Lattice Hadron Physics

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CSSM Summer School, February 11 - 15, 2013, CSSM, Adelaide, Australia
Excellent agreement between different collaborations/lattice formulations
Spectroscopy

• Rich spectrum of particles observed in experiment

• Some deviate from standard quark model predictions

• Other “missing states” are yet to be observed

• Can we understand the full hadron spectrum from QCD?
Experiments

- Many experiments dedicated to study meson and baryon spectroscopy
  - GlueX and CLAS12 [Jefferson Lab]
  - Compass [CERN]
  - BES III [Beijing]
Spectroscopy

- Recall: Masses (energies) are extracted from (Euclidean) time-dependence of lattice correlators
  \[ G(\vec{p}, t) = \sum_{n=1}^{N} \frac{e^{-E_n t}}{2E_n} \langle \Omega | O_f (0) | n, p \rangle \langle n, p | O_i^\dagger (0) | \Omega \rangle \]

- In general, works well for ground states (for large enough \( t \), fit a single exponential)

- Extracting excited states from multi-exponential fits difficult

- Try to optimise \( O \) to isolate state of interest. i.e. make
  \[ Z_n(\vec{p}) \equiv \frac{|\langle \Omega | O | n \rangle|^2}{2E_n(\vec{p})} \]

- large for state, \( n \), of interest as small for other states
Operators

• Hadrons are extended objects \( \sim 1 \text{fm} \)

• Using propagators computed from a point source perhaps only have small overlap with states of interest

• Gauge-invariant Gaussian smearing starts with a point source

\[
\psi_0^a \alpha = \delta^{ac} \delta_{\alpha \gamma} \delta_{\vec{x}, \vec{x}_0} \delta_{t, t_0}
\]

• and proceeds by the iterative scheme

\[
\psi_i(\vec{x}, t) = \sum_{\vec{x}'} F(\vec{x}, \vec{x}') \psi_{i-1}(\vec{x}', t)
\]

\[
F(\vec{x}, \vec{x}') = \frac{1}{(1 + \alpha)} \left( \delta_{\vec{x}, \vec{x}'} + \frac{\alpha}{6} \sum_{\mu=1}^{3} \left[ U_{\mu}(\vec{x}, t) \delta_{\vec{x}', \vec{x} + \hat{\mu}} + U_{\mu}^\dagger(\vec{x} - \hat{\mu}, t) \delta_{\vec{x}', \vec{x} - \hat{\mu}} \right] \right)
\]

• Repeating \( N \) times gives the resulting fermion source

\[
\psi_N(\vec{x}, t) = \sum_{\vec{x}'} F^N(\vec{x}, \vec{x}') \psi_0(\vec{x}', t)
\]
Example Effective Masses

Improved overlap with ground states
Extracting Excited States

- Different operators/smearing have differing overlap strengths with a variety of states of interest

- Set up a Generalised Eigenvalue Problem to cleanly isolate individual states

- E.g., express \((p=0)\) baryon 2pt function as a sum over states \(\alpha\)

\[
G_{ij}^\pm(t) = \sum_x \text{Tr} \left\{ \Gamma_\pm \langle \Omega | \mathcal{O}_i(x) \mathcal{O}_j(0) | \Omega \rangle \right\}
\]

\[
= \sum_\alpha \lambda_i^\alpha \lambda_j^\alpha e^{-m_\alpha t}
\]

• Look for a linear combination \(\mathcal{O}_j u_j^\alpha\) that cleanly isolates state \(\alpha\) with overlap \(z_\alpha\) at time \(t_0\)

\[
G_{ij}(t_0) u_j^\alpha = \lambda_i^\alpha z_\alpha e^{-m_\alpha t_0}
\]

• which leads to

\[
G_{ij}(t_0 + \Delta t) u_j^\alpha = e^{-m_\alpha \Delta t} G_{ij}(t_0) u_j^\alpha
\]
Extracting Excited States

• Multiplying on the left by $[G_{ij}(t_0)]^{-1}$ gives the generalised eigenvalue equation

$$[(G(t_0))^{-1}G(t_0 + \Delta t)]_{ij} u_j^\alpha = c^\alpha u_i^\alpha$$

• with eigenvalues

$$c^\alpha = e^{-m_\alpha \Delta t}$$

• and similarly

$$v_i^\alpha [G(t_0 + \Delta t)(G(t_0))^{-1}]_{ij} = c^\alpha v_j^\alpha$$

• We can diagonalise the correlation matrix

$$v_i^\alpha G_{ij}(t) u_j^\beta \propto \delta^{\alpha\beta}$$

• allowing fits at earlier times
Extracting Excited States

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• We can diagonalise the correlation matrix

$$ v_i^\alpha G_{ij}(t) u_j^\beta \propto \delta^{\alpha\beta} $$

• allowing fits at earlier times

• and cleanly isolate excited states

Figure 2: (Color online). Masses of the nucleon, $N_1$ and $N_2$ states, from the projected correlation functions as shown in Eq. (2.7). Each set of ground (g.s) and excited (e.s) states masses correspond to the diagonalization of the correlation matrix for each set of variational parameters $t_0$ (shown in major tick marks) and $\Delta t$ (shown in minor tick marks). Figure corresponds to $k_{ud} = 13754$ and for the 3rd basis.

The agreement among the three lowest lying eigenstates is remarkable and verifies that our approach successfully isolates true eigenstates [6, 14]. Basis number 3 has good diversity including both lower and higher smearings which is necessary for the extraction of masses over the entire heavy to light quark mass range. As a result, basis 5 is desirable to work with.
Excited States

FIG. 4: (Color online). Masses of the low-lying positive-parity spectrum of dynamical QCD (full symbols) and quenched QCD results (open symbols) from Ref. [26].

FIG. 5: (Color online). A comparison of the low-lying significant di-particle (full symbols) and non-particle (open symbols) for the Roper (filled triangles) and P-wave N+π (dashed line) and S-wave N+π+π (dotted line).
Spectroscopy

• Meson states allowed by the quark model

\[ J^{PC} = 0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 2^{--}, 2^{+-}, \ldots \]

• States with \( P=(-1)^J \) but \( CP=-1 \) forbidden

\[ J^{PC} = 0^{+-}, 0^{--}, 1^{-+}, 2^{+-}, 3^{--}, \ldots \]

• The are exotic states: not just a \( q\bar{q} \) pair

• Need to consider a large basis of operators to isolate higher spin states and access exotic quantum numbers

• E.g.

\[ \bar{\psi}\Gamma\psi \quad \bar{\psi}\Gamma\vec{D}\psi \quad \bar{\psi}\Gamma\vec{D}\vec{D}\psi \quad \bar{\psi}\Gamma F_{\mu\nu}\psi \]

\[ \vec{D} = \frac{1}{2}(\vec{D} - \vec{D}^\dagger) \]

• Solve Generalised Eigenvalue Problem to isolate individual states
The Excited Hadron Spectrum

Isoscalar Mesons

- Includes high spin and light exotic states

- Most states identically flavour mixed strange-light mixing

\[ \alpha_{\eta-\eta'} = 42(1)^{\circ} \quad \alpha_{\omega-\phi} = 1.7(2)^{\circ} \]

\[ m_\pi = 396 \text{ MeV} \]
The Excited Hadron Spectrum

Baryons

[Edwards et al. 1212.5236]

$E \approx 390$ MeV
Resonances

- Many states are actually resonances

- If quarks are light enough, state can decay strongly

  - e.g. \( \rho \rightarrow \pi \pi \quad \Delta^{++} \rightarrow p\pi^+ \)

- How can we be sure that the state we measure is the true resonance state and not some superposition of states?

  - [see Ross’ talk yesterday]

- Include two particle operators into your simulations

- Map out volume-dependence of energy levels

  - Wiese [Lat88] "free" 2-particle discrete spectrum

- Volume
Two Particle States

• Example $\rho \rightarrow \pi \pi$

• If you are below threshold (particle can decay), need to consider two particle system

• Not only need $\rho \rightarrow \rho$ two-point function

• But also

\[
\begin{align*}
\rho & \quad \rightarrow \quad \pi \\
\pi & \quad \rightarrow \quad \rho \\
\end{align*}
\]
Two Particle States

- Recall meson 2pt function
  \[ \langle \Omega | \mathcal{O}(x) \mathcal{O}^\dagger(0) | \Omega \rangle = -\langle \Omega | \bar{\psi}_a(x) \Gamma \psi_b(x) \bar{\psi}_b(0) \Gamma^\dagger \psi_a(0) | \Omega \rangle \]

- For two-particle system \( \pi^+ \pi^- \rightarrow \pi^+ \pi^- \)

\[ \langle \Omega | \bar{d}(x) \Gamma u(x) \bar{u}(x) \Gamma \bar{d}(x) \bar{u}(0) \Gamma^\dagger \bar{d}(0) \bar{u}(0) \Gamma^\dagger u(0) | \Omega \rangle \]
Recall meson 2pt function

\[ \langle \Omega | \mathcal{O}(x) \mathcal{O}^\dagger(0) | \Omega \rangle = - \langle \Omega | \bar{\psi}_a(x) \Gamma \psi_b(x) \bar{\psi}_b(0) \Gamma^\dagger \psi_a(0) | \Omega \rangle \]

For two-particle system \( \pi^+ \pi^- \rightarrow \pi^+ \pi^- \)

\[ \langle \Omega | \bar{d}(x) \Gamma u(x) \bar{u}(x) \Gamma d(x) \bar{u}(0) \Gamma^\dagger d(0) \bar{d}(0) \Gamma^\dagger u(0) | \Omega \rangle \]

\[ \pi \quad \pi \]

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Two Particle States

- Recall meson 2pt function

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\langle \Omega | \mathcal{O}(x) \mathcal{O}^\dagger(0) | \Omega \rangle = -\langle \Omega | \bar{\psi}_a(x) \Gamma \psi_b(x) \bar{\psi}_b(0) \Gamma^\dagger \psi_a(0) | \Omega \rangle
\]

- For two-particle system \( \pi^+ \pi^- \rightarrow \pi^+ \pi^- \)

\[
\langle \Omega | \bar{d}(x) \Gamma u(x) \bar{u}(x) \Gamma d(x) \bar{u}(0) \Gamma^\dagger d(0) \bar{d}(0) \Gamma^\dagger u(0) | \Omega \rangle
\]
Two Particle States

• Recall meson 2pt function

\[ \langle \Omega | \mathcal{O}(x) \mathcal{O}^\dagger(0) | \Omega \rangle = -\langle \Omega | \bar{\psi}_a(x) \Gamma \psi_b(x) \bar{\psi}_b(0) \Gamma^\dagger \psi_a(0) | \Omega \rangle \]

• For two-particle system $\pi^+ \pi^- \rightarrow \pi^+ \pi^-$

\[ \langle \Omega | \bar{d}(x) \Gamma u(x) \bar{u}(x) \Gamma d(x) \bar{u}(0) \Gamma^\dagger d(0) \bar{d}(0) \Gamma^\dagger u(0) | \Omega \rangle \]

Requires “all-to-all” propagator

\[ S(x, x) \]
Extracting Resonance

• Construct correlation matrix

\[
\begin{pmatrix}
G_{\rho \to \rho} & G_{\rho \to \pi \pi} \\
G_{\pi \pi \to \rho} & G_{\pi \pi \to \pi \pi}
\end{pmatrix}
\]

• Diagonalise to extract two energy levels

\[
\begin{pmatrix}
G_1 & 0 \\
0 & G_2
\end{pmatrix}
\]

\[E_1, E_2\]
Extracting Resonance

Dudek et al. [1212.0830]
Spectroscopy

- Light hadron spectrum is in good shape
- Significant advancement in the determination of excited states
- Now confronted with new challenges:
  - Isospin breaking
    (See plenary talk by T. Izubuchi at Lattice 2012)
  - QED effects
  - Resonances
    (See plenary talk by D. Mohler at Lattice 2012)
Isospin Breaking \((m_d - m_u)\)

- To date, all Lattice simulations have been performed with degenerate up and down quark masses
  \[ N_f = 2, \quad N_f = 2 + 1, \quad N_f = 2 + 1 + 1 \]
- Progress by several collaborations in determining the \((m_d - m_u)\) effect on some observables

QCDSF (1206.3156):
\[
M_n - M_p = 3.13(55) \\
M_{\Sigma^-} - M_{\Sigma^+} = 8.10(136) \\
M_{\Xi^-} - M_{\Xi^0} = 4.98(85)
\]

Rome 123 (1110.6294):
\[
m_d - m_u(\bar{MS}, 2GeV) = 2.35(25) \\
\frac{F_{K^+}/F_{\pi^+}}{F_K/F_{\pi}} - 1 = -0.0039(4) \\
M_n - M_p = 2.8(7)
\]
QED Effects

(See plenary talk by T. Izubuchi at Lattice 2012)

• Many Lattice QCD results are achieving high precision
  \[ \Delta f_\pi, \Delta f_K \sim 1\%, \Delta (f_\pi/f_K) \sim 0.5\% \]

• QED effects may not be negligible and should be included

• Although in some cases, QED can be treated perturbatively, this is not always the case

QCD+QED Lattice simulation

• Currently two main methods employed:
  • Quenched QED
  • Dynamical QED via reweighting
QED Effects

(See plenary talk by T. Izubuchi at Lattice 2012)

• E.g. Quark masses from QCD+QED simulation [PRD82, 094508 (2010)]

\begin{align*}
m_u &= 2.24 \pm 0.10 \pm 0.34 \text{ MeV} \\
m_d &= 4.65 \pm 0.15 \pm 0.32 \text{ MeV} \\
m_s &= 97.6 \pm 2.9 \pm 5.5 \text{ MeV} \\
m_d - m_u &= 2.411 \pm 0.065 \pm 0.476 \text{ MeV} \\
m_{ud} &= 3.44 \pm 0.12 \pm 0.22 \text{ MeV} \\
m_u/m_d &= 0.4818 \pm 0.0096 \pm 0.0860 \\
m_s/m_{ud} &= 28.31 \pm 0.29 \pm 1.77,
\end{align*}

• and for n-p: \((M_n - M_p)^{QED} = -0.54(24)\)

• Combine with previous QCD result

\((M_n - M_p) = 2.14(42) \text{ MeV}\)

c.f. experiment: 1.2933321(4) MeV
Calculating Matrix Elements

\[ \langle H' | \mathcal{O} | H \rangle \]

\( H, H' : \pi, Kp, n, \ldots \)

\( \mathcal{O} : V_\mu, A_\mu, \ldots \)
Calculating Matrix Elements

Spin-0
\[ \langle \pi(p')|J^\mu(q)|\pi(p) \rangle = P^\mu F_\pi(q^2) \]

Spin-1/2
\[ \langle N(p', s')|J^\mu(q)|N(p, s) \rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(q^2) + i\sigma^{\mu\nu} \frac{q_\nu}{2m} F_2(q^2) \right] u(p, s) \]

Spin-1
\[ \langle \rho(p', s')|J^\mu(q)|\rho(p, s) \rangle = \]
\[ - (\epsilon^{\prime*} \cdot \epsilon) P^\mu G_1(Q^2) - [(\epsilon^{\prime*} \cdot q)\epsilon^\mu - (\epsilon \cdot q)\epsilon^{\prime*\mu}] G_2(Q^2) + (\epsilon \cdot q)(\epsilon^{\prime*} \cdot q) \frac{P^\mu}{(2m_\rho)^2} G_3(Q^2) \]

Spin-3/2
\[ \langle \Delta(p', s')|J^\mu(q)|\Delta(p, s) \rangle = \]
\[ \bar{u}_\alpha(p', s') \left\{ - g^{\alpha\beta} \left[ \gamma^\mu a_1(Q^2) + \frac{P^\mu}{2M_\Delta} a_2(Q^2) \right] - \frac{q^\alpha q^\beta}{(2M_\Delta)^2} \left[ \gamma^\mu c_1(Q^2) + d \frac{P^\mu}{2M_\Delta} c_2(Q^2) \right] \right\} u_\beta(p, s) \]

\[ q^2 = -Q^2 = (p' - p)^2 \]
\[ P^\mu = p'^\mu + p^\mu \]
Lattice 3pt Functions

\[ \langle \Omega | T (\bar{\chi}_\beta(0)) | \Omega \rangle \]

- Create a state (with quantum numbers of the proton) at time \( t=0 \)
Lattice 3pt Functions

\[ \langle \Omega | T (\mathcal{O}(\vec{x}_1, \tau) \bar{\chi}_\beta(0)) | \Omega \rangle \]

- Create a state (with quantum numbers of the proton) at time \( t=0 \)
- Insert an operator, \( \mathcal{O} \), at some time \( \tau \)
Lattice 3pt Functions

\[ \langle \Omega | T(\chi_\alpha(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \chi_\beta(0)) | \Omega \rangle \]

- Create a state (with quantum numbers of the proton) at time \( t=0 \)
- Insert an operator, \( \mathcal{O} \), at some time \( \tau \)
- Annihilate state at final time \( t \)
Lattice 3pt Functions

\[
G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-ip' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-ip \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T (\chi_\alpha (\vec{x}_2, t) \mathcal{O} (\vec{x}_1, \tau) \overline{\chi}_\beta (0)) | \Omega \rangle
\]

- Create a state (with quantum numbers of the proton) at time \( t = 0 \)
- Insert an operator, \( \mathcal{O} \), at some time \( \tau \)
- Annihilate state at final time \( t \)
Lattice 3pt Functions

\[ G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta \alpha} \langle \Omega | T \left( \chi_{\alpha}(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \bar{\chi}_{\beta}(0) \right) | \Omega \rangle \]

- Insert complete set of states
  \[ I = \sum_{B', p', s'} |B', p', s'\rangle \langle B', p', s'| \quad I = \sum_{B, p, s} |B, p, s\rangle \langle B, p, s| \]

- Make use of translational invariance
  \[ \chi(\vec{x}, t) = e^{\hat{H}t} e^{-i\hat{P} \cdot \vec{x}} \chi(0) e^{i\hat{P} \cdot \vec{x}} e^{-\hat{H}t} \]

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{B, B'} \sum_{s, s'} e^{-E_B'(\vec{p})'(t-\tau)} e^{-E_B(\vec{p})\tau} \Gamma_{\beta \alpha} \]
\[ \times \langle \Omega | \chi_{\alpha}(0) | B', p', s'\rangle \langle B', p', s' | \mathcal{O}(\vec{q}) | B, p, s \rangle \langle B, p, s | \bar{\chi}_{\beta}(0) | \Omega \rangle \]

- Evolve to large Euclidean times to isolate ground state \( 0 \ll \tau \ll t \)

\[ G(t, \tau, \vec{p}, \vec{p}') = \sum_{s, s'} e^{-E_{B'}'(t-\tau)} e^{-E_B\tau} \Gamma_{\beta \alpha} \langle \Omega | \chi_{\alpha}(0) | N(p', s')\rangle \langle N(p', s') | \mathcal{O}(\vec{q}) | N(p, s) \rangle \langle Np, s | \bar{\chi}_{\beta}(0) | \Omega \rangle \]
Consider a pion 3pt function

\[
G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \langle \Omega | T (\chi(\vec{x}_2, t) \mathcal{O}(\vec{x}_1, \tau) \chi^\dagger(0)) | \Omega \rangle
\]

With interpolating operator \( \chi(x) = \bar{d}(x)\gamma_5 u(x) \)

And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \)

\[
-\bar{d}(x_2)\gamma_5 u(x_2)\bar{u}(x_1)\mathcal{O} u(x_1)\bar{u}(0)\gamma_5 d(0)
\]

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

\( u \)-quark
Lattice 3pt Functions

- Consider a pion 3pt function
  \[ G(t, \tau, p, p') = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \langle \Omega | T \left( \chi(\vec{x}_2, t) \, \mathcal{O}(\vec{x}_1, \tau) \, \chi^\dagger(0) \right) | \Omega \rangle \]

- With interpolating operator \( \chi(x) = \bar{d}(x) \gamma_5 u(x) \)

- And insert the local operator (quark bi-linear) \( \bar{q}(x) \mathcal{O} q(x) \)
  \[ -\bar{d}(x_2) \gamma_5 u(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}(0) \gamma_5 d(0) \]

\( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

u-quark
Lattice 3pt Functions

\[ -\bar{d}^a_{\beta}(x_2)\gamma_{5\beta\gamma}u^a_{\gamma}(x_2)\bar{u}^b_{\rho}(x_1)\Gamma_{\rho\delta}u^b_{\delta}(x_1)\bar{u}^c_{\xi}(0)\gamma_{5\xi\alpha}d^c_{\alpha}(0) \]

• all possible Wick contractions
Lattice 3pt Functions

- all possible Wick contractions

- connected

\[ -\bar{d}_\beta^a(x_2)\gamma_5\beta\gamma u_\gamma^a(x_2)\bar{u}_\rho^b(x_1)\Gamma_{\rho\delta} u_\delta^b(x_1)\bar{u}_\xi^c(0)\gamma_5\xi\alpha d_\alpha^c(0) \]

\[ S_{d\alpha\beta}^{ca}(0, x_2)\gamma_5\beta\gamma S_{u\gamma\rho}^{ab}(x_2, x_1)\Gamma_{\rho\delta} S_{u\delta\xi}^{bc}(x_1, 0)\gamma_5\xi\alpha \]
Lattice 3pt Functions

- all possible Wick contractions

- connected

\[ S_{d\alpha\beta}^{ca}(0, x_2) \gamma_{5\beta\gamma} S_{u\gamma\rho}^{ab}(x_2, x_1) \Gamma_{\rho\delta} S_{u\delta\xi}^{bc}(x_1, 0) \gamma_{5\xi\alpha} \]

- disconnected

\[ -S_{d\alpha\beta}^{ca}(0, x_2) \gamma_{5\beta\gamma} S_{u\gamma\xi}^{ac}(x_2, 0) \gamma_{5\xi\alpha} S_{u\delta\rho}^{bb}(x_1, x_1) \Gamma_{\rho\delta} \]
Lattice 3pt Functions

\[-\bar{d}^a_\beta(x_2)\gamma_5\beta\gamma u^a_\gamma(x_2)\bar{u}^b_\rho(x_1)\Gamma_{\rho\delta} u^b_\delta(x_1)\bar{u}^c_\xi(0)\gamma_5\xi\alpha d^c_\alpha(0)\]

- all possible Wick contractions

- connected

\[\text{Tr} \left[ S_d(0, x_2)\gamma_5 S_u(x_2, x_1)\Gamma S_u(x_1, 0)\gamma_5 \right] \]

- disconnected

\[\text{Tr} \left[ -S_d(0, x_2)\gamma_5 S_u(x_2, 0)\gamma_5 \right] \text{Tr} \left[ S_u(x_1, x_1)\Gamma \right] \]
Lattice 3pt Functions

- all possible Wick contractions

- connected

\[ \text{Tr} \left[ S_{d}^\dagger(x_2, 0) S_u(x_2, x_1) \Gamma S_u(x_1, 0) \right] \]

- disconnected

\[ \text{Tr} \left[ - S_{d}^\dagger(x_2, 0) S_u(x_2, 0) \right] \text{Tr} \left[ S_u(x_1, x_1) \Gamma \right] \]

- all-to-all propagators

\[ -\bar{d}_\beta(x_2) \gamma_{5\beta\gamma} u_\gamma(x_2) \bar{u}_\rho(x_1) \Gamma_{\rho\delta} u_\delta(x_1) \bar{u}_\xi(0) \gamma_{5\xi\alpha} d_\alpha(0) \]

\[ S^\dagger(x, 0) = \gamma_5 S(0, x) \gamma_5 \]
Lattice 3pt Functions

\[ G_\Gamma(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_\alpha(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_\beta(0) \right] | \Omega \rangle \]

- Use the following interpolating operator to create a proton

\[ \chi_\alpha(x) = \epsilon^{abc} \left( u^T \alpha(x) \ C \gamma_5 \ d^b(x) \right) u^c(x) \]

- And insert the local operator (quark bi-linear)

\[ \bar{q}(x) \mathcal{O} q(x) \quad \mathcal{O}: \text{Combination of } \gamma \text{ matrices and derivatives} \]

- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^T \alpha(x_2) \ C \gamma_5 \ d^b(x_2) \right) u^c_\alpha(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^{c'}(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^T \alpha'(0) \right) \]
Lattice 3pt Functions

\[ G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0) \right] \rangle \]

- Use the following interpolating operator to create a proton
  \[ \chi_{\alpha}(x) = \epsilon^{abc} \left( u^T a(x) \ C \gamma_5 \ d^b(x) \right) u^c(x) \]
- And insert the local operator (quark bi-linear) \[ \bar{q}(x) \mathcal{O} q(x) \]
  \[ \mathcal{O}: \text{Combination of } \gamma \text{ matrices and derivatives} \]
- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^T a(x_2) \ C \gamma_5 \ d^b(x_2) \right) u^c(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^T a'(0) \right) \]
Lattice 3pt Functions

\[ G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_\beta(0) \right] | \Omega \rangle \]

- Use the following interpolating operator to create a proton
  \[ \chi_{\alpha}(x) = \varepsilon^{abc} (u^T \gamma^a(x) C \gamma_5 \gamma^b(x)) u^c(x) \]
- And insert the local operator (quark bi-linear) \[ \bar{q}(x) \mathcal{O} q(x) \] \( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives
- Perform all possible (connected) Wick contractions

\[ \varepsilon^{abc} \varepsilon^{a'b'c'} (u^T \gamma^a(x_2) C \gamma_5 \gamma^b(x_2)) u^c(x_2) \bar{u}(x_1) \mathcal{O} u(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^T \gamma^a'(0) \right) \]
Lattice 3pt Functions

\[ G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T [ \chi_{\alpha}(t, \vec{x}_2) O(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0) ] | \Omega \rangle \]

- Use the following interpolating operator to create a proton
  \[ \chi_{\alpha}(x) = \epsilon^{abc} (u^{Ta}(x) \, C\gamma_5 \, d^b(x)) \, u^c_\alpha(x) \]

- And insert the local operator (quark bi-linear)
  \[ \bar{q}(x)Oq(x) \quad O: \text{Combination of } \gamma \text{ matrices and derivatives} \]

- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} (u^{Ta}(x_2) \, C\gamma_5 \, d^b(x_2)) \, u^c_\alpha(x_2) \bar{u}(x_1)Ou(x_1)\bar{u}^c(0) \left( \bar{d}^{b'}(0)C\gamma_5 \bar{u}^{Ta'}(0) \right) \]
Lattice 3pt Functions

\[ G_\Gamma(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T \left[ \chi_\alpha(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_\beta(0) \right] | \Omega \rangle \]

- Use the following interpolating operator to create a proton
  \[ \chi_\alpha(x) = \epsilon^{abc} \left( u^T a(x) \ C \gamma_5 \ d^b(x) \right) u^c_\alpha(x) \]
- And insert the local operator (quark bi-linear) \[ \bar{q}(x) \mathcal{O} q(x) \]
  \( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives
- Perform all possible (connected) Wick contractions

\[ \epsilon^{abc} \epsilon^{a'b'c'} \left( u^T a(x_2) \ C \gamma_5 \ d^b(x_2) \right) u^c_\alpha(x_2) \bar{d}(x_1) \mathcal{O} d(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^T a'(0) \right) \]

\( \bar{d} \)-quark (2 terms)
\( G_{\Gamma}(t, \tau; \vec{p}', \vec{p}) = \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}' \cdot (\vec{x}_2 - \vec{x}_1)} e^{-i\vec{p} \cdot \vec{x}_1} \Gamma_{\beta\alpha} \langle \Omega | T [\chi_{\alpha}(t, \vec{x}_2) \mathcal{O}(\tau, \vec{x}_1) \bar{\chi}_{\beta}(0)] | \Omega \rangle \)

- Use the following interpolating operator to create a proton
  \( \chi_{\alpha}(x) = \epsilon^{abc} (u^{Ta}(x) C \gamma_5 d^b(x)) u^c_{\alpha}(x) \)

- And insert the local operator (quark bi-linear)
  \( \bar{q}(x)\mathcal{O}q(x) \)
  \( \mathcal{O} \): Combination of \( \gamma \) matrices and derivatives

- Perform all possible (connected) Wick contractions

\( \epsilon^{abc} \epsilon^{a'b'c'} (u^{Ta}(x_2) C \gamma_5 d^b(x_2)) u^c_{\alpha}(x_2) \bar{d}(x_1)\mathcal{O}d(x_1) \bar{u}^c(0) \left( \bar{d}^{b'}(0) C \gamma_5 \bar{u}^{Ta'}(0) \right) \)
Lattice 3pt Functions

• Pictorially:

  • u-quark
  ![Diagram](image1)

  • d-quark
  ![Diagram](image2)

  • s-quark
  ![Diagram](image3)

• quark-line disconnected contributions drop out in isovector quantities \((u-d)\) if isospin is exact \((m_u=m_d)\)