Lattice Hadron Physics

James Zanotti
The University of Adelaide

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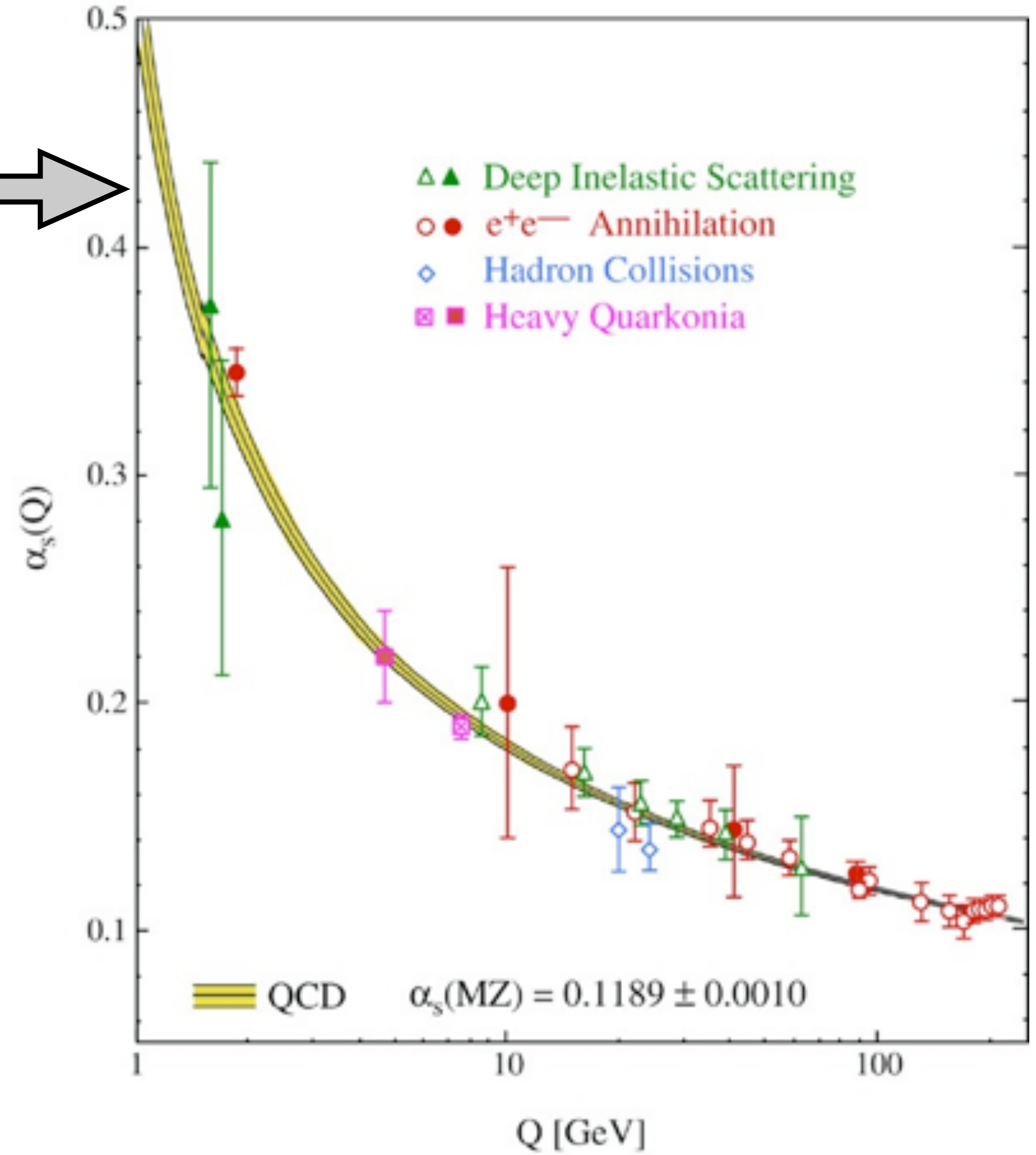
Outline

- Lecture 1: Introduction to Lattice QCD
- Lecture 2: Spectroscopy and related issues
- Lecture 3: Hadronic matrix elements on the lattice, Nucleon form factors
- Lecture 4: Moments of parton distribution functions and generalised parton distribution functions

Why Lattice?

- Seen need for non-perturbative methods
- Strong coupling constant large at low-energies
- Quarks confined inside hadrons
- Perturbation theory no longer useful
- Need a nonperturbative method
    - Lattice QCD



Why Lattice?

- A lattice regularisation provides a non-perturbative tool for calculating quantities such as the hadron spectrum, form factors, ...
- Also used to address issues like the mechanism for confinement and chiral symmetry breaking
- Discretise space-time into a 4-dimensional grid
  - LGT can be simulated on a computer using methods similar to Statistical Mechanics
- Can tune input parameters (e.g. quark masses)
  - Make predictions on the dependence of quantities on these parameters

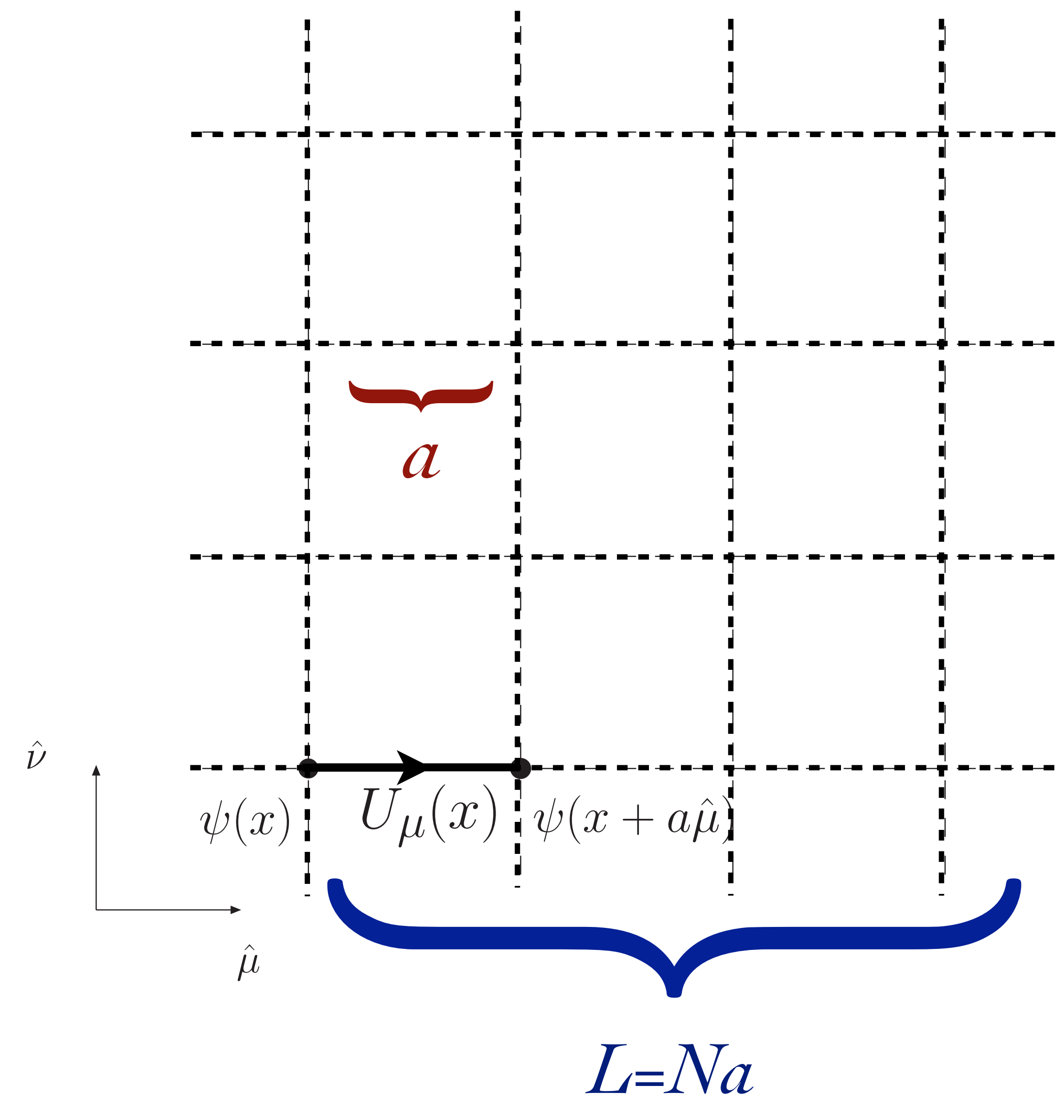
⇨ Make contact with Chiral Perturbation Theory

The Lattice

Ken Wilson (1974)

The Basics

- Discretise space-time with lattice spacing *a* volume *L*^{3}x*T*
- Quark fields reside on sites $\psi(x)$
- Gauge fields on the links $U_\mu(x) = e^{-iagA_\mu(x)}$
- Approximate the QCD path integral by Monte Carlo methods
- Use a big computer



The Lattice

Ken Wilson (1974)

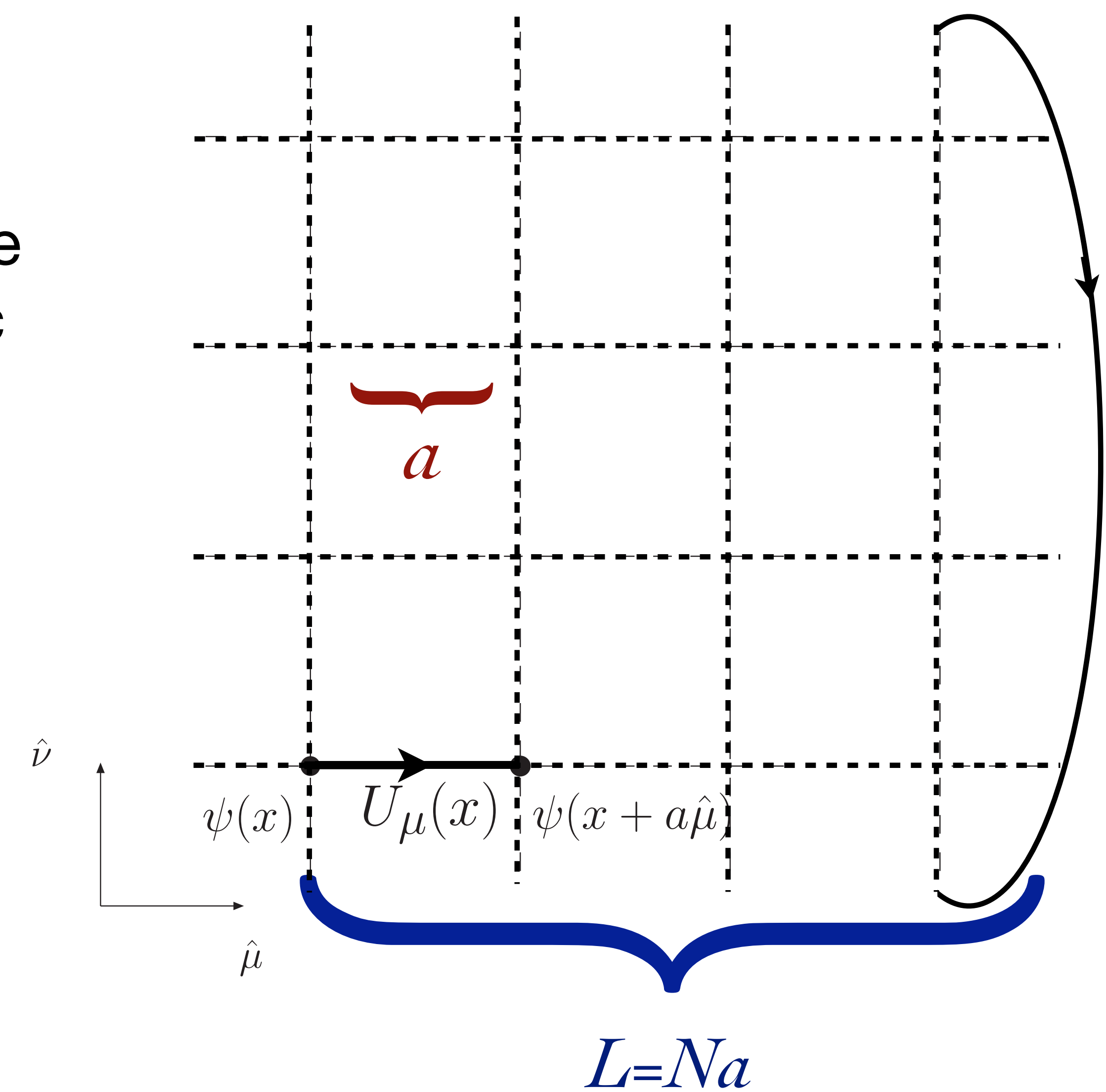
The Basics

- Work in Euclidean space $t \rightarrow i\tau$

- Given a lattice spacing, a, define the set of available space-time points to be restricted to the hypercubic lattice

$$\mathbb{L} \subset a\mathbb{Z}^4 = \{x | x^\mu = a n^\mu, \, n \in \mathbb{Z}^4\}$$

- If we have a finite lattice we usually introduce periodic boundary conditions ie formulate theory on the 4-torus



The Lattice

The Basics

- It is not possible to consider infinitesimal distances on the lattice

- replace derivatives by finite difference operators

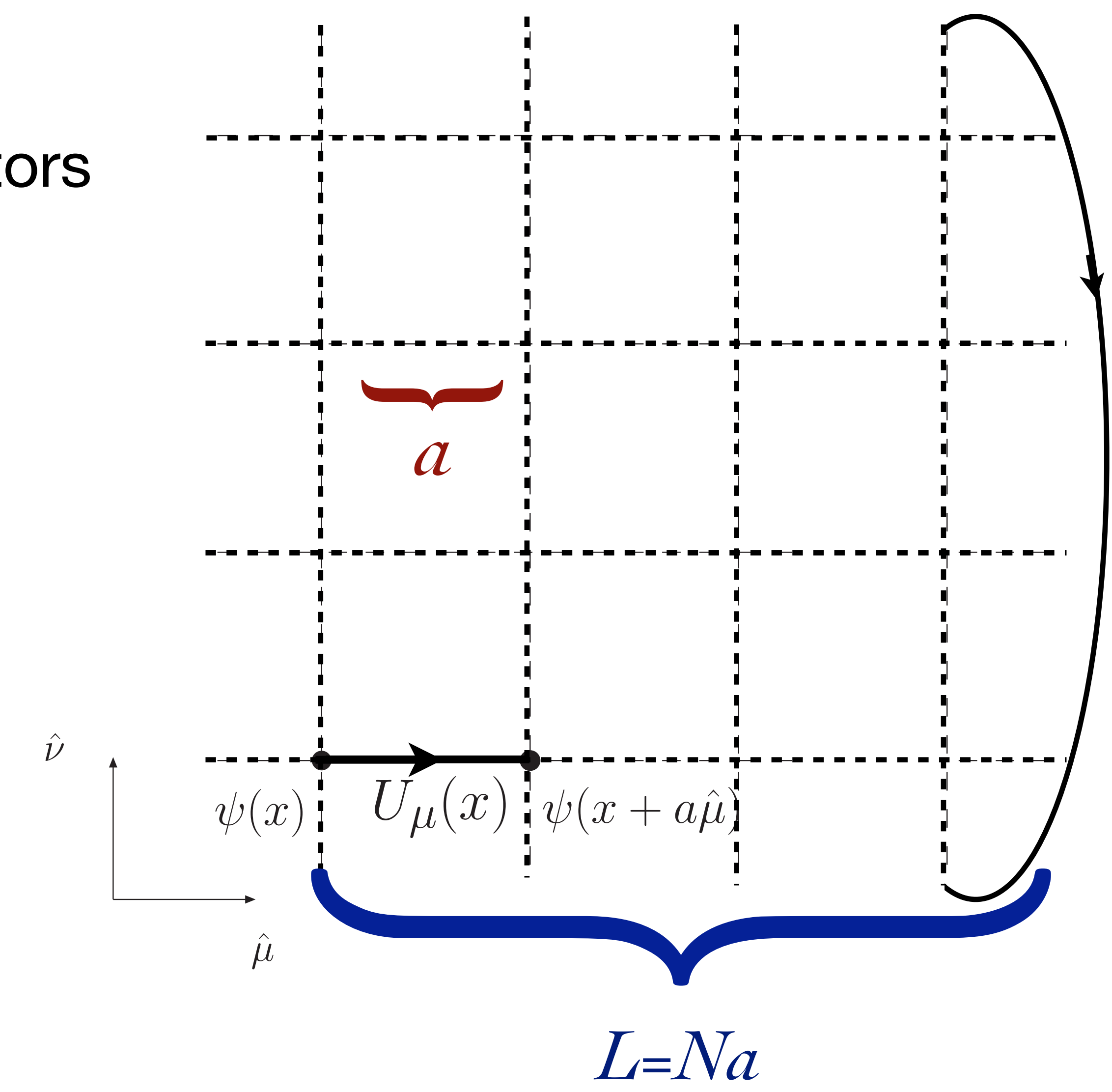
$$\partial_\mu \psi(x) \to \Delta_\mu \psi(x) = \frac{1}{a}(\psi(x+\hat{\mu}) - \psi(x))$$
$$\Delta^*_\mu \psi(x) = \frac{1}{a}(\psi(x) - \psi(x-\hat{\mu}))$$

- and integrals with sums $\int dx_i \to a \sum_{n_i}$

- As might be expected, when we introduce a minimum distance *a*, the corresponding generator of translations, momentum, is also affected

- Each component of 4-momentum is now restricted to the Brillouin zone

$$p_\mu \in (-\frac{\pi}{a}, \frac{\pi}{a}]$$



The Lattice

The Basics

- On a periodic lattice with spatial volume L[3], quark fields satisfy

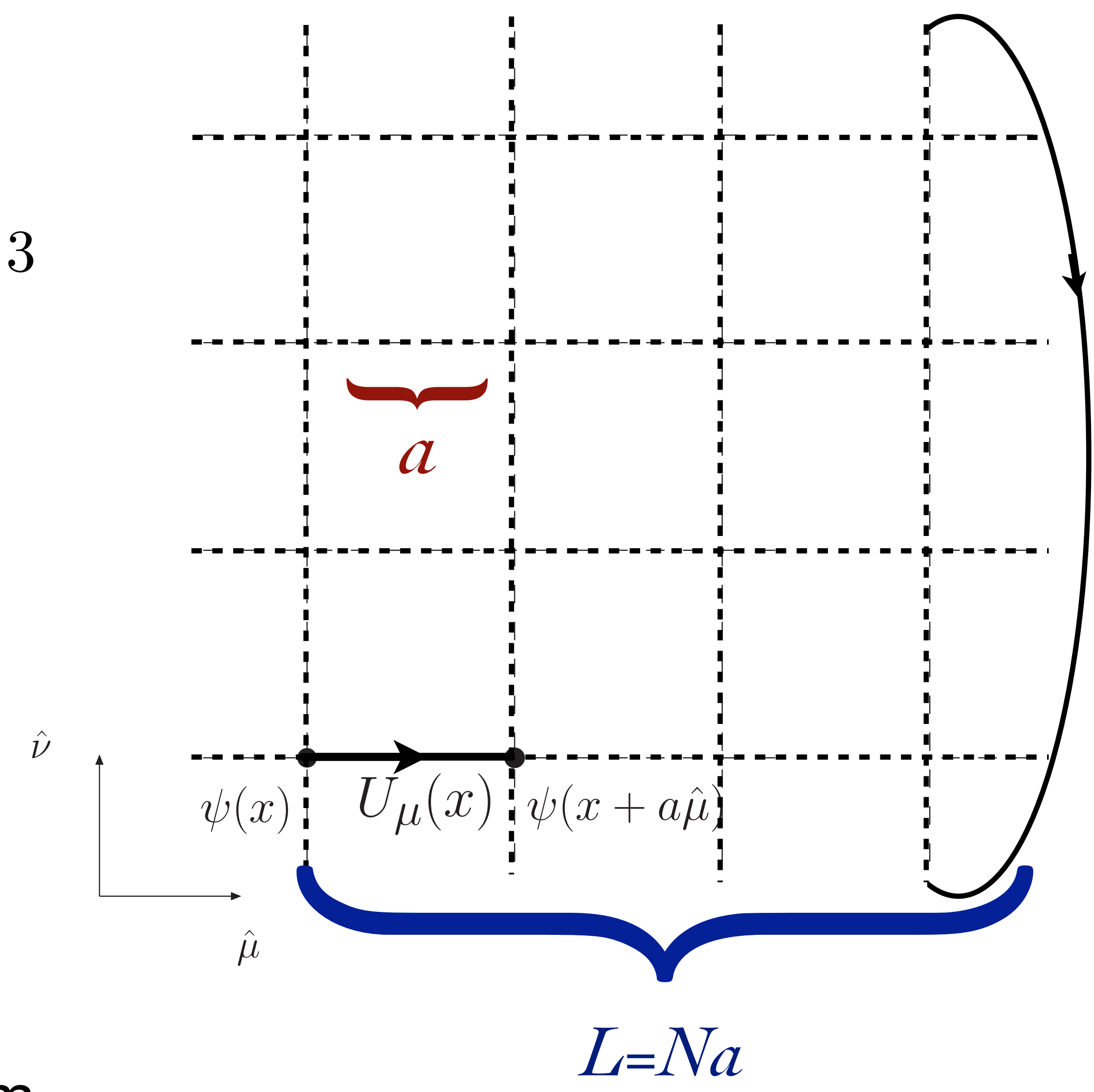
$$\psi(x+\vec{e}_i\,L) = \psi(x), \quad i=1,\,2,\,3$$

$$\int \mathrm{d}^4p\, e^{-ip(x+\vec{e}_i\,L)}\tilde{\psi}(p) = \int \mathrm{d}^4p\, e^{-ipx}\tilde{\psi}(p), \quad i=1,\,2,\,3$$

- so we see that momenta are discretised in units of

$$p_i = \frac{2\pi}{L}n_i, \quad i=1,\,2,\,3$$

- For typical lattices, smallest non-zero momentum ~400-500 MeV



- Poor momentum resolution

- Can affect phenomenological observables e.g. form factors

The Lattice

Actions

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(x)(\not{D} + m)\psi(x) \right)$$

- When constructing a lattice action, the most important feature to consider is local gauge symmetry

- Under gauge transformation $\Lambda(x)$ quark and gauge fields transform as

  - Quarks $\quad \psi(x) \rightarrow \Lambda(x)\psi(x)$

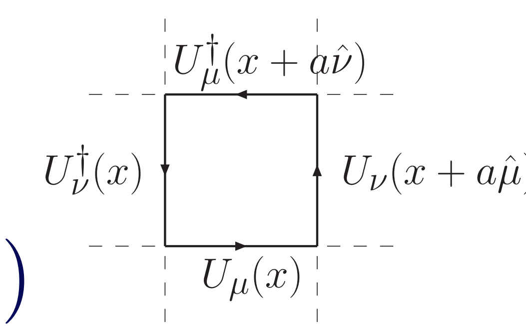
  - Gauge $\quad U_\mu(x) \rightarrow \Lambda(x) U_\mu(x) \Lambda^\dagger(x + a\hat{\mu})$

$$\Lambda \in SU(3)$$

- Invariant quantities:

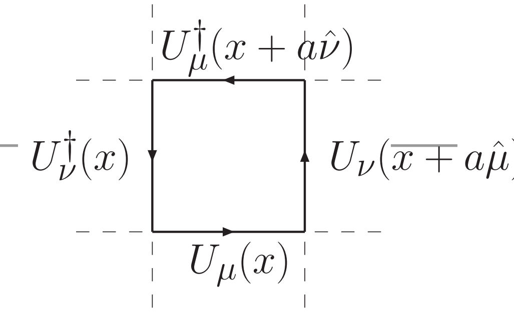
  - Include gauge fields in derivative $\quad \bar{\psi}(x)\psi(x+\hat{\mu}) \rightarrow \bar{\psi}(x) U_\mu(x) \psi(x + \hat{\mu})$

  - Trace of closed loops gauge invariant, e.g. Plaquette



$$P_{\mu\nu} = \frac{1}{3} \mathcal{R}e\mathrm{Tr} U_{\mathrm{Plaq}} = \frac{1}{3} \mathcal{R}e\mathrm{Tr} \left(U_\mu(x) U_\nu(x + a\hat{\mu}) U^\dagger_\mu(x + a\hat{\nu}) U^\dagger_\nu(x) \right)$$

Lattice Gauge Action



- Rewrite Plaquette

$$P_{\mu\nu} = \frac{1}{3}\mathcal{R}e\,\mathrm{Tr}\,\mathcal{P}e^{ig\oint_\square A\cdot \mathrm{d}x}$$

$$= \frac{1}{3}\mathcal{R}e\,\mathrm{Tr}\,\mathcal{P}\left[1 + ig\oint_\square A\cdot \mathrm{d}x - \frac{1}{2}\left(g\oint_\square A\cdot \mathrm{d}x\right)^2 + \mathcal{O}(A^3)\right]$$

- Stokes' Theorem implies

$$P_{\mu\nu} = 1 - \frac{1}{6}g^2\mathrm{Tr}\,F_{\mu\nu}^2 - \frac{1}{72}g^2a^2\mathrm{Tr}\,F_{\mu\nu}(\partial_\mu^2 + \partial_\nu^2)F_{\mu\nu} + \mathcal{O}(a^4) + \mathcal{O}(g^4a^2)$$

- where $F_{\mu\nu}$ is the non-abelian field strength tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

- The Wilson Gauge Action

$$\boxed{S_{\mathrm{Wil}} = \beta \sum_{\mathrm{Plaq}} \frac{1}{3}\mathrm{Re}\,\mathrm{Tr}(1 - U_{\mathrm{Plaq}}), \quad \beta = \frac{6}{g^2}}$$

Lattice Gauge Action

- Wilson gauge action differs from the continuum action at $\mathcal{O}(a^2)$ and $\mathcal{O}(g^2 a^2)$
- Remove a^2 errors via Symanzik improvement scheme
- Use 1x2 and 2x1 Rectangular Loops, e.g.

$$R_{\mu\nu}^{1\times 2} = 1 - \frac{4}{6} g^2 a^4 \mathrm{Tr}\, F_{\mu\nu}^2 - \frac{4}{72} g^2 a^6 \mathrm{Tr}\, \left(F_{\mu\nu} (4\partial_\mu^2 + \partial_\nu^2) F_{\mu\nu} \right) - \ldots$$

- to remove $\mathcal{O}(a^2)$ errors gives the tree-level improved Symanzik gluon action

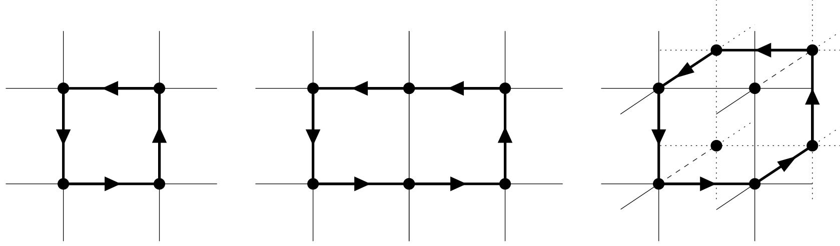
$$S_{\mathrm{Imp}} = \beta \left\{ c_0 \sum_{\mathrm{Plaq}} \frac{1}{3} \mathrm{Re\,Tr}(1 - U_{\mathrm{Plaq}}) + c_1 \sum_{\mathrm{Rect}} \frac{1}{3} \mathrm{Re\,Tr}(1 - U_{\mathrm{Rect}}) \right\}$$

- more generally, can include parallelogram (or "chair") diagrams with

$$c_0 + 8c_1 + 8c_2 = 1$$

Iwasaki: $c_1 = -0.331$, $c_2=0$

DBW2: $c_1 = -1.4069$, $c_2=0$



Lattice Quark Action

- The continuum Dirac operator $$\not{D} = \gamma^{\mu}(\partial_{\mu} + i\, g\, A_{\mu})$$

- is discretised by:

  - Replacing the derivative with a discrete difference, and

  - Including gauge links which $\begin{cases} \text{Encode the gluon field, } A_{\mu}\text{, and} \\ \text{Maintain gauge invariance} \end{cases}$

$$\bar{\psi}\,\not{\nabla}\psi = \frac{1}{2a}\bar{\psi}(x)\sum_{\mu}\gamma_{\mu}\Big[U_{\mu}(x)\psi(x+a\hat{\mu}) - U_{\mu}^{\dagger}(x-a\hat{\mu})\psi(x-a\hat{\mu})\Big]$$

$$\begin{aligned}\nabla_{\mu}\psi(x) &= \frac{1}{2}(\Delta_{\mu} + \Delta_{\mu}^{*})\psi(x) \\ &= \frac{1}{2a}[U_{\mu}(x)\psi(x+a\hat{\mu}) - U_{\mu}^{\dagger}(x-a\hat{\mu})\psi(x-a\hat{\mu})]\end{aligned}$$

Lattice Quark Action

- The continuum Dirac action is recovered in the limit $a \longrightarrow 0$ by Taylor expanding the links and $\psi(x+a\hat{\mu})$ in powers of the lattice spacing *a*

$$
\begin{aligned}
&\frac{1}{2a}\bar{\psi}(x)\gamma_\mu\big[(1+iagA_\mu(x+\frac{a\hat{\mu}}{2})+\ldots)(\psi(x)+a\psi'(x)+\ldots)- \\
&\qquad\qquad (1-iagA_\mu(x-\frac{a\hat{\mu}}{2})+\ldots)(\psi(x)-a\psi'(x)+\ldots)\big] \\
&= \bar{\psi}(x)\gamma_\mu(\partial_\mu+\mathcal{O}(a^2))\psi(x)+ig\bar{\psi}(x)\gamma_\mu\big[A_\mu+\mathcal{O}(a^2)\big]\psi(x)
\end{aligned}
$$

- Hence we arrive at the simplest ("naive") lattice fermion action

$$
\boxed{
\begin{aligned}
S_N &= m_q\sum_x \bar{\psi}(x)\psi(x) \\
&\quad + \frac{1}{2a}\sum_x \bar{\psi}(x)\gamma_\mu\Big[U_\mu(x)\psi(x+a\hat{\mu}) - U^\dagger_\mu(x-a\hat{\mu})\psi(x-a\hat{\mu})\Big]
\end{aligned}
}
$$

Naive Quark Action

- While preserving chiral symmetry, encounters the fermion - doubling problem
- ∇_μ only couples sites that are separated by *2a*
- In one dimension, coupling sites spaced by *2a* means that even sites are coupled only to even sites, and odd to odd
- This is equivalent to having two lattice fermion fields ψ_{even} and ψ_{odd}
- This situation is not ameliorated by taking the continuum limit
- While we discretised a theory with only one fermion species, when we extrapolate back to the continuum our results are contaminated by additional fermions (2^d - 1 in d dimensions)

Naive Quark Action

- This doubling problem is demonstrated in momentum space.
- The momentum space representation of ∂_μ is ip_μ
- this function has only one zero, at $p_\mu = 0$
- However in momentum space $\nabla_\mu \to \frac{i}{a}\sin(ap_\mu)$
- which has 16 zeros within the Brillouin zone,

$$p_\mu = (0,0,0,0),\ \frac{1}{a}(\pi,0,0,0),\ \frac{1}{a}(\pi,\pi,0,0)\ldots$$

Wilson Quark Action

- Wilson introduced an irrelevant (energy) dimension - five operator (the so-called Wilson term) to fix this problem

$$M_W = m_0 + \sum_\mu \left(\gamma_\mu \nabla_\mu - \frac{1}{2} r a \Delta_\mu \right)$$

- where

$$\Delta_\mu \psi(x) = \frac{1}{a^2} [U_\mu(x)\psi(x+\hat{\mu}) + U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu}) - 2\psi(x)]$$

- Δ couples sites that are only one lattice spacing apart

- In momentum space $\Delta \rightarrow \frac{2}{a^2} \sum_\mu (1 - \cos(ap_\mu))$, which clearly has only a zero at $p_\mu = 0$

- Rescaling quark fields, $\psi \rightarrow \psi/\sqrt{2\kappa}$,the Wilson action is (in terms of $U_\mu(x)$)

$$S_W = \sum_x \bar{\psi}(x)\psi(x)$$

$$+ \kappa \sum_{x,\mu} \bar{\psi}(x) \Big[(\gamma_\mu - r) U_\mu(x) \psi(x+\hat{\mu}) - (\gamma_\mu + r) U_\mu^\dagger(x-\hat{\mu}) \psi(x-\hat{\mu}) \Big]$$

$$am_q = \frac{1}{2}\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right)$$

Wilson Quark Action

- In the continuum limit we find

$$S_W = \int d^4x \bar{\psi}(x)(\not{D} + m_q + m_{0c} - \frac{ar\,\not{D}^2}{2})\psi(x) + \mathcal{O}(a^2)$$

- where $am_{0c} = \frac{1}{2}\left(\frac{1}{\kappa_c} - 8\right)$ and $\kappa_c \neq \frac{1}{8}$ at finite lattice spacing

- By lifting the mass of the unwanted doublers with a second derivative, we have
    - introduced $\mathcal{O}(a)$ discretisation errors $\longrightarrow$ bad scaling
    - broken chiral symmetry at $\mathcal{O}(a)$

- The scaling properties of this Wilson action at finite *a* can be improved by introducing any number of irrelevant operators of increasing dimension which vanish in the continuum limit

- In this manner, one can improve fermion actions at finite *a* by combining operators to eliminate $\mathcal{O}(a)$ and perhaps $\mathcal{O}(a^2)$ errors etc

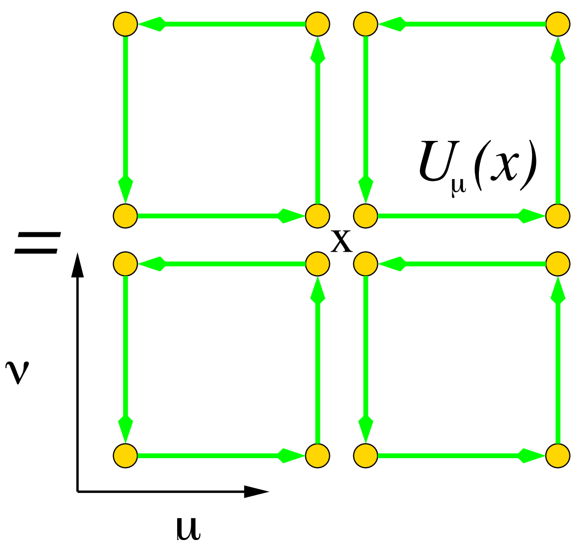
Clover Quark Action

- The Wilson term adds an $\mathcal{O}(a)$ error. How can we remove that?

- The are 5 basic dimension-5 operators

$$
\begin{array}{ll}
\mathcal{O}_1 & \bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi \\
\mathcal{O}_2 & \bar{\psi}D_\mu D_\nu \psi \\
\mathcal{O}_3 & m g_0^2 \mathrm{Tr}(F_{\mu\nu}^2) \\
\mathcal{O}_4 & m\bar{\psi}(D_\mu - D_\mu^*)\psi \\
\mathcal{O}_5 & m^2\bar{\psi}\psi
\end{array}
$$

- Only the first is needed: Clover term



$$
\boxed{S_{SW} = S_W - \frac{iaC_{SW}r}{4}\bar{\psi}(x)\sigma_{\mu\nu}F_{\mu\nu}\psi(x)}
$$

$C_{SW} = 1 + \mathcal{O}(g^2) \longleftarrow$ Tune to remove all $\mathcal{O}(a)$ errors

No Go Theorem

- The Nielsen-Ninomiya No-Go theorem states that it is not possible to find a lattice Dirac operator D_a that simultaneously satisfies the following four conditions:

  - Correct continuum limit: In the limit $a \to 0, \ D_a \to \not{D}$, where D_μ is the covariant derivative in the continuum, giving rise to a single fermion species of zero or finite mass

  - No doublers: All other modes of D_a are of order $1/a$, i.e., all other fermion species decouple in the continuum limit (grow infinitely heavy)

  - Locality: D_a is local, i.e., the matrix elements D_{xy} decay exponentially as $|x-y|$ grows large

  - Chirality: D_a does not explicitly break chiral symmetry, i.e., $D_a \gamma_5 + \gamma_5 D_a = 0$

Ginsbarg-Wilson Relation

- Ginsparg and Wilson proposed a lattice deformed version of chiral symmetry

$$\gamma_5 D_\Lambda + D_\Lambda \gamma_5 = 2a D_\Lambda \gamma_5 D_\Lambda$$

- Lattice Dirac operators that satisfy this are called Lattice Chiral Fermions

- Neuberger's Overlap operator is an example of such a lattice Dirac operator

Very computationally demanding!!

- Only one group (JLQCD) seriously tackling real (dynamical) overlap simulations

Other Fermion Actions

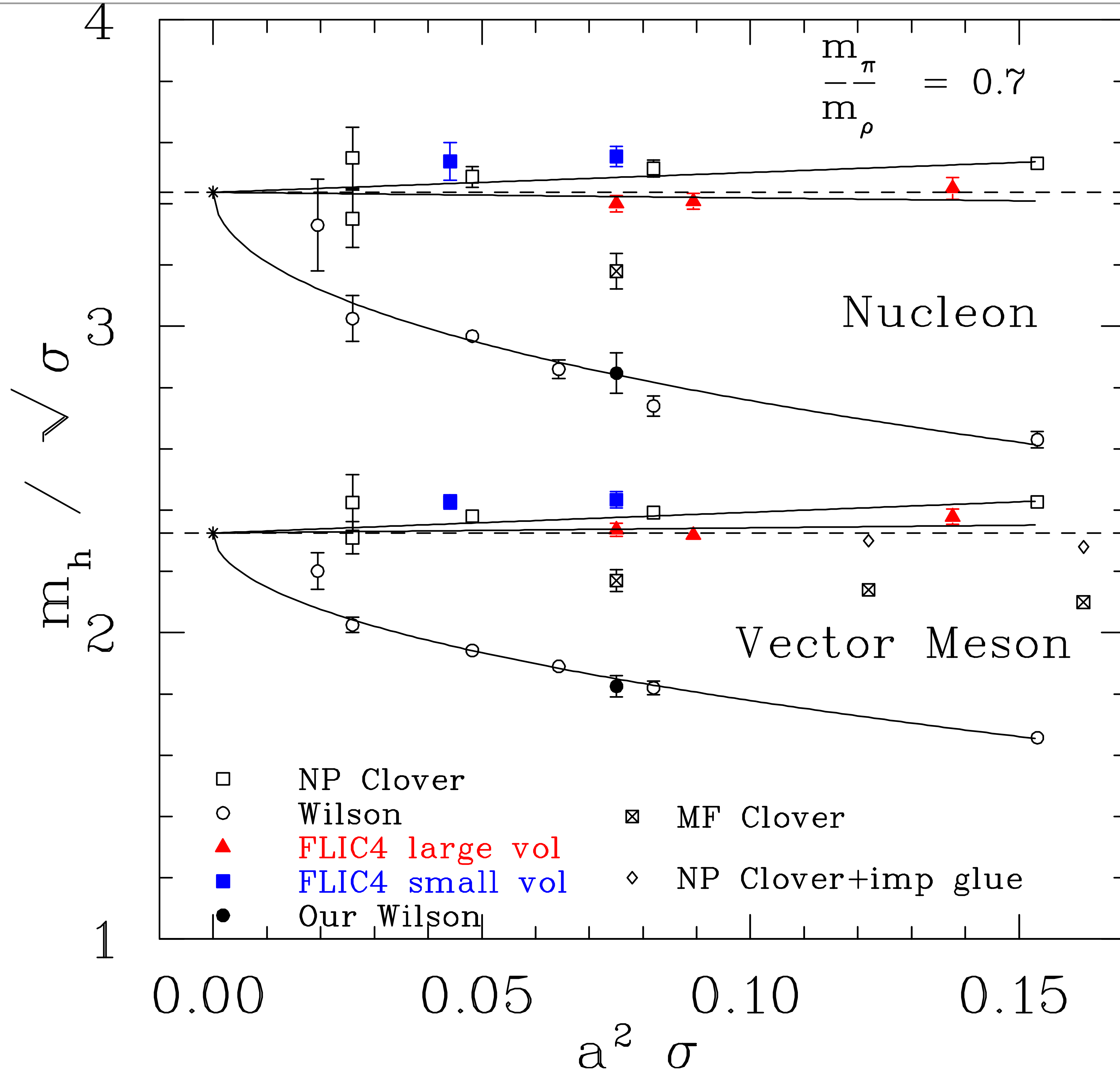
- Domain Wall: [RBC/UKQCD]
  - Both chiral and flavour symmetric at finite a
  - Computationally expensive
  - Residual chiral symmetry breaking m_{res}
- (Improved) Staggered Fermions: [MILC & HPQCD]
  - Good chiral properties
  - Efficient access to light quark masses / large volumes
  - Small discretisation errors
  - Remnant doubling problem
    - Each fermion flavour comes in four “tastes” (“fourth-root trick”)

Other Fermion Actions

- Twisted Mass [European Twisted Mass Collaboration]
    - Automatic $\mathcal{O}(a)$ improvement at maximal twist
    - Breaks flavour symmetry
- Fat-Link (Clover) Fermions (FLIC, SLIC, SLiNC, HEX) [CSSM, WMB, QCDSF, HSC,...]
    - Smear (smooth) gauge links appearing in irrelevant operators
    - Excellent scaling (discretisation effects small)
    - Efficient access to light quark mass regime

Common Continuum Limit

CSSM [hep-lat/0110216]



Path Integral

- Start with the partition function in Euclidean space ($t \rightarrow i\tau$)

$$Z = \int \mathcal{D}A_\mu \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, e^{-S}$$

- QCD action: $$S = \int d^4x \, (\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi} M \psi)$$
  - *M* - Dirac fermion matrix
  - $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ - field strength tensor
  - Gauge fields represented by A_μ
- Using the following identity for Grassmannian fields

$$\det M = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, e^{-\int d^4x \, \bar{\psi}(x) M \psi(x)}$$

- the fermion fields can be integrated out

$$Z = \int \mathcal{D}A_\mu \, \det M \, e^{\int d^4x \, (\frac{1}{4} F_{\mu\nu} F^{\mu\nu})}$$

Path Integral

- Fermionic contribution to the action is now contained in *detM* and *Z* is now only an integral over background gauge configurations

- The QCD action can now be written

$$S = S_{\text{gauge}} + S_{\text{quarks}} = \int d^4x \, (\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \sum_i \log \det M_i)$$

- where the sum in the last term is over quark flavours

- Quenched Approximation - *detM=0*
  - Neglect sea quark loops
  - cheap
  - No longer necessary

Expectation Values of Observables

- Calculation of physical observables are obtained via expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \mathcal{O}[A, \bar{\psi}, \psi] \, e^{-S[A, \bar{\psi}, \psi]}$$

- After applying the lattice regularisation

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \prod_{x \in \mathbb{L}} \prod_{\mu=1}^{4} dU_\mu(x) \mathcal{O}[U] e^{-S[U]}$$

- We must evaluate a multi-dimensional integral where for each point *x* we have to integrate over the available degrees of freedom

- The gauge field has eight degrees of freedom per link, and four links per site. So on a L^4 lattice, integration space has dimension $d=32L^4$

- If we sample *N* points per dimension to evaluate the integral, then the complexity of the functional integral is $O(N^d)$ - impossible!

- However, the weighting of $e^{-S[U]}$ in the integrand above means we are only interested in a small portion of the available configuration space (that with small action), as the remainder is exponentially suppressed

Expectation Values of Observables

Importance Sampling

- A statistical technique that takes advantage of this situation is importance sampling

- Given a set of *N* representative bosonic field configurations U_i distributed according to $e^{-S[U]}$, the functional integral $$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{O}[U] e^{-S[U]}$$

- will be approximated by $$\langle \mathcal{O} \rangle \simeq \frac{1}{N} \sum_{i}^{N} \mathcal{O}([U^{[i]}])$$

- with statistical errors that decrease as $1/\sqrt{N}$

- So, to calculate observables on the lattice, we first generate a set of gauge field configurations randomly chosen with probability $e^{-S[U]}$

- Then evaluate the desired quantity on each of these configurations and calculate the ensemble average

Expectation Values of Observables

- $\mathcal{O}$ can be any given combination of operators expressed in terms of time-ordered products of gauge and quark fields
- Using Wick's Theorem for contracting fields, it is possible to re-express quark fields in terms of quark propagators, removing any dependence on the quark fields as dynamical variables
- Quark propagator is calculated by inverting the Dirac operator on any given background field

$$S_F(y,j,b;x,i,a) = (M_F^{-1})^{y,j,b}_{x,i,a} = \langle \psi^b_j(y)\bar{\psi}^a_i(x)\rangle_F$$

- which gives the amplitude for the propagation of a quark from site *x* with spin-colour *i,a* to site-spin-colour *y,j,b* and can be computed from

$$\sum_w M_F(y;w)S_F(w;x) = \delta_{yx}$$

- Use your favourite matrix inverter
    - Eg. Conjugate Gradient

Expectation Values of Observables

Example

- Consider the momentum-space two-point function ($t > 0$)

$$G(\vec{p},t) = \int d^3x\, e^{-i\vec{p}.\vec{x}} \langle \Omega | T[\mathcal{O}_f(\vec{x},t)\mathcal{O}_i^\dagger(0)] | \Omega \rangle$$

- let $\mathcal{O}_f = \mathcal{O}_i = A_4 = \bar{\psi}\gamma_4\gamma_5\psi$, which has a large coupling to the pion

- 2-point function $\equiv$

  - $\mathcal{O}_i^\dagger$ creates a state with quantum numbers of the pion from the vacuum

  - evolution via QCD Hamiltonian to $(\vec{x},t)$

  - annihilation of this state by $\mathcal{O}_f$ back to the vacuum

- insert complete set of states, *n*, with quantum numbers of the pion, with momentum, *p'*

$$G(\vec{p},t) = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \int \frac{d^3p'}{(2\pi)^3 4E_n^2} \sum_{n=1}^{N} \langle \Omega | \mathcal{O}_f(\vec{x},t) | n, p' \rangle \langle n, p' | \mathcal{O}_i^\dagger(0) | \Omega \rangle$$

Expectation Values of Observables

Example

- We can make use of translational invariance to write

$$G(\vec{p},t) = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}} \int \frac{d^3p'}{(2\pi)^3 4E_n^2} \sum_{n=1}^{N} \langle \Omega | e^{\hat{H}t} e^{-i\hat{\vec{P}}\cdot\vec{x}} \mathcal{O}_f(0) e^{i\hat{\vec{P}}\cdot\vec{x}} e^{-\hat{H}t} | n, p' \rangle \langle n, p' | \mathcal{O}_i^\dagger(0) | \Omega \rangle$$

$$= \int d^3x \int \frac{d^3p'}{(2\pi)^3 4E_n^2}\, e^{-i(\vec{p}-\vec{p'})\cdot\vec{x}} \sum_{n=1}^{N} e^{-E_n t} \langle \Omega | \mathcal{O}_f(0) | n, p' \rangle \langle n, p' | \mathcal{O}_i^\dagger(0) | \Omega \rangle$$

$$= \sum_{n=1}^{N} \frac{e^{-E_n t}}{2E_n} \langle \Omega | \mathcal{O}_f(0) | n, p \rangle \langle n, p | \mathcal{O}_i^\dagger(0) | \Omega \rangle$$

Expectation Values of Observables

Example

- At $\vec{p} = 0, \;\; E_n \to M_n$ and masses are extracted

- If $\mathcal{O}$ has overlap with more than one state, then as a result of exponential damping, the ground (lowest mass) state can be isolated by examining the large *t* behaviour of

$$C_\pi(\vec{p} = \vec{0}, t) \stackrel{t \to \infty}{=} \frac{|Z(\vec{p} = \vec{0})|^2}{2E_\pi(\vec{p})} \; e^{-M_\pi t}$$

- where

$$\begin{aligned} Z(\vec{p}) &= \langle \Omega | \mathcal{O}(0) | \pi(\vec{p}) \rangle \\ &= E_\pi(\vec{p}) f_\pi \qquad \text{for } \mathcal{O} = A_4 \end{aligned}$$

- (Anti-)Periodic time-boundary conditions:

- $$\begin{aligned} C_\pi(\vec{p}, t) &\stackrel{t \to \infty}{=} \frac{|Z(\vec{p})|^2}{2E_\pi(\vec{p})} \left(e^{-E_p t} + e^{-E_p (T-t)} \right) \\ &= \frac{|Z(\vec{p})|^2}{2E_\pi(\vec{p})} \mathrm{e}^{-E_p T/2} \cosh[E_p(T/2 - t)] + \ldots \end{aligned}$$

Wick Contractions

- We are going to make use of Wick's Theorem to contract quark fields to make propagators:
  - E.g. consider four quark field insertions $\langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle$
  - two quark-antiquark contractions $\langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle \qquad \langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle$
  - giving propagator combinations $S_{ij} S_{kl} - S_{jk} S_{il}$
  - minus-sign from fermion anti-commutation
  - More quark fields $\Longrightarrow$ more complicated correlation functions

Expectation Values of Observables

Example

$$\langle\Omega|\mathcal{O}(x)\mathcal{O}^\dagger(0)|\Omega\rangle = -\langle\Omega|\bar\psi_a(x)\Gamma\psi_b(x)\;\bar\psi_b(0)\Gamma^\dagger\psi_a(0)|\Omega\rangle$$

- Now consider all the possible Wick contractions of the two fermion fields

- The correlation function can now be written in terms of a product of two quark propagators, S_F

$$\langle\mathcal{O}(x)\mathcal{O}^\dagger(0)\rangle = \langle\mathrm{Tr}\left(S_a(0,x)\Gamma S_b(x,0)\Gamma^\dagger\right)\rangle + \delta_{ab}\langle\mathrm{Tr}\big(\Gamma S_a(x,x)\big)\mathrm{Tr}\big(\Gamma^\dagger S_b(0,0)\big)\rangle$$
$$= \frac{1}{N}\sum_{\{U\}}\left\{\mathrm{Tr}\left(S_a(0,x,[U])\Gamma S_b(x,0,[U])\Gamma\right) + \delta_{ab}\mathrm{Tr}\big(\Gamma S_a(x,x,[U])\big)\mathrm{Tr}\big(\Gamma^\dagger S_b(0,0,[U])\big)\right\}$$

- For flavour non-singlets (*a*≠*b*) and using γ_5-hermiticity $\boxed{S^\dagger(x,0) = \gamma_5 S(0,x)\gamma_5}$

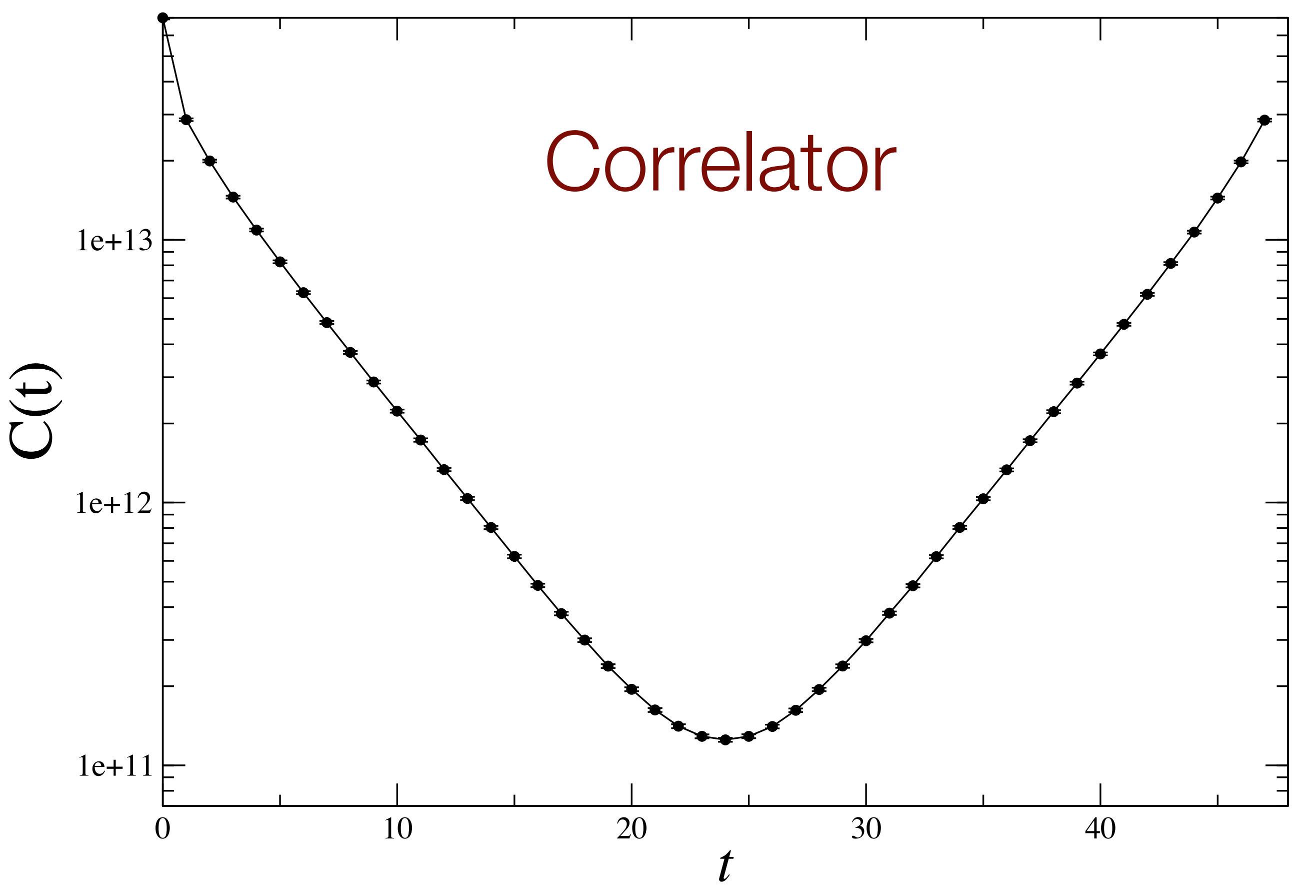
$$\langle\Omega|\mathcal{O}(x)\mathcal{O}(0)|\Omega\rangle = -\frac{1}{N}\sum_{\{U\}}\mathrm{Tr}\left\{\gamma_5 S_a^\dagger(x,0,[U])\gamma_5\Gamma S_b(x,0,[U])\Gamma^\dagger\right\}$$

- where the trace is only over the colour indices, and on each configuration the fermion propagator is computed by inverting the fermion matrix numerically

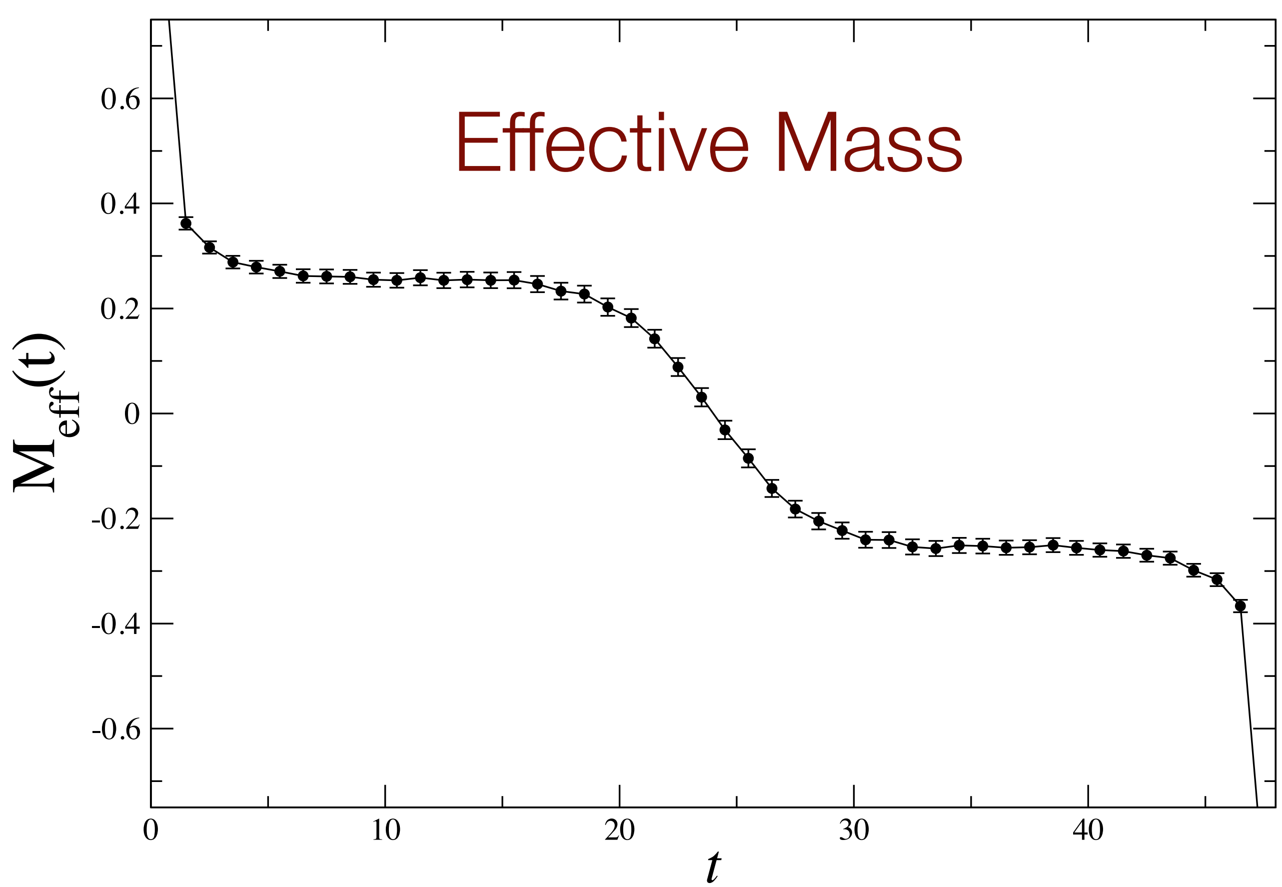
Expectation Values of Observables

Example

$$C_\pi(t,p) = \frac{\langle\Omega|\eta_\pi(\vec{p})|p\rangle\langle p|\eta_\pi^\dagger(\vec{p})|\Omega\rangle}{2E_p} \mathrm{e}^{-E_pT/2}\cosh[E_p(T/2-t)] + \dots$$



$$M_{eff}(t) = \log\left(\frac{C_\pi(t+1)}{C_\pi(t)}\right)$$



Resampling Techniques

- Two methods used: Jackknife and Bootstrap
- Jackknife:
  - Consider *N* measurements with a fit to the full dataset giving fit parameters α
  - Remove the first, leaving a set of *N-1* *resampled* measurements. Fitting this set gives parameters α_1
  - Repeat resampling of *N-1* *resampled* measurements, this time removing second, then third, etc measurements, giving fit parameters $\alpha_i, \;\; i = 2, \ldots N$
  - A Jackknife estimate of the errors in your fit parameters α are then

$$\sigma^2 = \frac{(N-1)}{N} \sum_{i=1}^{N} (\alpha_i - \alpha)^2$$

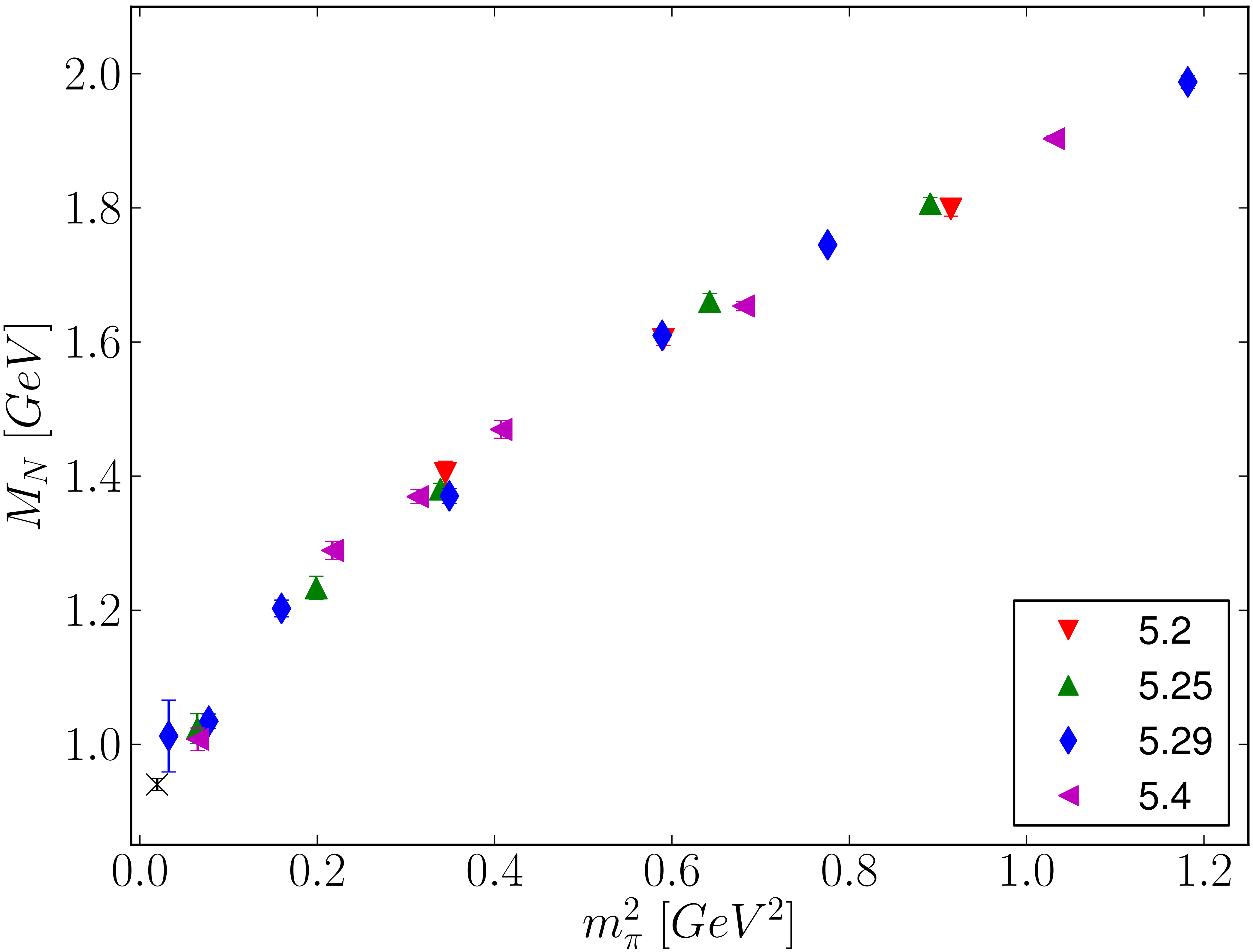
Resampling Techniques

- Two methods used: Jackknife and Bootstrap
- Bootstrap:
  - Consider *N* measurements with a fit to the full dataset giving fit parameters α
  - Create a new dataset by randomly selecting *N* datapoints with replacement (some points can occur more than once) from the original dataset
  - Determine fit parameters α_1 on this new dataset
  - Repeat *M* times, each time with a different random set, giving fit parameters $\alpha_i, \;\; i = 2, \ldots M$
  - A Bootstrap estimate of the errors in your fit parameters α are then

$$\sigma^2 = \frac{1}{M} \sum_{i=1}^{M} (\alpha_i - \alpha)^2$$

Example Lattice Results

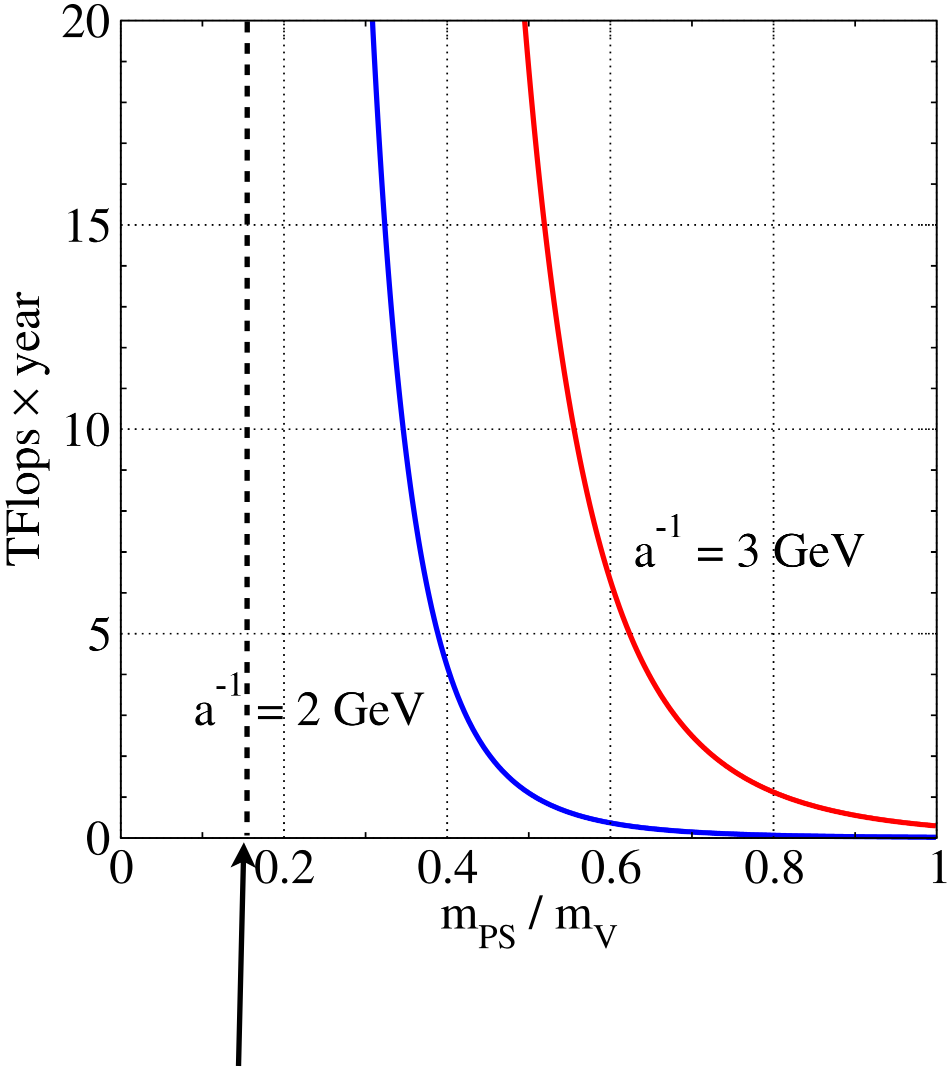
Nucleon Mass



$m_\pi^2 \propto m_q$

Speed of a Lattice Calculation

1000 configurations with L=2fm
[Ukawa (Berlin, 2001)]



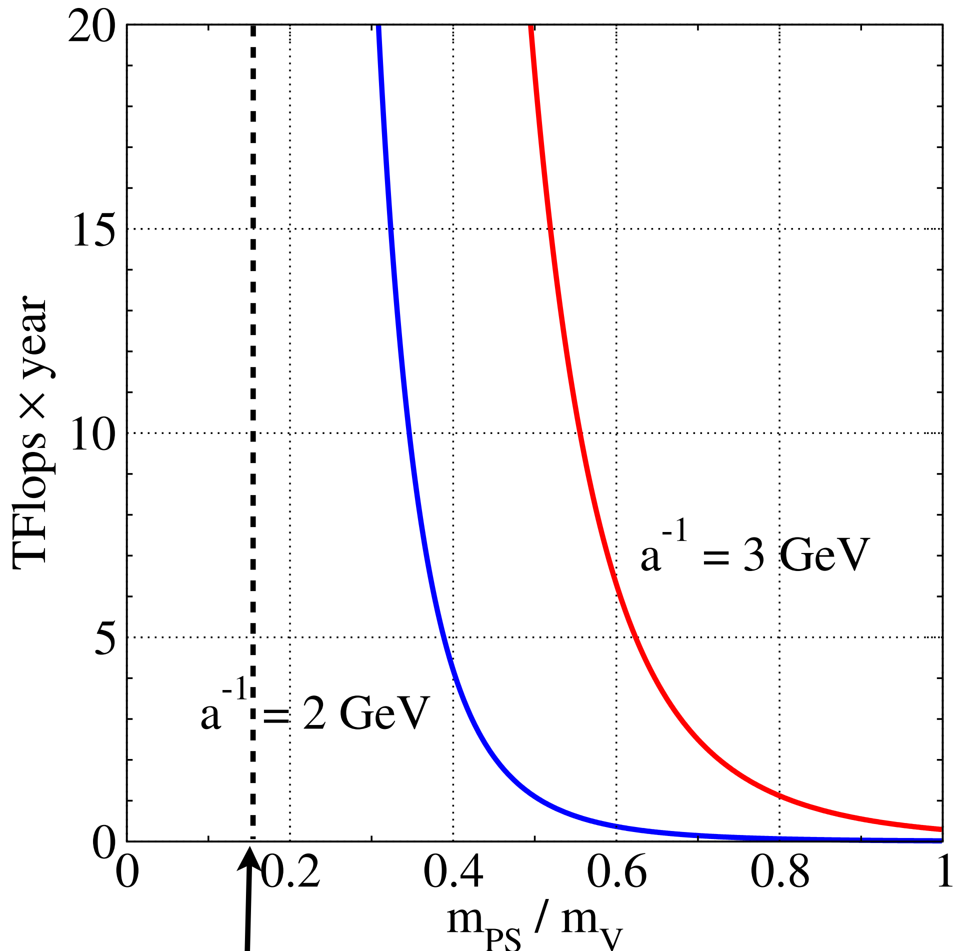
Physical point

$$C = K \left(\frac{m_{\mathrm{PS}}}{m_{\mathrm{V}}}\right)^{-6} L^5 a^{-7}$$

Speed of a Lattice Calculation

1000 configurations with L=2fm
[Ukawa (Berlin, 2001)]

"Berlin Wall" of Lattice QCD



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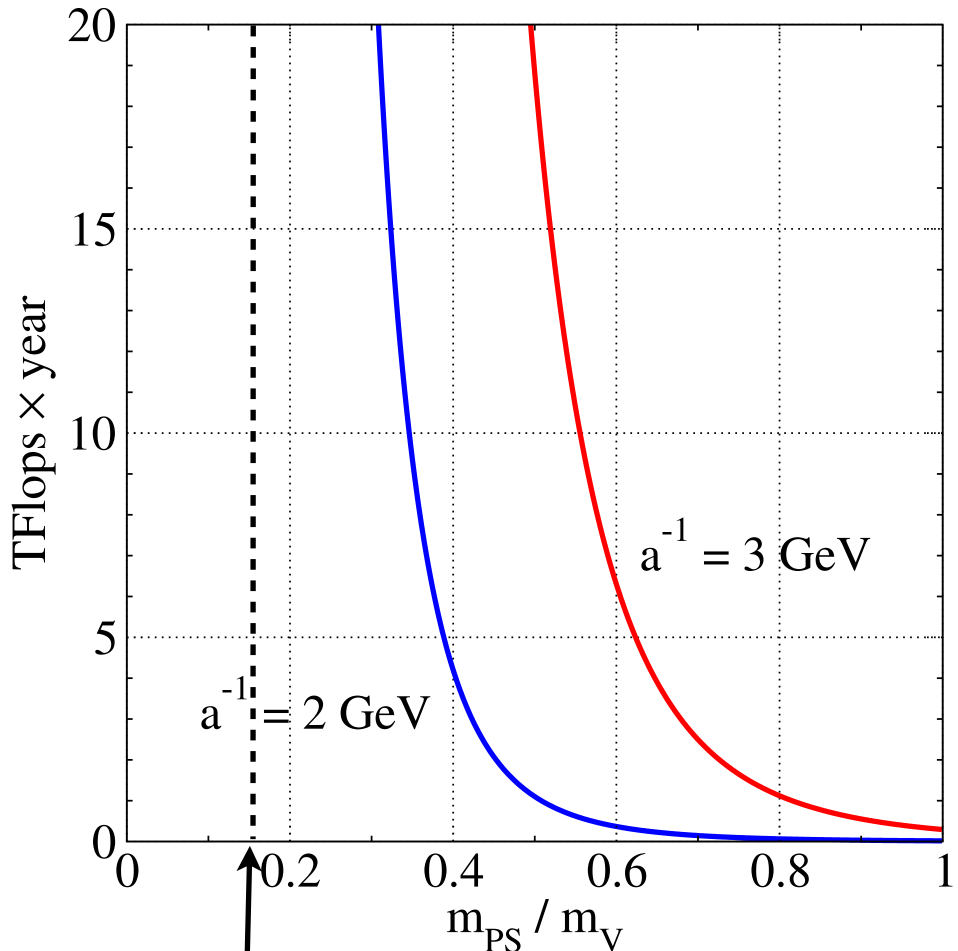

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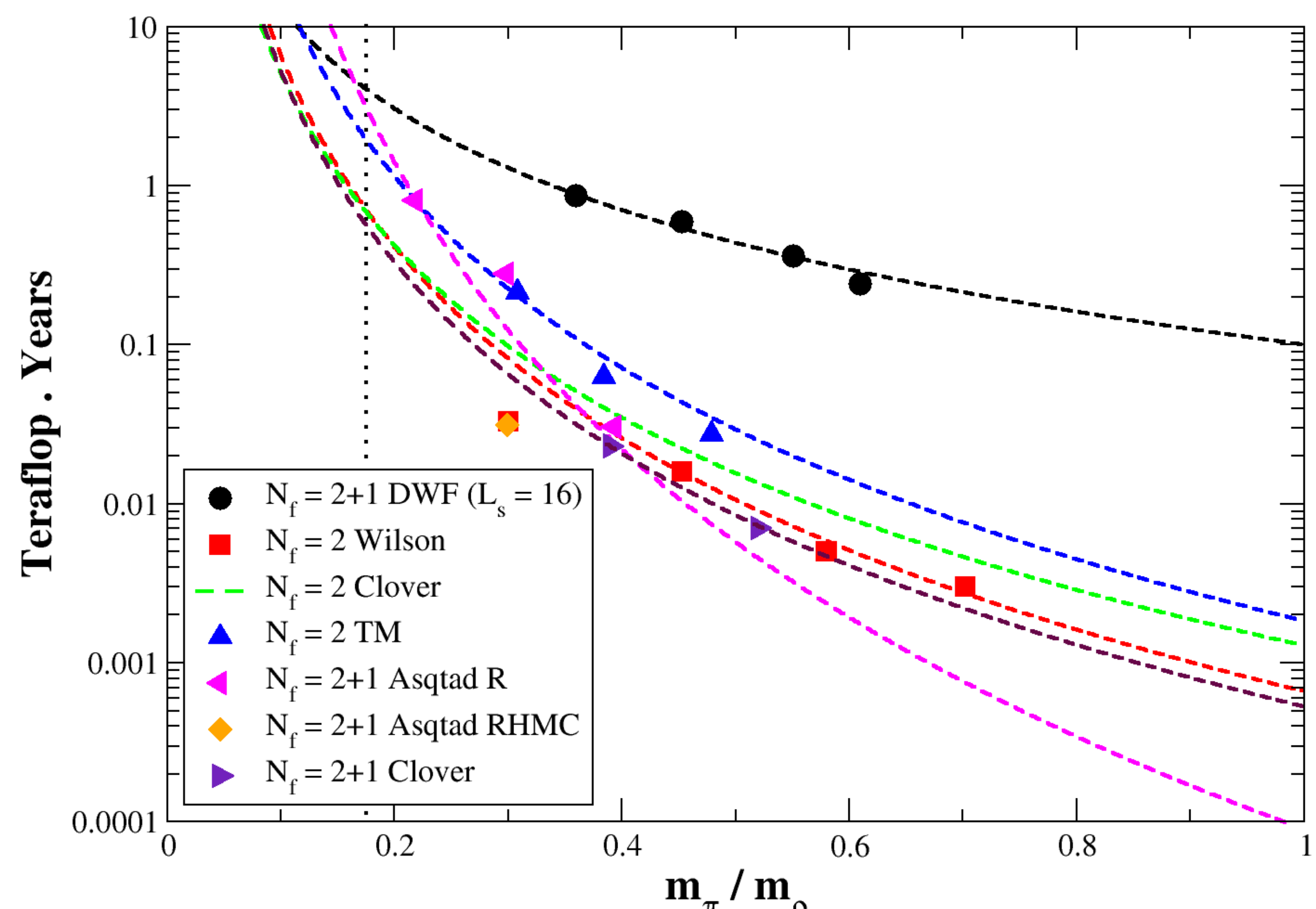
Many algorithmic improvements
[Clark (Tucson, 2006)]



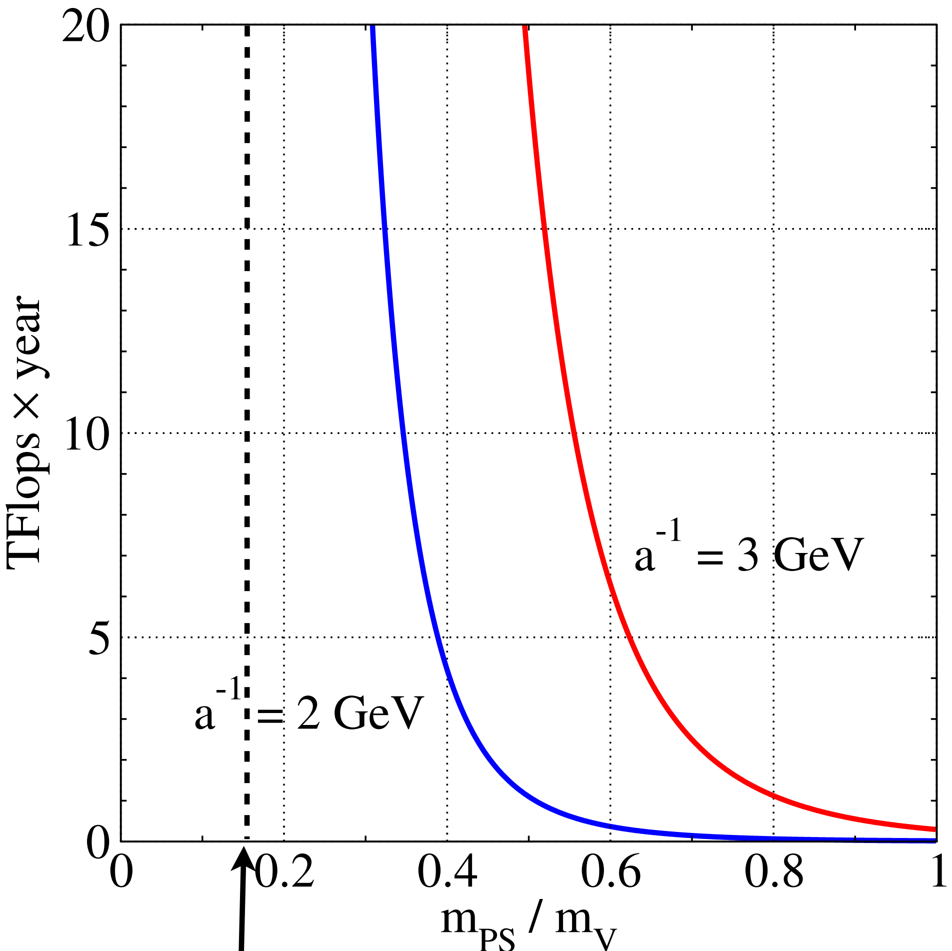
Speed of a Lattice Calculation

2001 2006

1000 configurations with L=2fm

[Ukawa (Berlin, 2001)]

“Berlin Wall” of Lattice QCD



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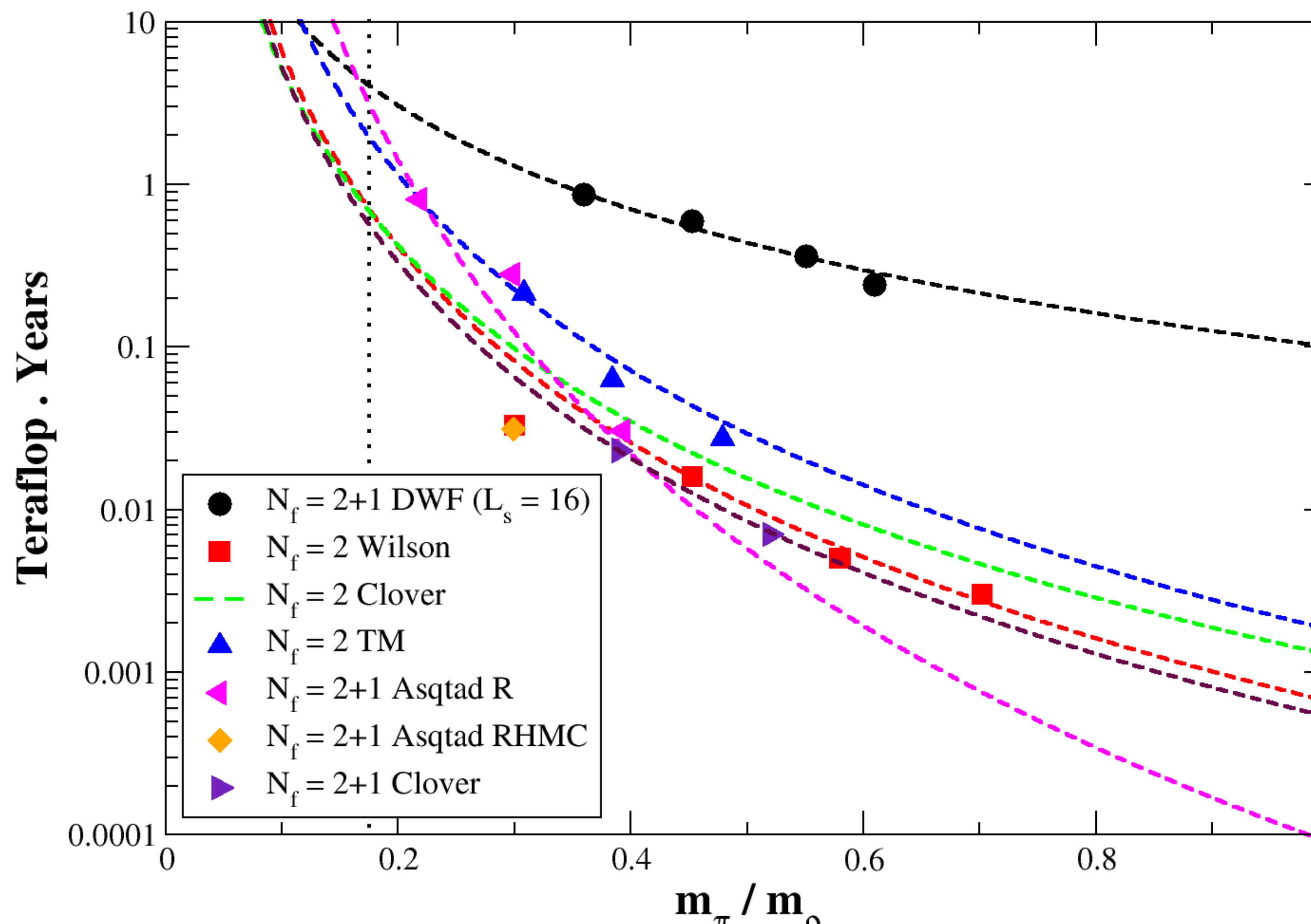
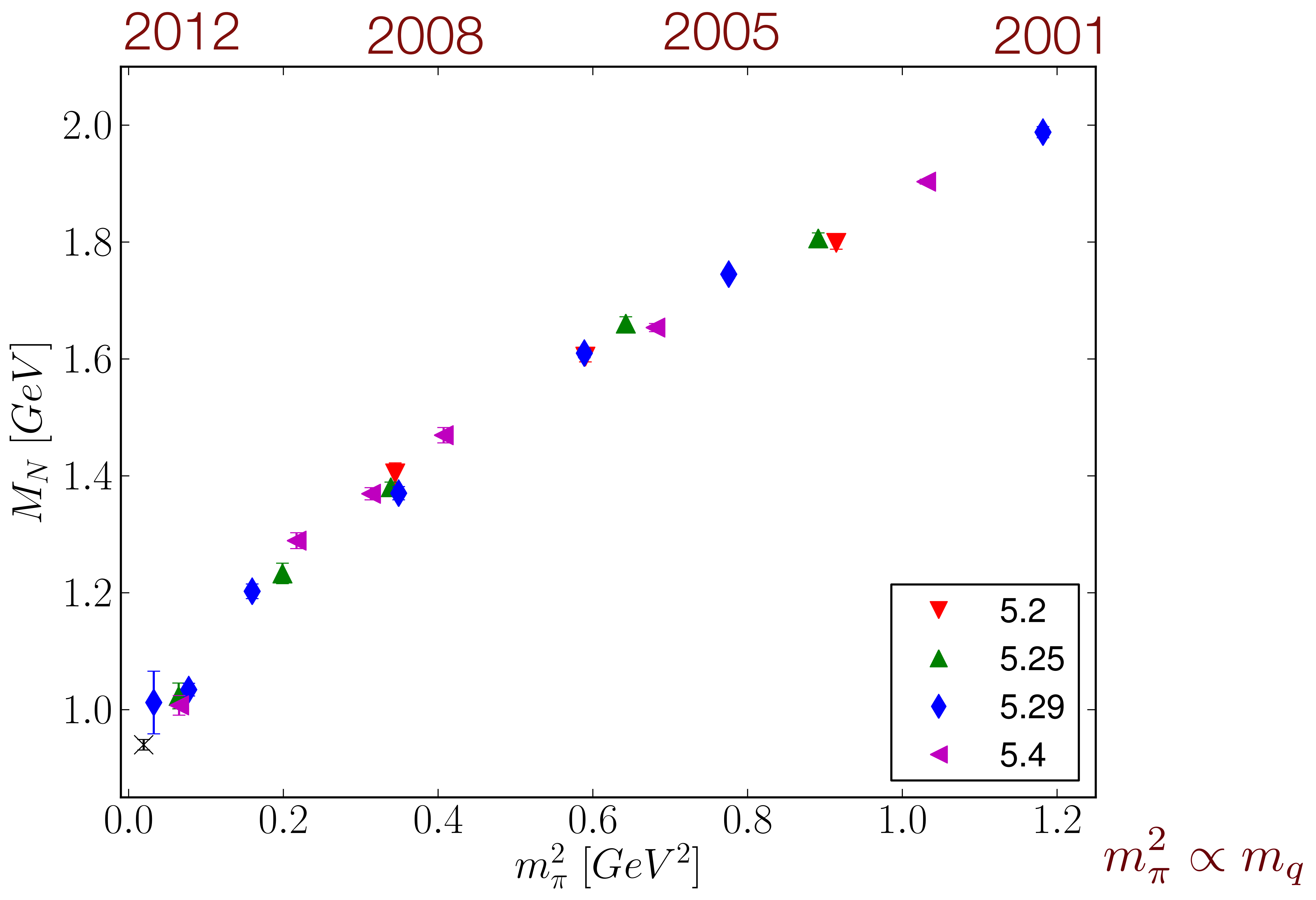


The Power of Computers

- Progress in Lattice QCD aligned with the dramatic increase in supercomputing power
  - From Top 500 list (http://www.top500.org)
    - June 2002: Earth Simulator, 36 TFlops
    - June 2012: Sequoia (BG/Q), 16 PFlops
- Lattice code performance on BG/Q:
  - 3.07 Petaflop/s sustained on half of Sequoia
  - 32% of peak for highly optimised routines (BAGEL - Peter Boyle, Edinburgh)

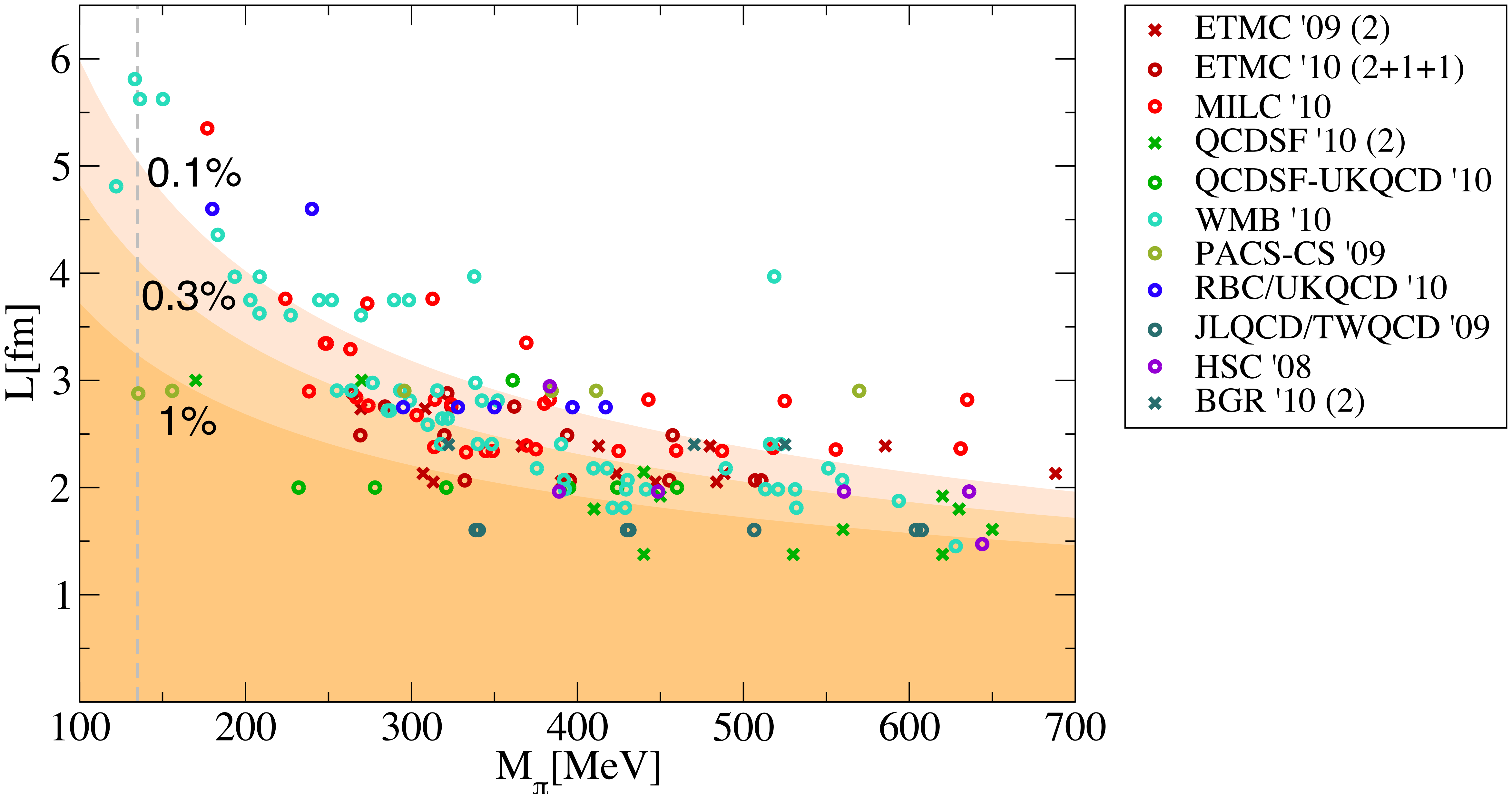
Real-Time Evolution of Lattice Results

Nucleon Mass



The Lattice Landscape

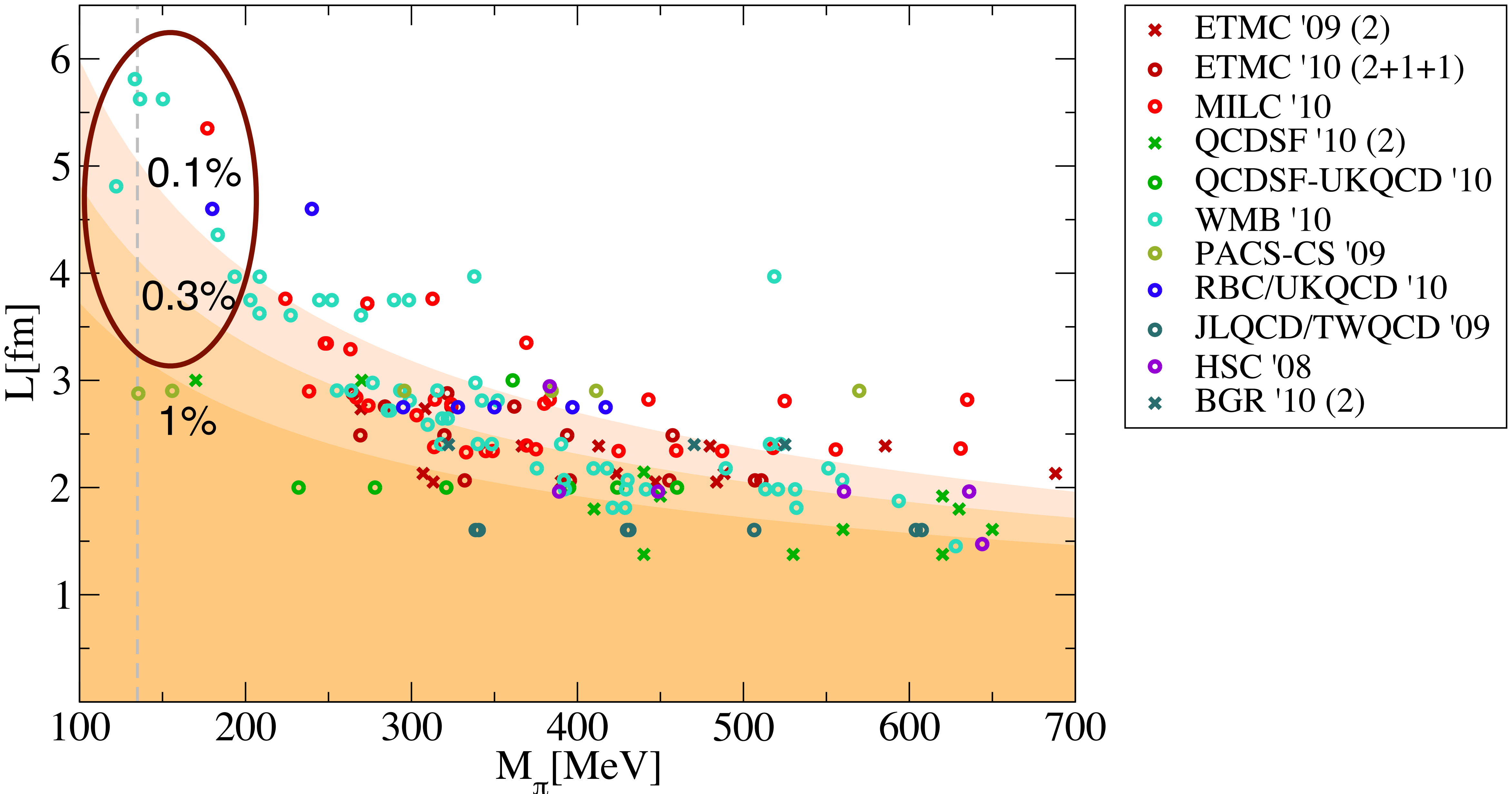
[Hoebling (Lattice 2010) 1102.0410]



- Leading sources of error:
  - Unphysically large quark masses
  - Finite Volume

The Lattice Landscape

[Hoebling (Lattice 2010) 1102.0410]



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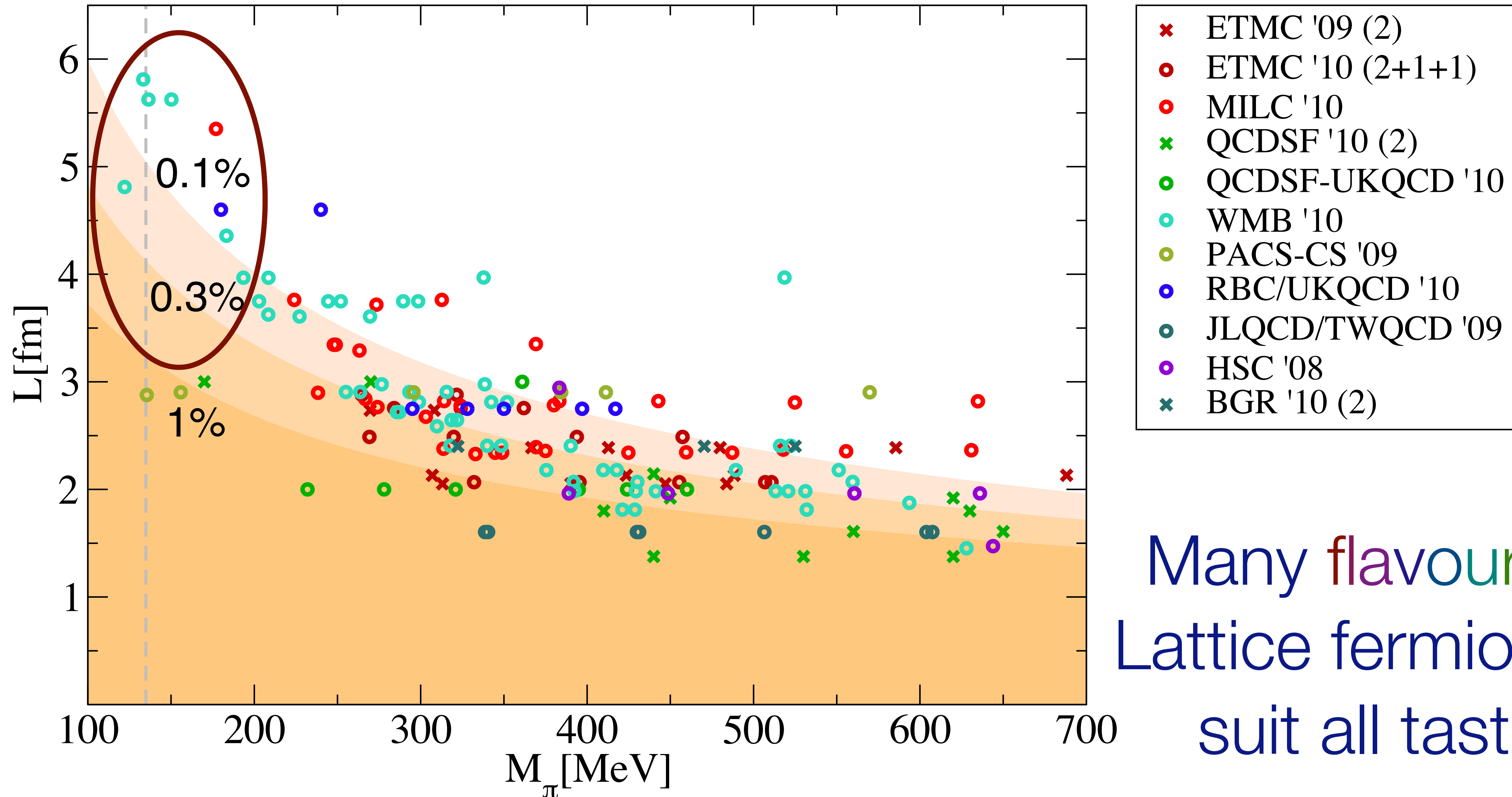


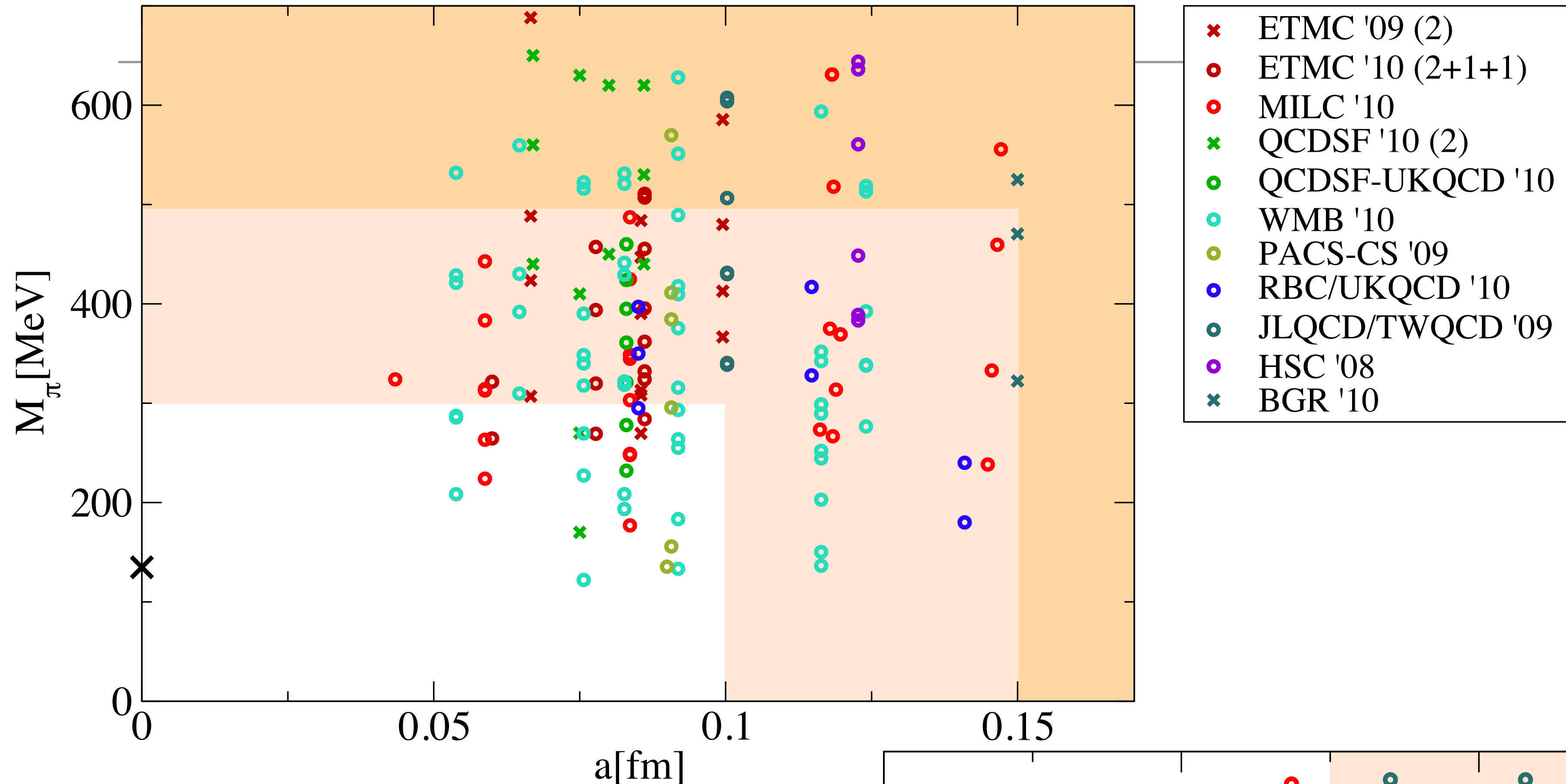
Many flavours of Lattice fermions to suit all tastes

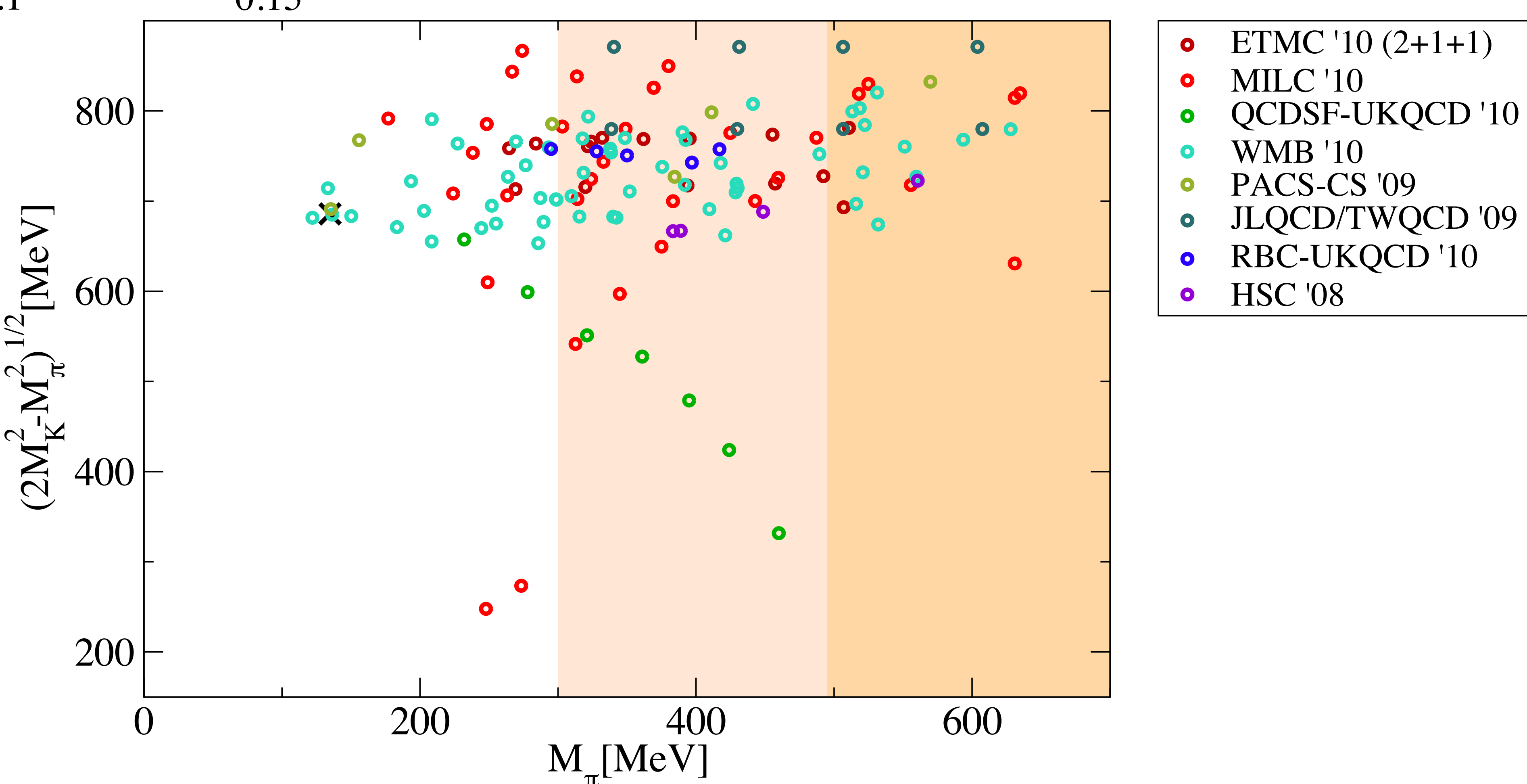
- Leading sources of error:
  - Unphysically large quark masses
  - Finite Volume

All agree in the continuum limit - important cross checks

The Lattice Landscape

[Hoebling (Lattice 2010) 1102.0410]

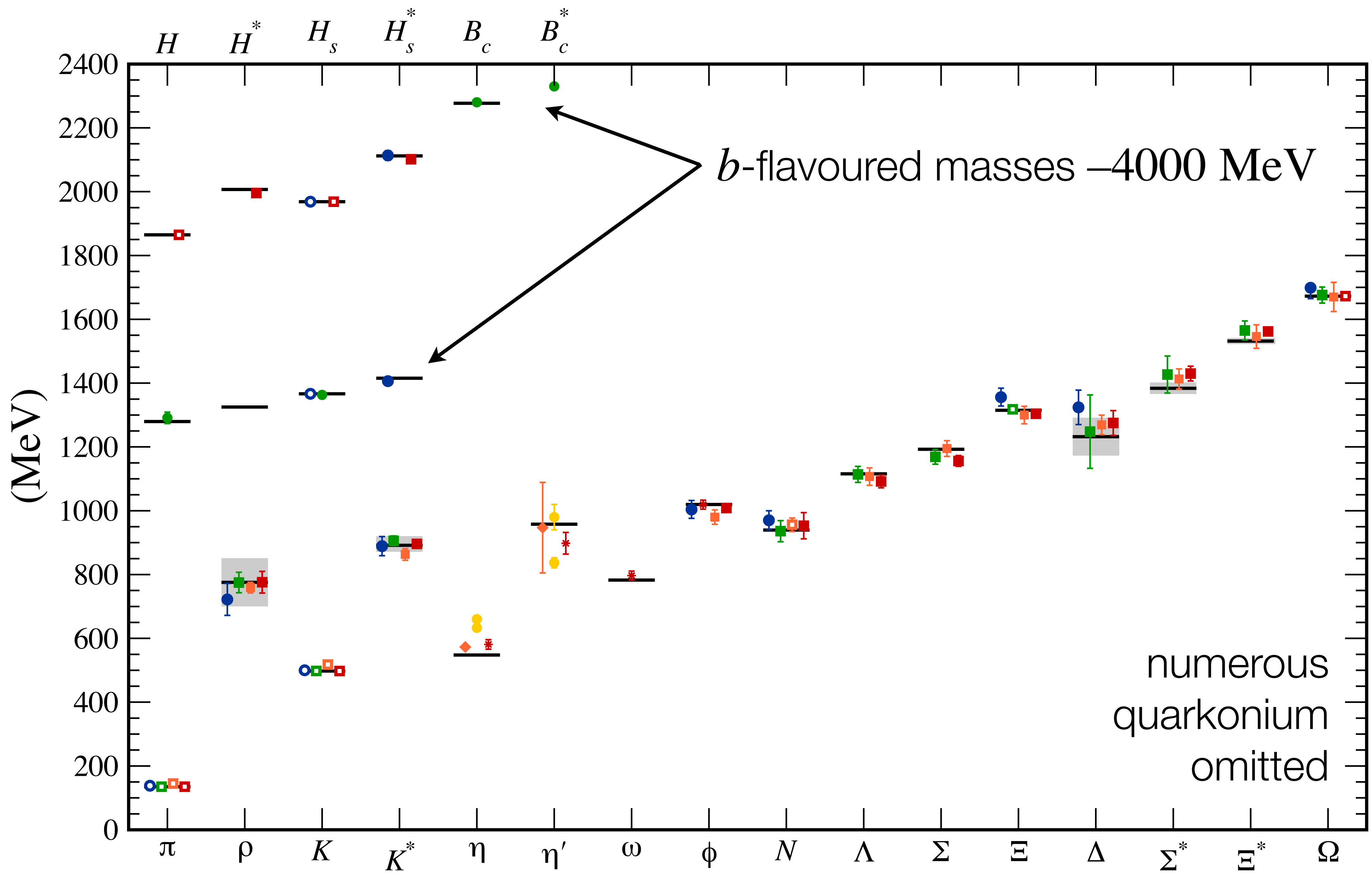




QCD Hadron Spectrum

Plot from A. Kronfeld [1203.1204]

$\pi \ldots \Omega$: BMW, MILC, PACS-CS, QCDSF;
η-η': RBC, UKQCD, Hadron Spectrum (ω);
D, B: Fermilab, HPQCD, Mohler-Woloshyn



Excellent agreement between different collaborations/lattice formulations