some deviate from standard quark model predictions
- Other “missing states” are yet to be observed
- Can we understand the full hadron spectrum from QCD?



Experiments

- Many experiments dedicated to study meson and baryon spectroscopy
  - GlueX and CLAS12 [Jefferson Lab]
  - Compass [CERN]
  - BES III [Beijing]





Jo Dudek [Lattice 2012]

Spectroscopy

- Recall: Masses (energies) are extracted from (Euclidean) time-dependence of lattice correlators

$$G(\vec{p},t) = \sum_{n=1}^{N} \frac{e^{-E_n t}}{2E_n} \langle \Omega | \mathcal{O}_f(0) | n, p \rangle \langle n, p | \mathcal{O}_i^\dagger(0) | \Omega \rangle$$

- In general, works well for ground states (for large enough *t*, fit a single exponential)

- Extracting excited states from multi-exponential fits difficult

- Try to optimise $\mathcal{O}$ to isolate state of interest. i.e. make

$$Z_n(\vec{p}) \equiv \frac{|\langle \Omega | \mathcal{O} | n \rangle|^2}{2E_n(\vec{p})}$$

- large for state, *n*, of interest as small for other states

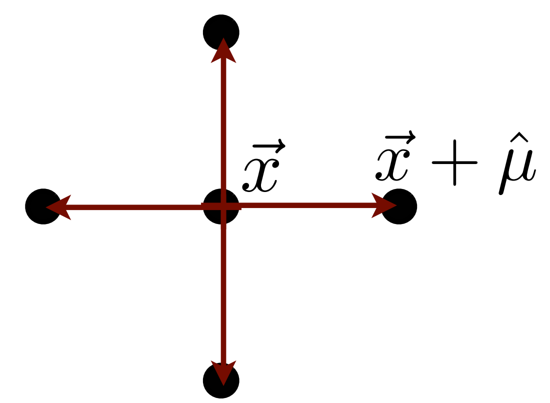
Operators

- Hadrons are extended objects ~1fm

- Using propagators computed from a point source perhaps only have small overlap with states of interest

- Gauge-invariant Gaussian smearing starts with a point source

$$\psi_0{}^{a}_{\alpha} = \delta^{ac}\delta_{\alpha\gamma}\delta_{\vec{x},\vec{x}_0}\delta_{t,t_0}$$

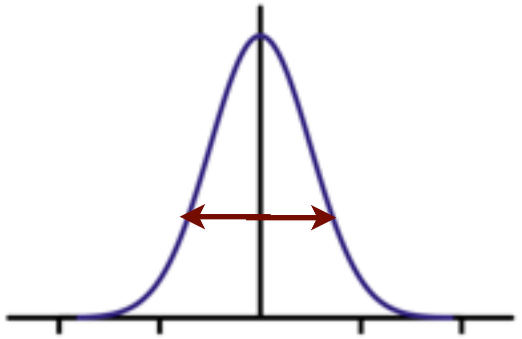


- and proceeds by the iterative scheme

$$\psi_i(\vec{x},t) = \sum_{\vec{x}\,'} F(\vec{x},\vec{x}\,')\psi_{i-1}(\vec{x}\,',t)$$

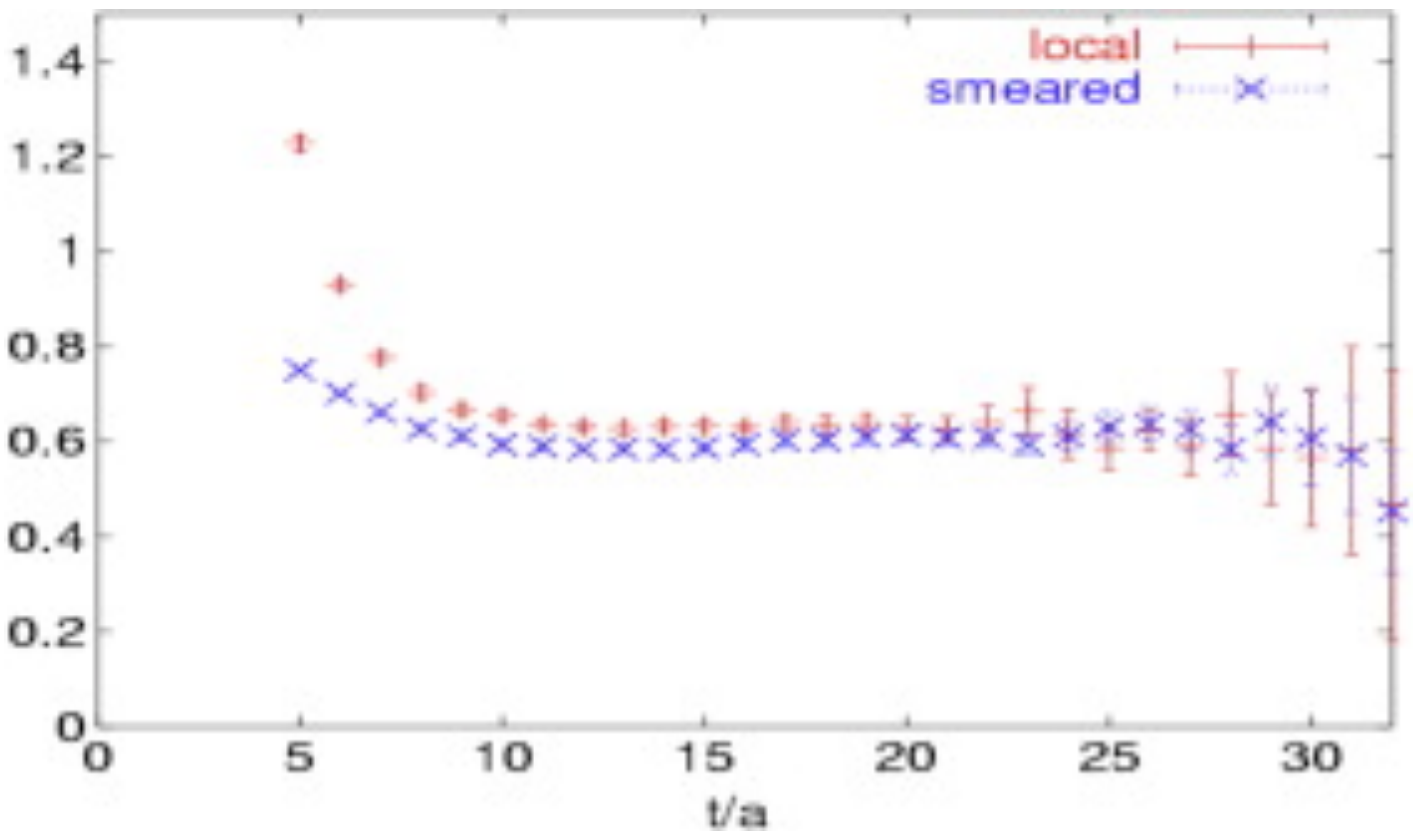
$$F(\vec{x},\vec{x}\,') = \frac{1}{(1+\alpha)}\left(\delta_{\vec{x},\vec{x}\,'} + \frac{\alpha}{6}\sum_{\mu=1}^{3}\left[U_\mu(\vec{x},t)\delta_{\vec{x}\,',\vec{x}+\hat{\mu}} + U^\dagger_\mu(\vec{x}-\hat{\mu},t)\delta_{\vec{x}\,',\vec{x}-\hat{\mu}}\right]\right)$$

- Repeating *N* times gives the resulting fermion source

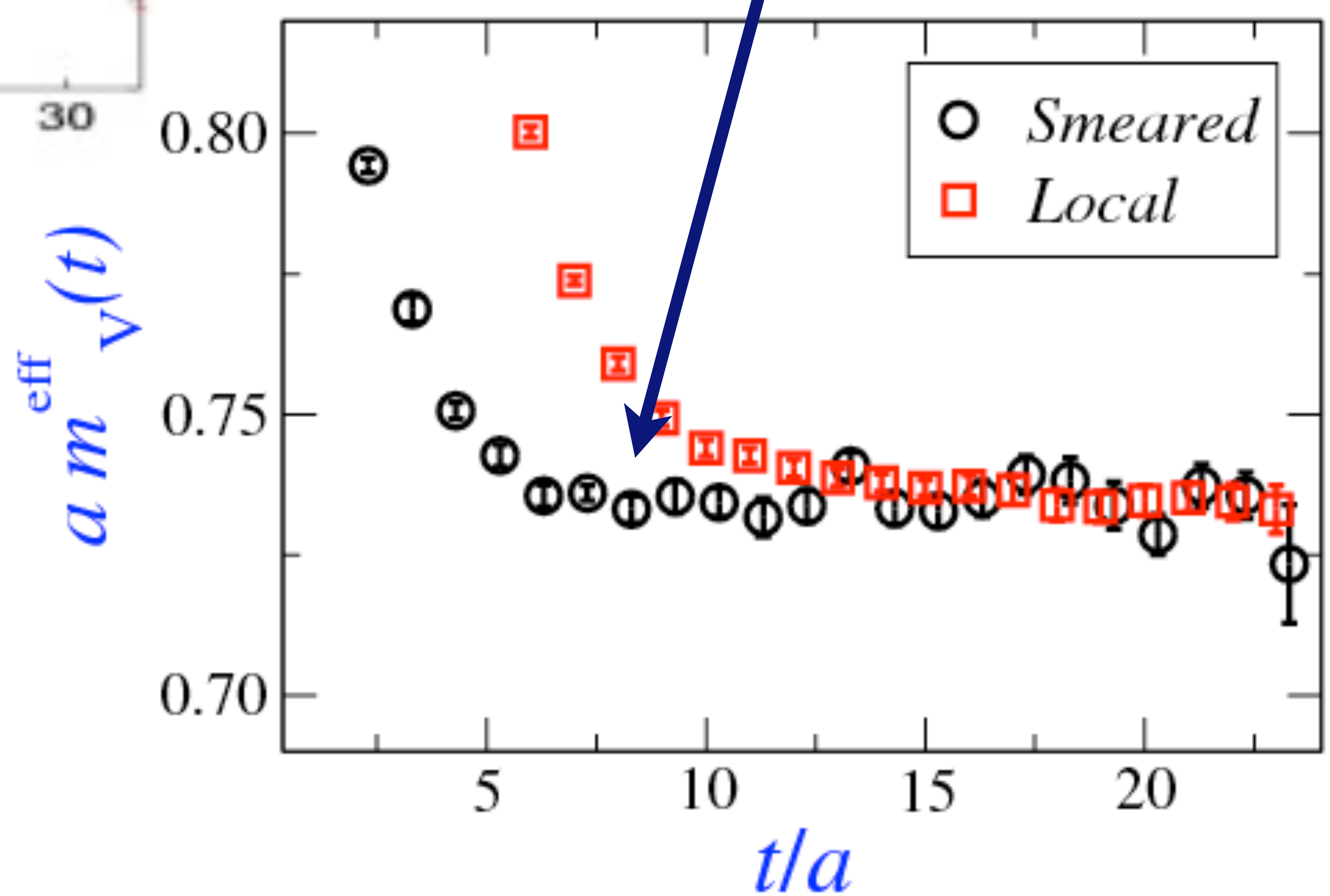


$$\psi_N(\vec{x},t) = \sum_{\vec{x}\,'} F^N(\vec{x},\vec{x}\,')\psi_0(\vec{x}\,',t)$$

Example Effective Masses



Improved overlap with ground states



Extracting Excited States

- Different operators/smearing have differing overlap strengths with a variety of states of interest

- Set up a Generalised Eigenvalue Problem to cleanly isolate individual states

- E.g., express (*p=0*) baryon 2pt function as a sum over states α

$$G^{\pm}_{ij}(t) = \sum_{\vec{x}} \mathrm{Tr}\Big\{\Gamma_{\pm}\langle\Omega|\mathcal{O}_i(x)\mathcal{O}_j(0)|\Omega\rangle\Big\}$$

Parity projectory

$$= \sum_{\alpha} \lambda^{\alpha}_i \lambda^{\alpha}_j e^{-m_\alpha t}$$

- Look for a linear combination $\mathcal{O}_j u^{\alpha}_j$ that cleanly isolates state α with overlap z^{α} at time t_0

$$G_{ij}(t_0)u^{\alpha}_j = \lambda^{\alpha}_i z^{\alpha} e^{-m_\alpha t_0}$$

- which leads to

$$G_{ij}(t_0+\Delta t)u^{\alpha}_j = e^{-m_\alpha \Delta t} G_{ij}(t_0)u^{\alpha}_j$$

Extracting Excited States

- Multiplying on the left by $[G_{ij}(t_0)]^{-1}$ gives the generalised eigenvalue equation

$$[(G(t_0))^{-1}G(t_0+\Delta t)]_{ij}u_j^\alpha = c^\alpha u_i^\alpha$$

- with eigenvalues

$$c^\alpha = e^{-m_\alpha \Delta t}$$

- and similarly

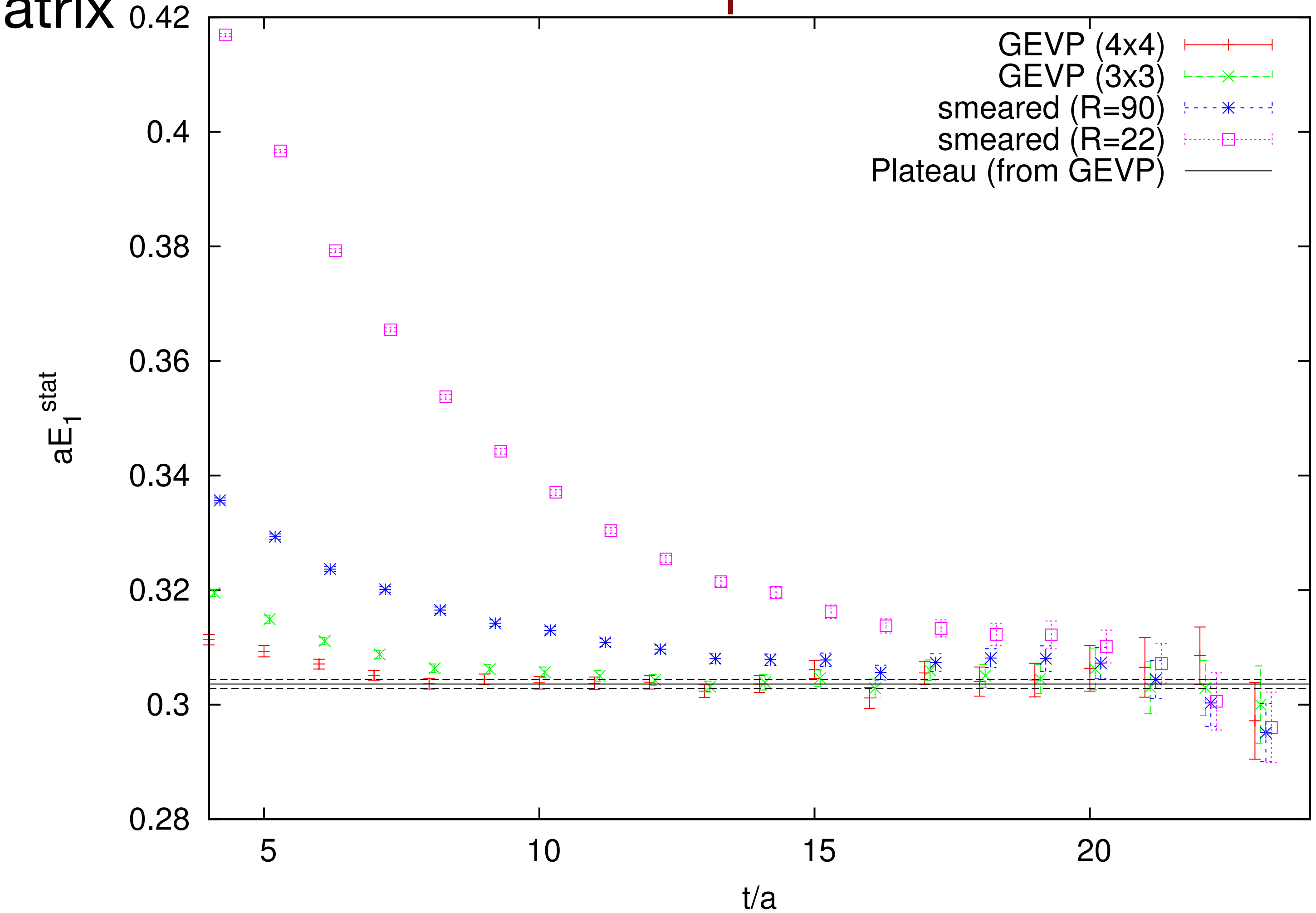
$$v_i^\alpha[G(t_0+\Delta t)(G(t_0))^{-1}]_{ij} = c^\alpha v_j^\alpha$$

- We can diagonalise the correlation matrix

$$v_i^\alpha G_{ij}(t)u_j^\beta \propto \delta^{\alpha\beta}$$

- allowing fits at earlier times

Alpha: 0902.1265



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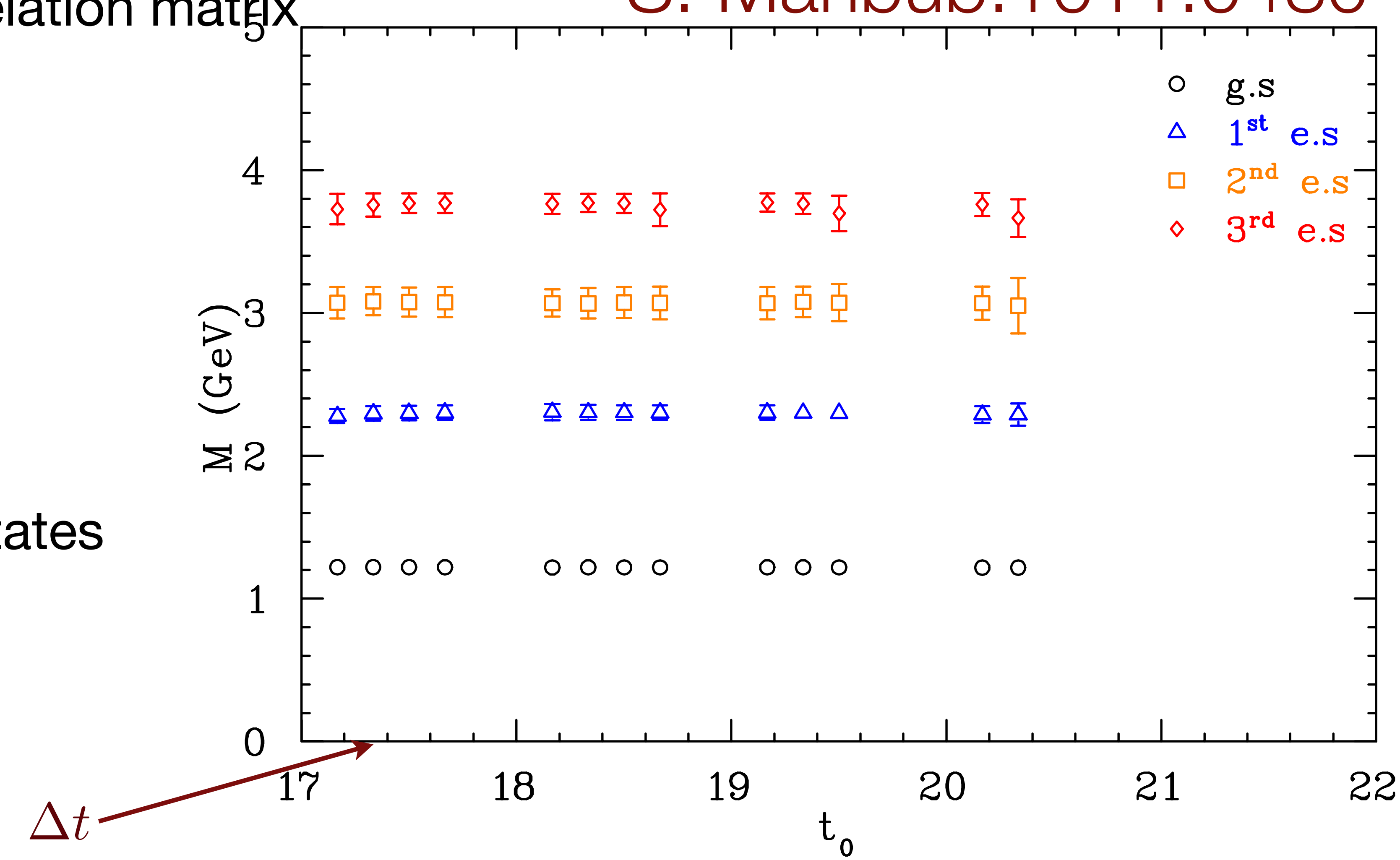
- allowing fits at earlier times

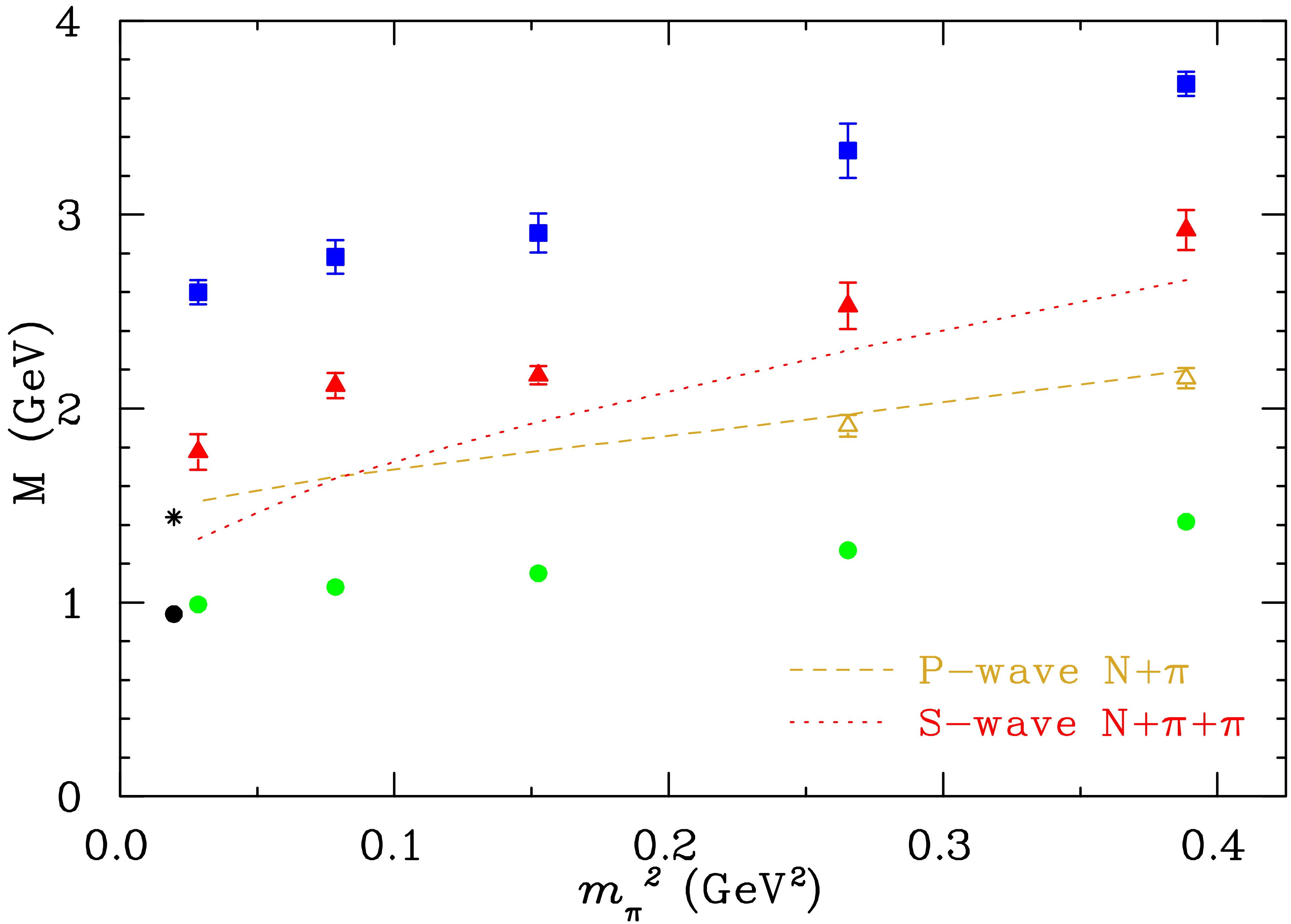
- and cleanly isolate excited states

S. Mahbub:1011.0480



Excited States

S. Mahbub:1011.5724



Spectroscopy

- Meson states allowed by the quark model

$$J^{PC} = 0^{-+},\, 0^{++},\, 1^{--},\, 1^{+-},\, 2^{--},\, 2^{-+},\, \ldots$$

- States with *P=(-1)^J* but *CP=-1* forbidden

$$J^{PC} = 0^{+-},\, 0^{--},\, 1^{-+},\, 2^{+-},\, 3^{-+},\, \ldots$$

- The are exotic states: not just a $q\bar{q}$ pair

- Need to consider a large basis of operators to isolate higher spin states and access exotic quantum numbers

- E.g. $\bar{\psi}\Gamma\psi \quad \bar{\psi}\Gamma\overleftrightarrow{D}\psi \quad \bar{\psi}\Gamma\overleftrightarrow{D}\overleftrightarrow{D}\psi \quad \bar{\psi}\Gamma F_{\mu\nu}\psi$

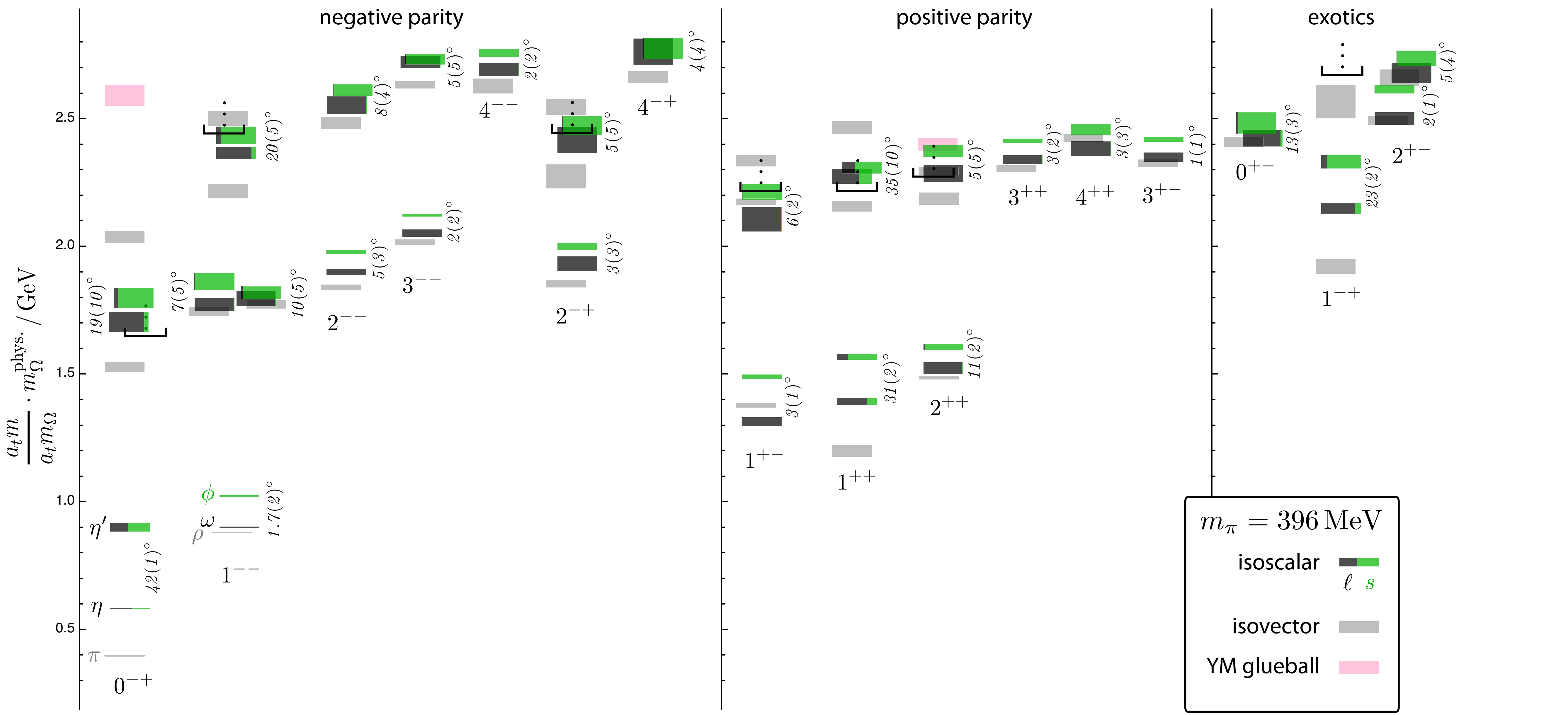
$$\overleftrightarrow{D} = \frac{1}{2}(\overrightarrow{D} - \overleftarrow{D})$$

- Solve Generalised Eigenvalue Problem to isolate individual states

The Excited Hadron Spectrum

Isoscalar Mesons

[Dudek et al. 1102.4299]



- Includes high spin and light exotic states

$$m_\pi = 396\ \mathrm{MeV}$$

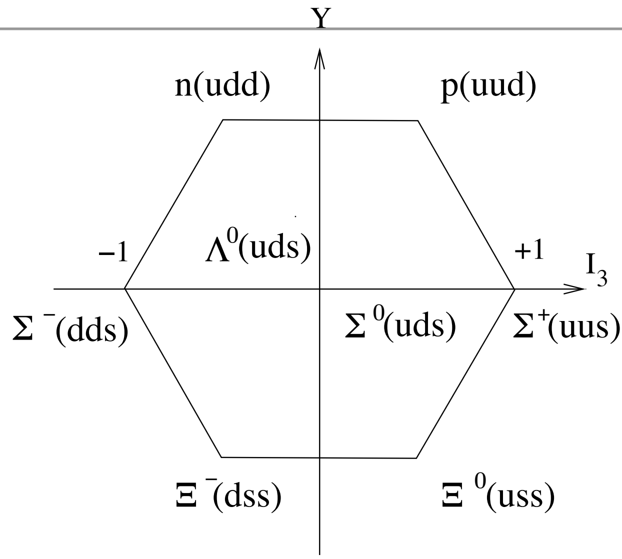
- Most states identically flavour mixed strange-light mixing

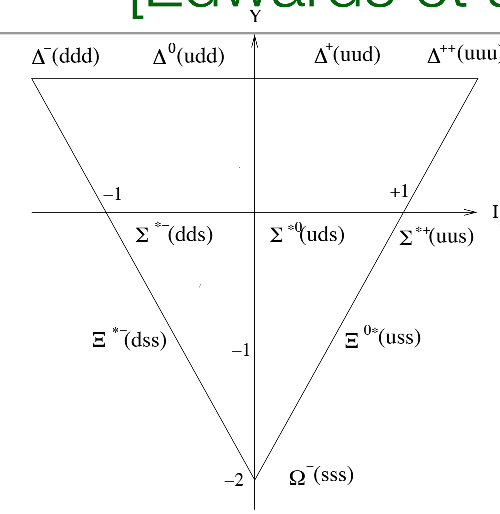
$$\alpha_{\eta-\eta'} = 42(1)^{\mathrm{o}} \qquad \alpha_{\omega-\phi} = 1.7(2)^{\mathrm{o}}$$

The Excited Hadron Spectrum

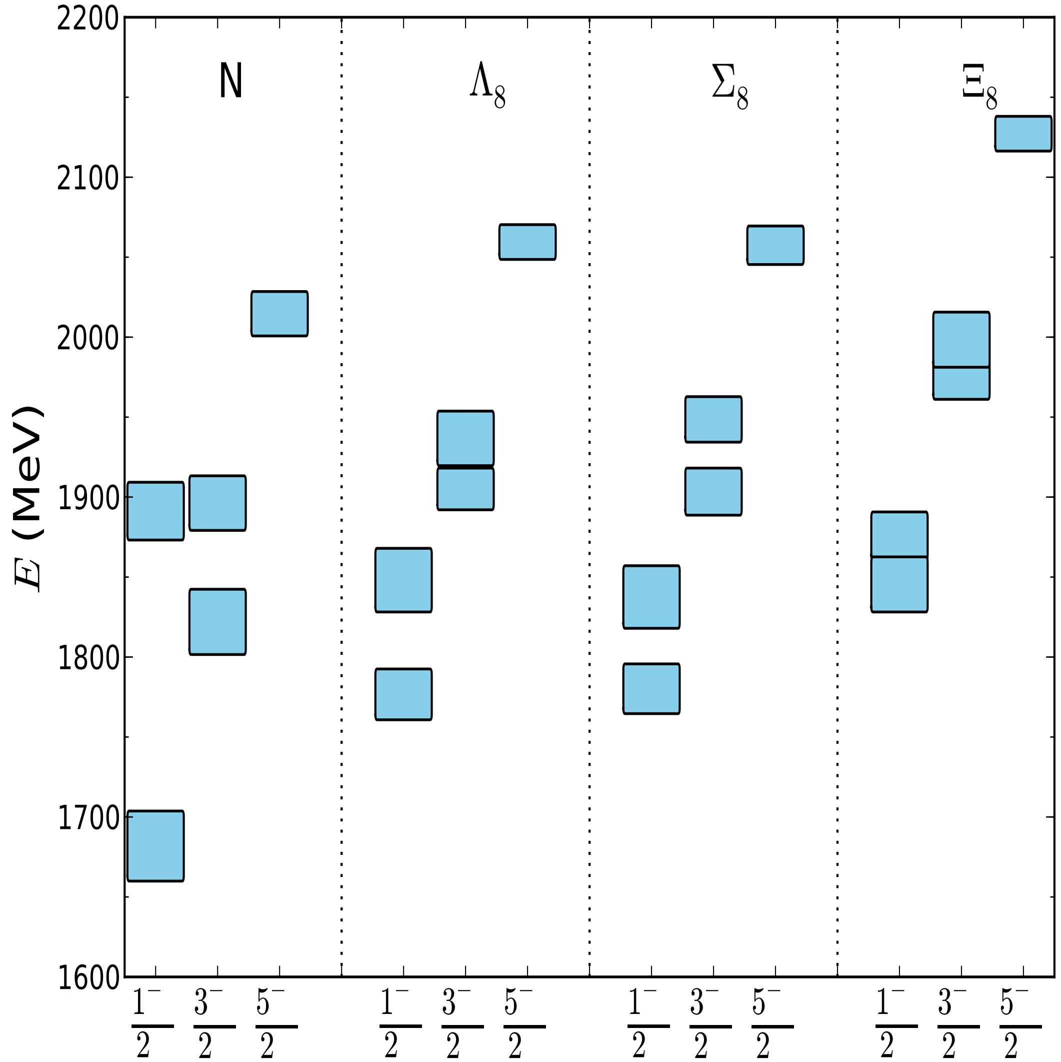
Baryons

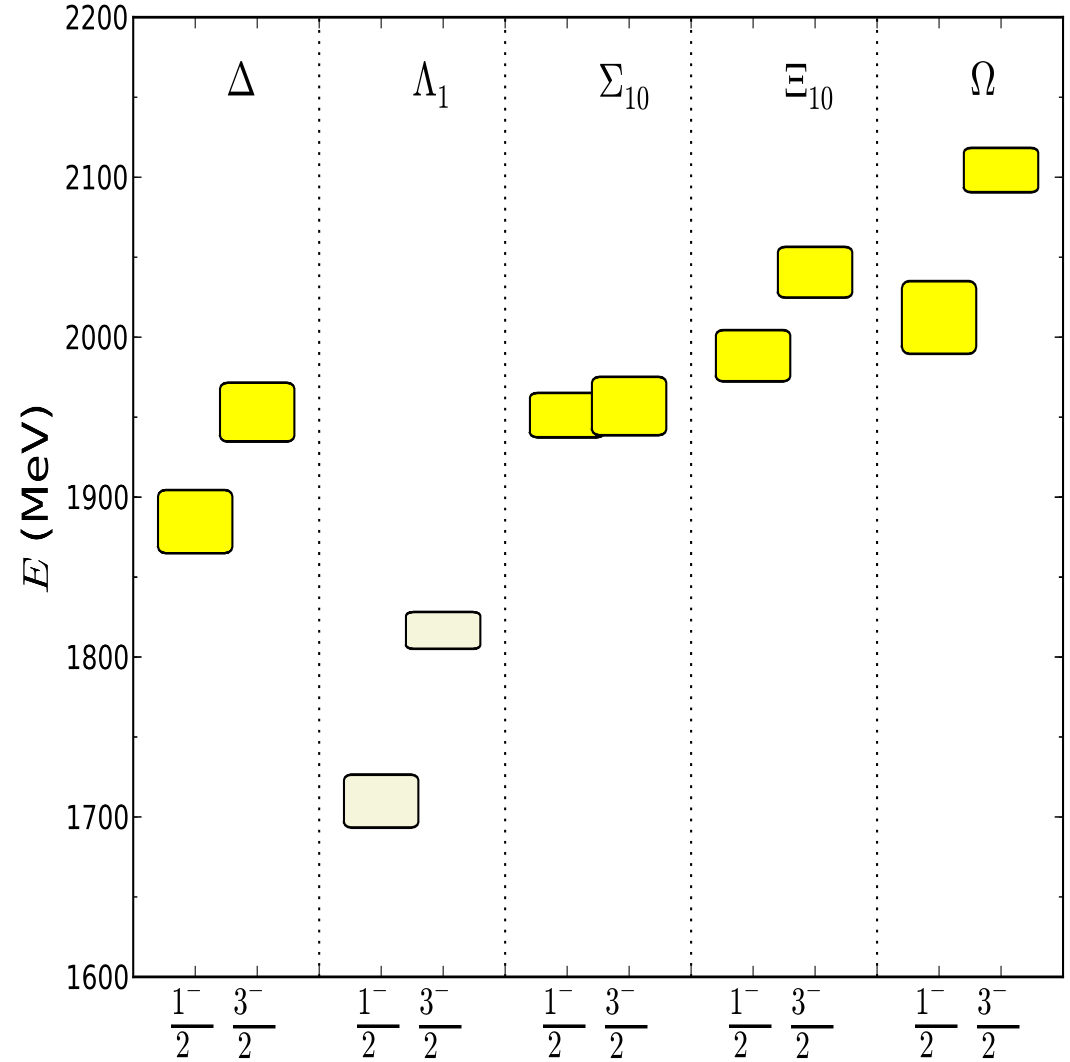
[Edwards et al. 1212.5236]





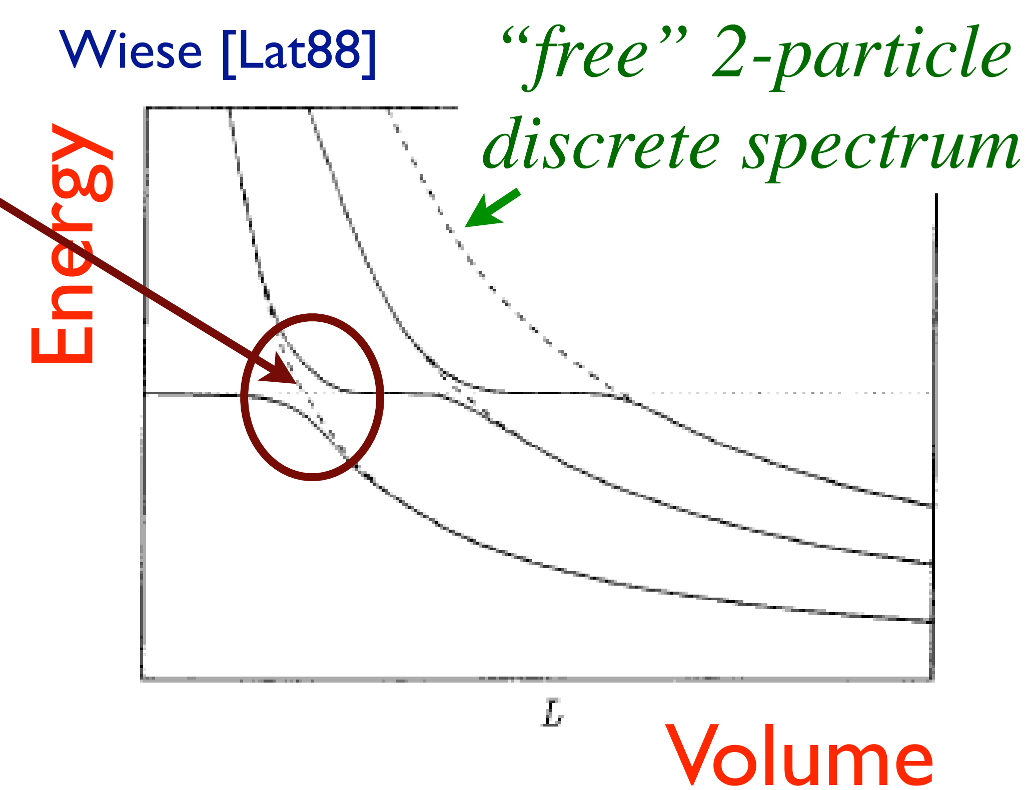
$m_\pi \approx 390$ MeV





Resonances

- Many states are actually resonances
  - If quarks are light enough, state can decay strongly
    - e.g. $\rho \longrightarrow \pi\pi \quad \Delta^{++} \rightarrow p\pi^{+}$
- How can we be sure that the state we measure is the true resonance state and not some superposition of states?
- [see Ross' talk yesterday]



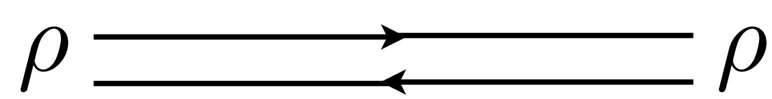
- Include two particle operators into your simulations

Two Particle States

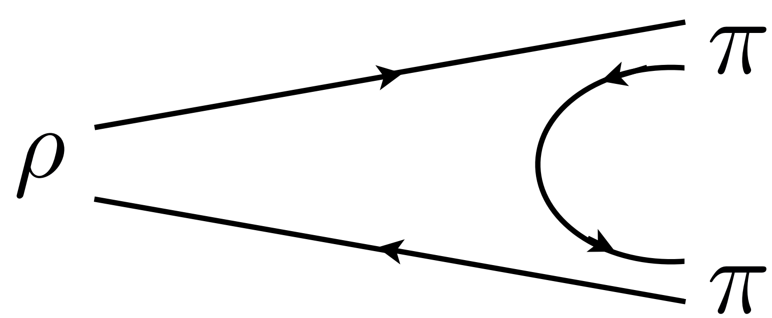
- Example $\rho \rightarrow \pi\pi$

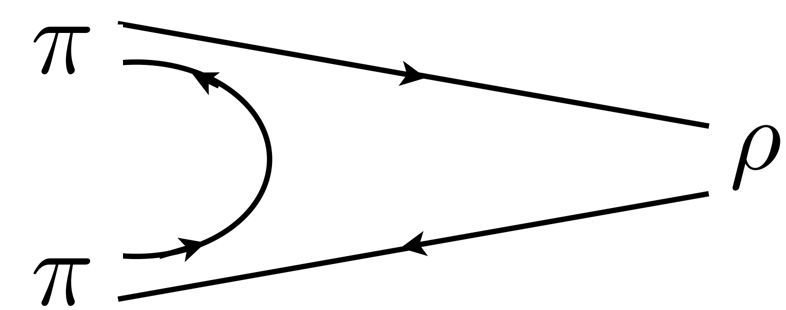
  - If you are below threshold (particle can decay), need to consider two particle system

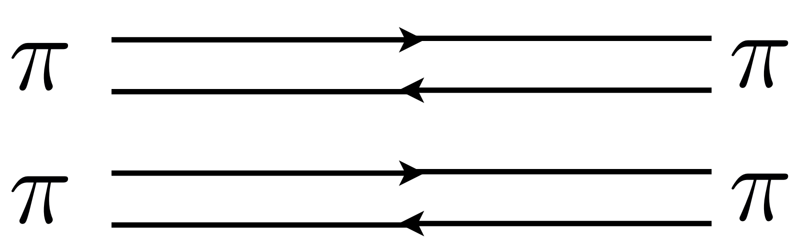
  - Not only need $\rho \rightarrow \rho$ two-point function

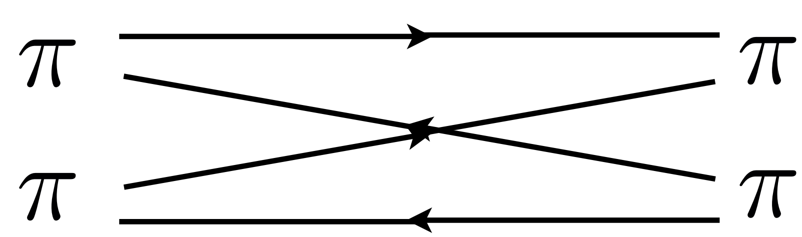


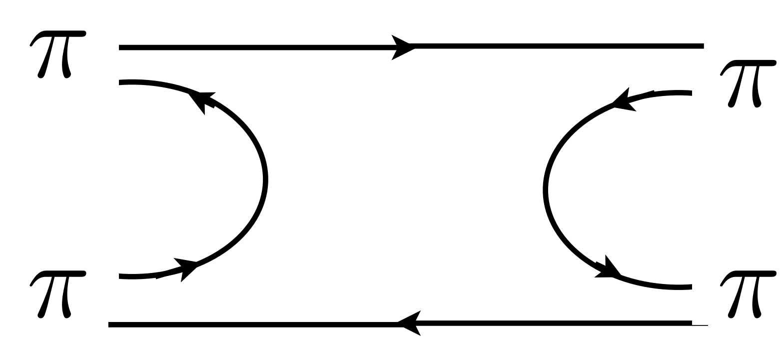
  - But also











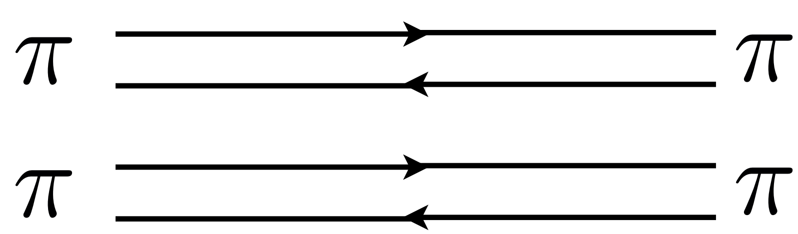
Two Particle States

- Recall meson 2pt function

$$\langle\Omega|\mathcal{O}(x)\mathcal{O}^\dagger(0)|\Omega\rangle = -\langle\Omega|\bar{\psi}_a(x)\Gamma\psi_b(x)\,\bar{\psi}_b(0)\Gamma^\dagger\psi_a(0)|\Omega\rangle$$

- For two-particle system $\pi^+\pi^- \rightarrow \pi^+\pi^-$

$$\langle\Omega|\bar{d}(x)\Gamma u(x)\bar{u}(x)\Gamma d(x)\;\bar{u}(0)\Gamma^\dagger d(0)\bar{d}(0)\Gamma^\dagger u(0)|\Omega\rangle$$



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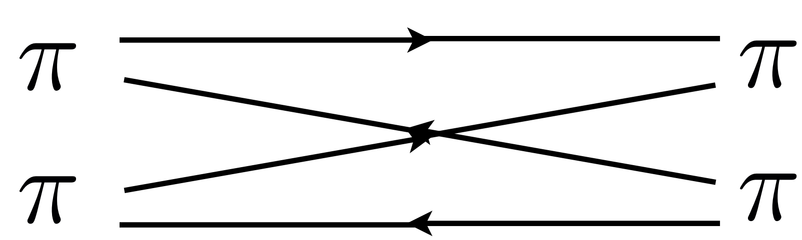

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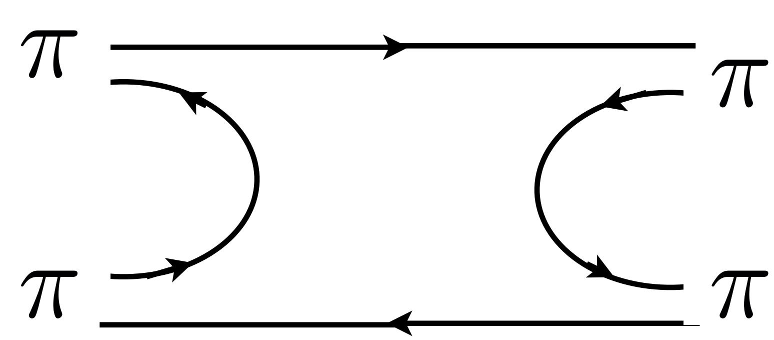

Two Particle States

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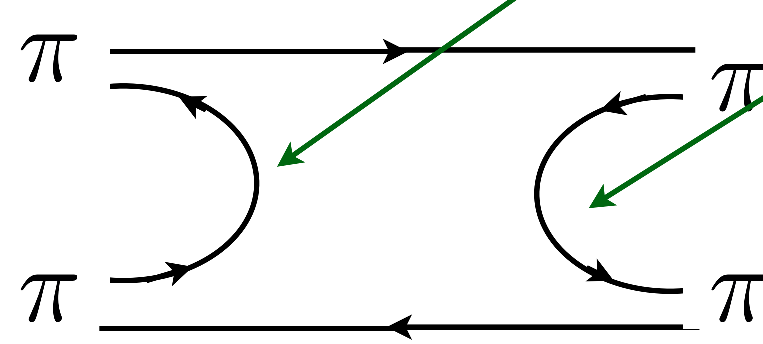
- For two-particle system $\pi^+\pi^- \rightarrow \pi^+\pi^-$

$$\langle \Omega | \bar{d}(x) \Gamma u(x) \bar{u}(x) \Gamma d(x)\ \bar{u}(0) \Gamma^\dagger d(0) \bar{d}(0) \Gamma^\dagger u(0) | \Omega \rangle$$

Requires "all-to-all" propagator

$S(x,x)$



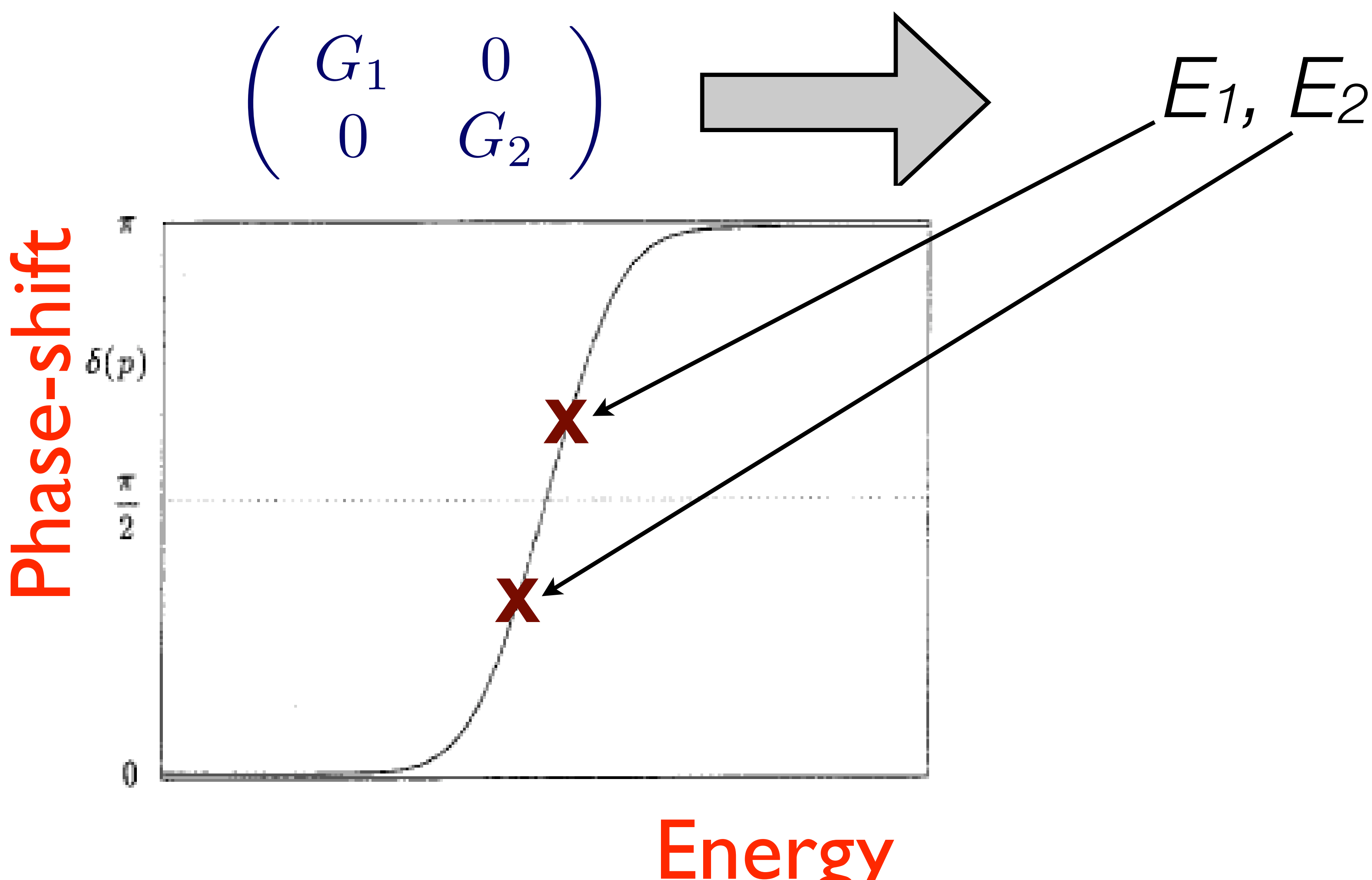
Extracting Resonance

- Construct correlation matrix

$$\begin{pmatrix} G_{\rho\to\rho} & G_{\rho\to\pi\pi} \\ G_{\pi\pi\to\rho} & G_{\pi\pi\to\pi\pi} \end{pmatrix}$$

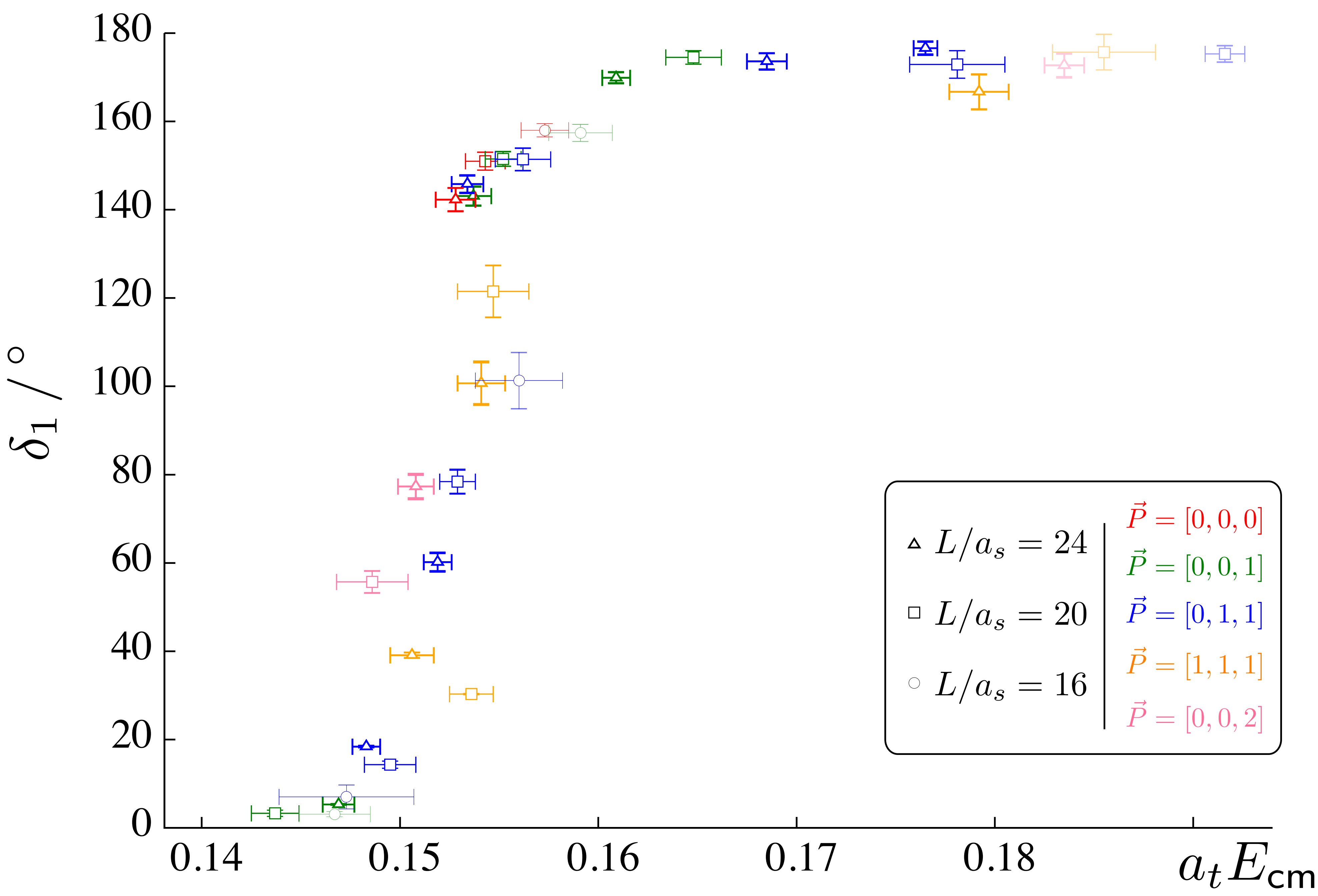
- Diagonalise to extract two energy levels

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$



Extracting Resonance

Dudek et al.[1212.0830]



Spectroscopy

- Light hadron spectrum is in good shape
- Significant advancement in the determination of excited states
- Now confronted with new challenges:
    - Isospin breaking

      (See plenary talk by T. Izubuchi at Lattice 2012)
    - QED effects
    - Resonances (See plenary talk by D. Mohler at Lattice 2012)

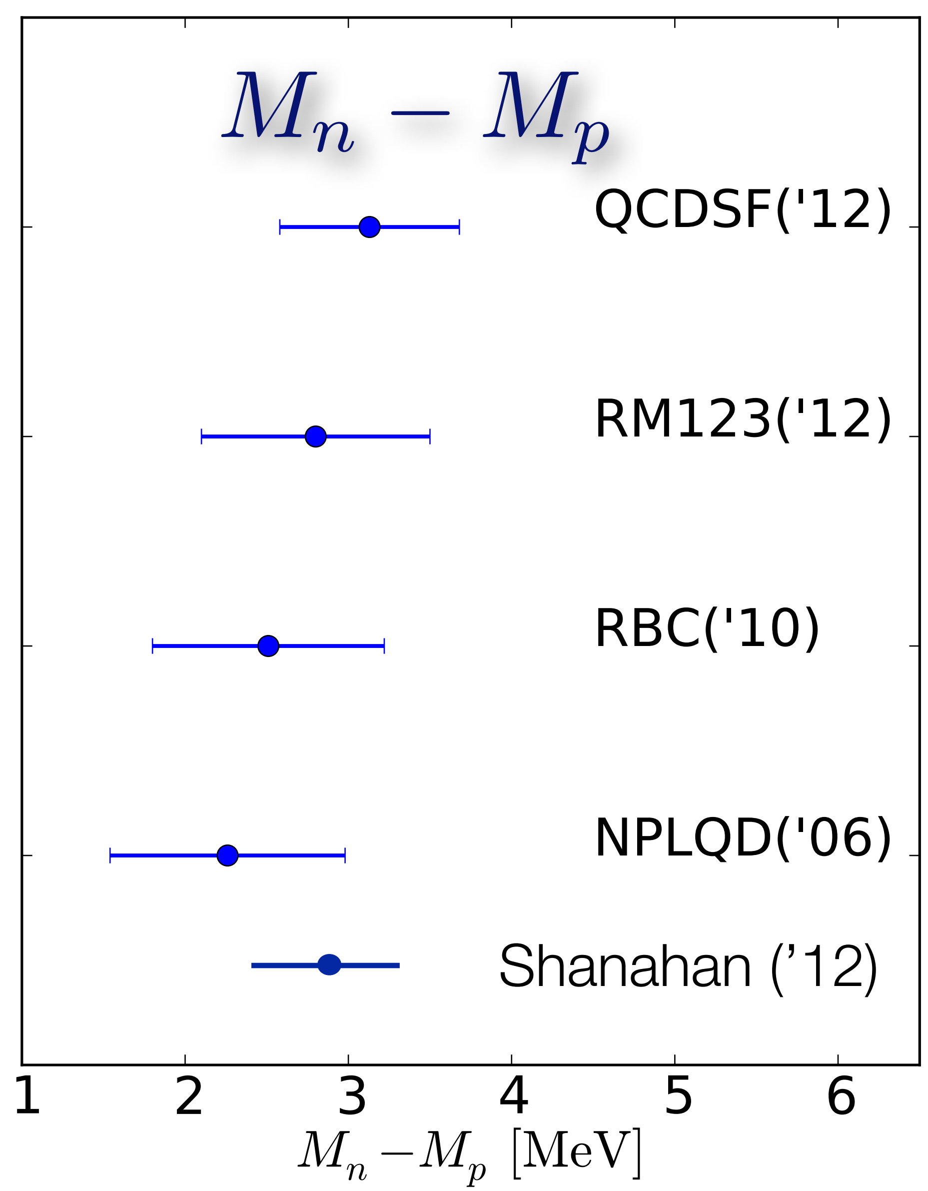
Isospin Breaking

$$(m_d - m_u)$$

- To date, all Lattice simulations have been performed with degenerate up and down quark masses

$$N_f = 2,\ N_f = 2+1,\ N_f = 2+1+1$$

- Progress by several collaborations in determining the $(m_d - m_u)$ effect on some observables



QCDSF (1206.3156):

$$M_n - M_p = 3.13(55)$$

$$M_{\Sigma^-} - M_{\Sigma^+} = 8.10(136)$$

$$M_{\Xi^-} - M_{\Xi^0} = 4.98(85)$$

Rome 123 (1110.6294):

$$m_d - m_u(\bar{\mathrm{MS}}, 2GeV) = 2.35(25)$$

$$\frac{F_{K^+}/F_{\pi^+}}{F_K/F_\pi} - 1 = -0.0039(4)$$

$$M_n - M_p = 2.8(7)$$

QED Effects

(See plenary talk by T. Izubuchi at Lattice 2012)

- Many Lattice QCD results are achieving high precision

$$\Delta f_\pi,\ \Delta f_K \sim 1\%,\ \Delta(f_\pi/f_K) \sim 0.5\%$$

- QED effects may not be negligible and should be included

- Although in some cases, QED can be treated perturbatively, this is not always the case

QCD+QED Lattice simulation

- Currently two main methods employed:
  - Quenched QED
  - Dynamical QED via reweighting

QED Effects

(See plenary talk by T. Izubuchi at Lattice 2012)

- E.g. Quark masses from QCD+QED simulation [PRD82, 094508 (2010)]

$$\begin{aligned} m_u &= 2.24 \pm 0.10 \pm 0.34 \text{ MeV} \\ m_d &= 4.65 \pm 0.15 \pm 0.32 \text{ MeV} \\ m_s &= 97.6 \pm 2.9 \pm 5.5 \text{ MeV} \\ m_d - m_u &= 2.411 \pm 0.065 \pm 0.476 \text{ MeV} \\ m_{ud} &= 3.44 \pm 0.12 \pm 0.22 \text{ MeV} \\ m_u/m_d &= 0.4818 \pm 0.0096 \pm 0.0860 \\ m_s/m_{ud} &= 28.31 \pm 0.29 \pm 1.77, \end{aligned}$$

- and for n-p: $(M_n - M_p)^{QED} = -0.54(24)$
  - Combine with previous QCD result

$$(M_n - M_p) = 2.14(42) \text{ MeV}$$

c.f. experiment: 1.2933321(4) MeV

Calculating Matrix Elements

$$\langle H' | \mathcal{O} | H \rangle$$

$$H, H' : \;\; \pi,\; K\, p,\; n,\; \ldots$$

$$\mathcal{O} : \;\; V_\mu,\; A_\mu,\; \ldots$$

Calculating Matrix Elements

$$q^2 = -Q^2 = (p' - p)^2$$

$$P^\mu = p'^\mu + p^\mu$$

Spin-0

$$\langle \pi(p')|J^\mu(\vec{q})|\pi(p)\rangle = P^\mu F_\pi(q^2)$$

Spin-1/2

$$\langle N(p', s')|J^\mu(\vec{q})|N(p, s)\rangle = \bar{u}(p', s')\left[\gamma^\mu F_1(q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2m}F_2(q^2)\right]u(p, s)$$

Spin-1

$$\langle \rho(p', s')|J^\mu(\vec{q})|\rho(p, s)\rangle =$$

$$-(\epsilon'^* \cdot \epsilon)P^\mu G_1(Q^2) - [(\epsilon'^* \cdot q)\epsilon^\mu - (\epsilon \cdot q)\epsilon'^{*\mu}]G_2(Q^2) + (\epsilon \cdot q)(\epsilon'^* \cdot q)\frac{P^\mu}{(2m_\rho)^2}G_3(Q^2)$$

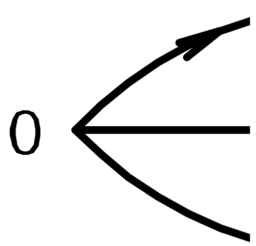
Spin-3/2

$$\langle \Delta(p', s')|J^\mu(\vec{q})|\Delta(p, s)\rangle =$$

$$\bar{u}_\alpha(p', s')\left\{-g^{\alpha\beta}\left[\gamma^\mu a_1(Q^2) + \frac{P^\mu}{2M_\Delta}a_2(Q^2)\right] - \frac{q^\alpha q^\beta}{(2M_\Delta)^2}\left[\gamma^\mu c_1(Q^2) + d\frac{P^\mu}{2M_\Delta}c_2(Q^2)\right]\right\}u_\beta(p, s)$$

Lattice 3pt Functions

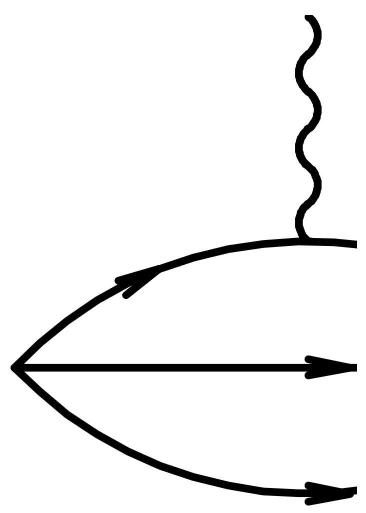
$$\langle \Omega | T \Big(\qquad\qquad \overline{\chi}_\beta(0) \Big) | \Omega \rangle$$



- Create a state (with quantum numbers of the proton) at time *t=0*

Lattice 3pt Functions

$$\langle \Omega | T \left(\qquad \mathcal{O}(\vec{x}_1, \tau)\, \overline{\chi}_\beta(0) \right) | \Omega \rangle$$

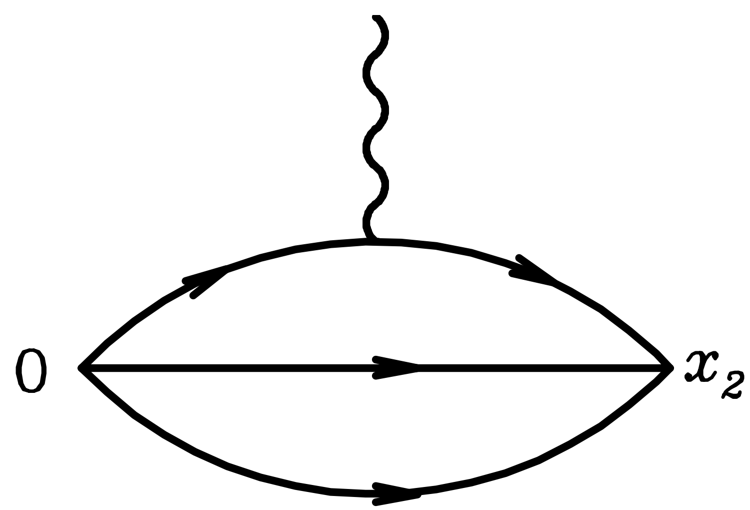


- Create a state (with quantum numbers of the proton) at time *t=0*

- Insert an operator, $\mathcal{O}$, at some time τ

Lattice 3pt Functions

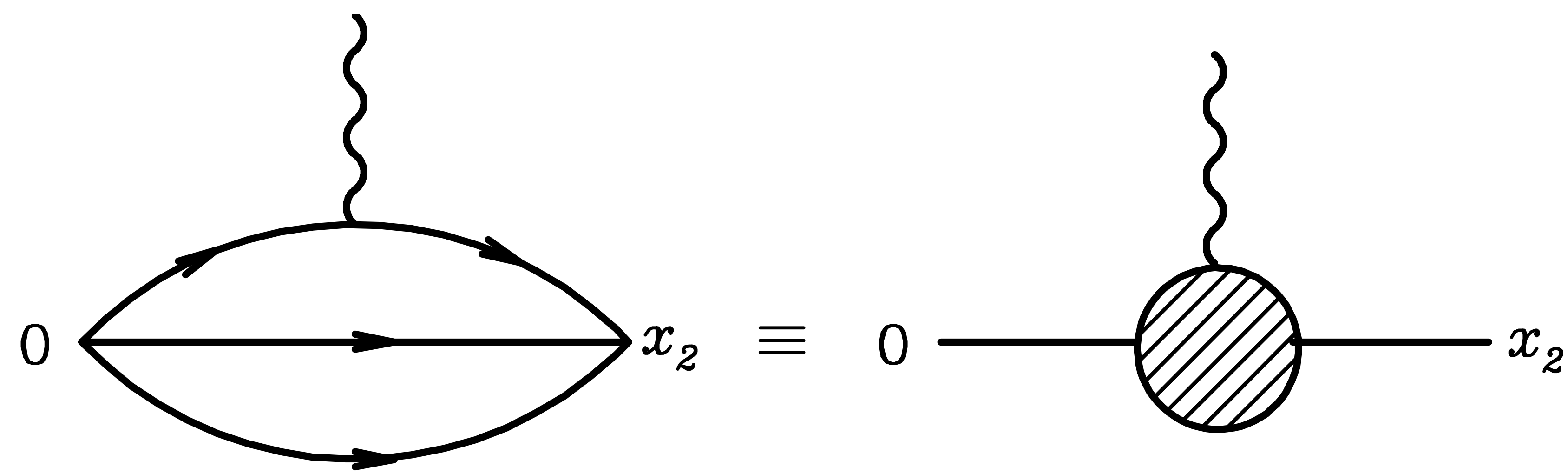
$$\langle\Omega|T\left(\chi_\alpha(\vec{x}_2,t)\,\mathcal{O}(\vec{x}_1,\tau)\,\overline{\chi}_\beta(0)\right)|\Omega\rangle$$



- Create a state (with quantum numbers of the proton) at time *t=0*
- Insert an operator, $\mathcal{O}$, at some time τ
- Annihilate state at final time *t*

Lattice 3pt Functions

$$G(t,\tau,p,p') = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1} \Gamma_{\beta\alpha} \langle\Omega| T\left(\chi_\alpha(\vec{x}_2,t)\,\mathcal{O}(\vec{x}_1,\tau)\,\overline{\chi}_\beta(0)\right)|\Omega\rangle$$



- Create a state (with quantum numbers of the proton) at time *t=0*
- Insert an operator, $\mathcal{O}$, at some time τ
- Annihilate state at final time *t*

Lattice 3pt Functions

$$G(t,\tau,p,p') = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1} \Gamma_{\beta\alpha} \langle\Omega\big| T\left(\chi_\alpha(\vec{x}_2,t)\,\mathcal{O}(\vec{x}_1,\tau)\,\overline{\chi}_\beta(0)\right)\big|\Omega\rangle$$

- Insert complete set of states $\quad I = \sum_{B',p',s'} |B',p',s'\rangle\langle B',p',s'| \quad I = \sum_{B,p,s} |B,p,s\rangle\langle B,p,s|$
- Make use of translational invariance $\quad \chi(\vec{x},t) = e^{\hat{H}t} e^{-i\hat{\vec{P}}\cdot\vec{x}} \chi(0) e^{i\hat{\vec{P}}\cdot\vec{x}} e^{-\hat{H}t}$

$$G(t,\tau,\vec{p},\vec{p}') = \sum_{B,B'} \sum_{s,s'} e^{-E_{B'}(\vec{p}')(t-\tau)} e^{-E_B(\vec{p})\tau} \Gamma_{\beta\alpha}$$

$$\times \langle\Omega\big|\chi_\alpha(0)\big|B',p',s'\rangle \langle B',p',s'\big|\mathcal{O}(\vec{q})\big|B,p,s\rangle \langle B,p,s\big|\overline{\chi}_\beta(0)\big|\Omega\rangle$$

- Evolve to large Euclidean times to isolate ground state $\quad 0 \ll \tau \ll t$

$$G(t,\tau,\vec{p},\vec{p}') = \sum_{s,s'} e^{-E_{\vec{p}'}(t-\tau)} e^{-E_{\vec{p}}\tau} \Gamma_{\beta\alpha} \langle\Omega|\chi_\alpha(0)|N(p',s')\rangle \langle N(p',s')|\mathcal{O}(\vec{q})|N(p,s)\rangle \langle Np,s)|\overline{\chi}_\beta(0)|\Omega\rangle$$

Lattice 3pt Functions

pion

- Consider a pion 3pt function

$$G(t,\tau,p,p') = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1} \langle\Omega\big|T\left(\chi(\vec{x}_2,t)\,\mathcal{O}(\vec{x}_1,\tau)\,\chi^\dagger(0)\right)\big|\Omega\rangle$$

- With interpolating operator $\chi(x) = \bar{d}(x)\gamma_5 u(x)$

- And insert the local operator (quark bi-linear) $\bar{q}(x)\mathcal{O}q(x)$

$\mathcal{O}$: Combination of γ matrices and derivatives

$$-\bar{d}(x_2)\gamma_5 u(x_2)\bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}(0)\gamma_5 d(0)$$

u-quark

Lattice 3pt Functions

pion

- Consider a pion 3pt function

$$G(t,\tau,p,p') = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1} \langle\Omega|T\left(\chi(\vec{x}_2,t)\,\mathcal{O}(\vec{x}_1,\tau)\,\chi^\dagger(0)\right)|\Omega\rangle$$

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u-quark

Lattice 3pt Functions

pion

u-quark

$$-\bar{d}^a_\beta(x_2)\gamma_{5\beta\gamma}u^a_\gamma(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta}u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_{5\xi\alpha}d^c_\alpha(0)$$

- all possible Wick contractions

Lattice 3pt Functions

pion

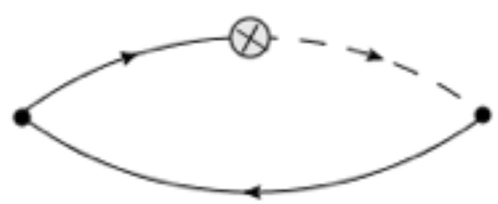
u-quark

$$-\bar{d}^a_\beta(x_2)\gamma_{5\beta\gamma}u^a_\gamma(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta}u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_{5\xi\alpha}d^c_\alpha(0)$$

- all possible Wick contractions

- connected

$$S^{ca}_{d\alpha\beta}(0,x_2)\gamma_{5\beta\gamma}S^{ab}_{u\gamma\rho}(x_2,x_1)\Gamma_{\rho\delta}S^{bc}_{u\delta\xi}(x_1,0)\gamma_{5\xi\alpha}$$



Lattice 3pt Functions

pion

u-quark

$$-\bar{d}^a_\beta(x_2)\gamma_{5\beta\gamma}u^a_\gamma(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta}u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_{5\xi\alpha}d^c_\alpha(0)$$

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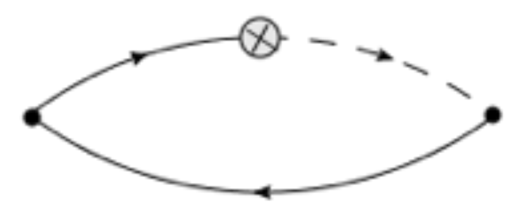
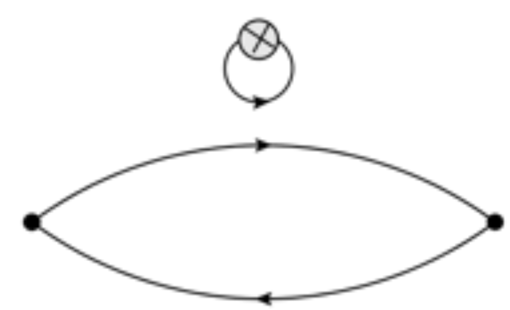


- disconnected

$$-S^{ca}_{d\alpha\beta}(0,x_2)\gamma_{5\beta\gamma}S^{ac}_{u\gamma\xi}(x_2,0)\gamma_{5\xi\alpha}S^{bb}_{u\delta\rho}(x_1,x_1)\Gamma_{\rho\delta}$$



Lattice 3pt Functions

pion

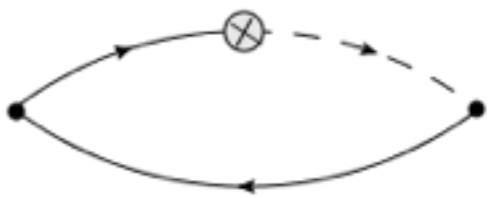
u-quark

$$-\bar{d}^a_\beta(x_2)\gamma_{5\beta\gamma}u^a_\gamma(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta}u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_{5\xi\alpha}d^c_\alpha(0)$$

- all possible Wick contractions

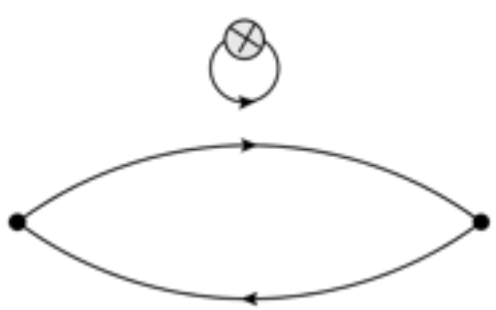
- connected

$$\mathrm{Tr}\left[S_d(0,x_2)\gamma_5 S_u(x_2,x_1)\Gamma S_u(x_1,0)\gamma_5\right]$$



- disconnected

$$\mathrm{Tr}\left[-S_d(0,x_2)\gamma_5 S_u(x_2,0)\gamma_5\right]\mathrm{Tr}\left[S_u(x_1,x_1)\Gamma\right]$$



Lattice 3pt Functions

pion

$$-\bar{d}^a_\beta(x_2)\gamma_{5\beta\gamma}u^a_\gamma(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta}u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_{5\xi\alpha}d^c_\alpha(0)$$

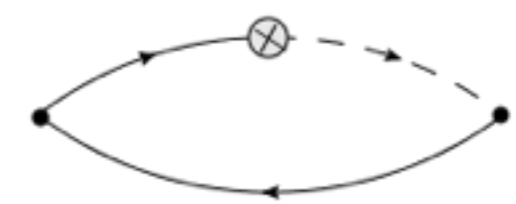
- all possible Wick contractions

γ_5-hermiticity

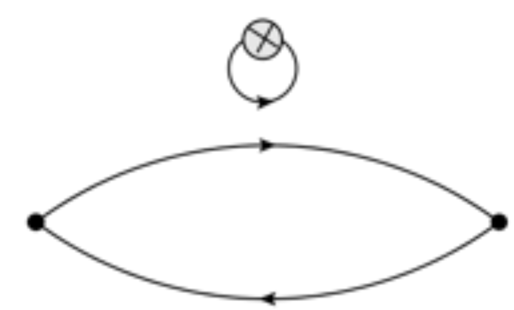
$$S^\dagger(x,0) = \gamma_5 S(0,x)\gamma_5$$

- connected

$$\mathrm{Tr}\left[S_d^\dagger(x_2,0)S_u(x_2,x_1)\Gamma S_u(x_1,0)\right]$$



- disconnected



$$\mathrm{Tr}\left[-S_d^\dagger(x_2,0)S_u(x_2,0)\right]\mathrm{Tr}\left[S_u(x_1,x_1)\Gamma\right]$$

- all-to-all propagators

Lattice 3pt Functions

proton

$$G_\Gamma(t,\tau;\vec{p}',\vec{p}) = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1}\, \Gamma_{\beta\alpha}\langle\Omega|T\left[\chi_\alpha(t,\vec{x}_2)\,\mathcal{O}(\tau,\vec{x}_1)\,\overline{\chi}_\beta(0)\right]|\Omega\rangle$$

- Use the following interpolating operator to create a proton

$$\chi_\alpha(x) = \epsilon^{abc}\left(u^{Ta}(x)\; C\gamma_5\; d^b(x)\right) u^c_\alpha(x)$$

- And insert the local operator (quark bi-linear) $\bar{q}(x)\mathcal{O}q(x)$

$\mathcal{O}$: Combination of γ matrices and derivatives

- Perform all possible (connected) Wick contractions

u-quark (4 terms)

$$\epsilon^{abc}\epsilon^{a'b'c'}\left(u^{Ta}(x_2)\; C\gamma_5\; d^b(x_2)\right) u^c_\alpha(x_2)\bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}^{c'}(0)\left(\bar{d}^{b'}(0)C\gamma_5\bar{u}^{Ta'}(0)\right)$$

Lattice 3pt Functions

proton

$$G_\Gamma(t,\tau;\vec{p}',\vec{p}) = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1}\, \Gamma_{\beta\alpha}\langle\Omega|T\left[\chi_\alpha(t,\vec{x}_2)\,\mathcal{O}(\tau,\vec{x}_1)\,\overline{\chi}_\beta(0)\right]|\Omega\rangle$$

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$$\epsilon^{abc}\epsilon^{a'b'c'}\left(u^{Ta}(x_2)\ C\gamma_5\ d^b(x_2)\right) u^c_\alpha(x_2)\bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}^{c'}(0)\left(\bar{d}^{b'}(0)C\gamma_5\bar{u}^{Ta'}(0)\right)$$

Lattice 3pt Functions

proton

$$G_\Gamma(t,\tau;\vec{p}',\vec{p}) = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1}\, \Gamma_{\beta\alpha}\langle\Omega|T\left[\chi_\alpha(t,\vec{x}_2)\,\mathcal{O}(\tau,\vec{x}_1)\,\overline{\chi}_\beta(0)\right]|\Omega\rangle$$

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$$\epsilon^{abc}\epsilon^{a'b'c'}\left(u^{Ta}(x_2)\; C\gamma_5\; d^b(x_2)\right) u^c_\alpha(x_2)\bar{u}(x_1)\mathcal{O}u(x_1)\bar{u}^{c'}(0)\left(\bar{d}^{b'}(0)C\gamma_5\bar{u}^{Ta'}(0)\right)$$

Lattice 3pt Functions

proton

$$G_\Gamma(t,\tau;\vec{p}',\vec{p}) = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1}\, \Gamma_{\beta\alpha}\langle\Omega|T\left[\chi_\alpha(t,\vec{x}_2)\,\mathcal{O}(\tau,\vec{x}_1)\,\overline{\chi}_\beta(0)\right]|\Omega\rangle$$

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Lattice 3pt Functions

proton

$$G_\Gamma(t,\tau;\vec{p}',\vec{p}) = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1}\, \Gamma_{\beta\alpha}\langle\Omega|T\left[\chi_\alpha(t,\vec{x}_2)\,\mathcal{O}(\tau,\vec{x}_1)\,\overline{\chi}_\beta(0)\right]|\Omega\rangle$$

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$\mathcal{O}$: Combination of γ matrices and derivatives

- Perform all possible (connected) Wick contractions

d-quark (2 terms)

$$\epsilon^{abc}\epsilon^{a'b'c'}\left(u^{Ta}(x_2)\; C\gamma_5\; d^b(x_2)\right) u^c_\alpha(x_2)\bar{d}(x_1)\mathcal{O}d(x_1)\bar{u}^{c'}(0)\left(\bar{d}^{b'}(0)C\gamma_5\bar{u}^{Ta'}(0)\right)$$

Lattice 3pt Functions

proton

$$G_\Gamma(t,\tau;\vec{p}',\vec{p}) = \sum_{\vec{x}_2,\vec{x}_1} e^{-i\vec{p}'\cdot(\vec{x}_2-\vec{x}_1)} e^{-i\vec{p}\cdot\vec{x}_1}\, \Gamma_{\beta\alpha} \langle\Omega| T\left[\chi_\alpha(t,\vec{x}_2)\,\mathcal{O}(\tau,\vec{x}_1)\,\overline{\chi}_\beta(0)\right]|\Omega\rangle$$

- Use the following interpolating operator to create a proton

$$\chi_\alpha(x) = \epsilon^{abc}\left(u^{Ta}(x)\; C\gamma_5\; d^b(x)\right) u^c_\alpha(x)$$

- And insert the local operator (quark bi-linear) $\bar{q}(x)\mathcal{O}q(x)$

$\mathcal{O}$: Combination of γ matrices and derivatives

- Perform all possible (connected) Wick contractions

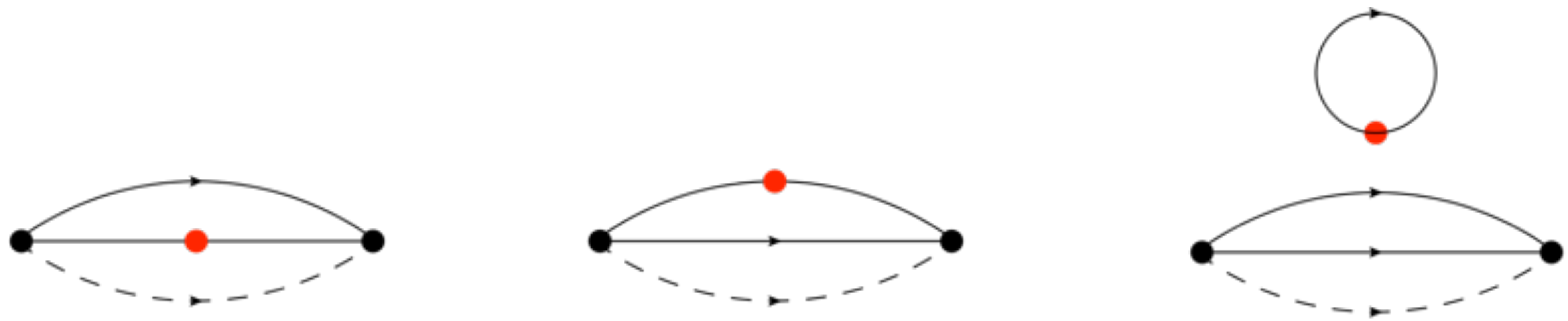
d-quark (2 terms)

$$\epsilon^{abc}\epsilon^{a'b'c'}\left(u^{Ta}(x_2)\; C\gamma_5\; d^b(x_2)\right) u^c_\alpha(x_2)\bar{d}(x_1)\mathcal{O}d(x_1)\bar{u}^{c'}(0)\left(\bar{d}^{b'}(0)C\gamma_5\bar{u}^{Ta'}(0)\right)$$

Lattice 3pt Functions

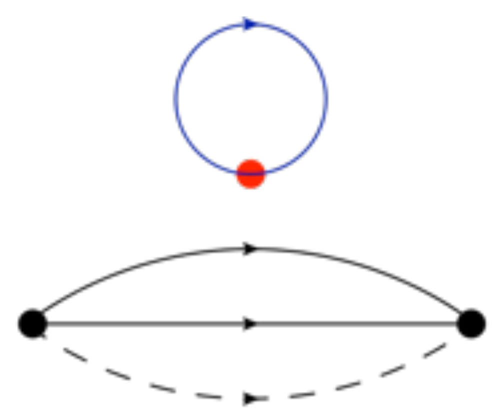
proton

- Pictorially:
  - u-quark



  - d-quark

  - s-quark



  - quark-line disconnected contributions drop out in isovector quantities (*u*-*d*) if isospin is exact ($m_u = m_d$)