*Lecture 1*

# QCD and the Nambu–Jona-Lasinio Model


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Argonne NATIONAL LABORATORY

SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

# Building Blocks of the Universe

**FERMIONS** matter constituents
spin = 1/2, 3/2, 5/2, ...

**Leptons** spin =1/2

| Flavor | Mass GeV/c$^2$ | Electric charge |
|---|---|---|
| $\nu_L$ lightest neutrino* | $(0-0.13)\times10^{-9}$ | 0 |
| e electron | 0.000511 | –1 |
| $\nu_M$ middle neutrino* | $(0.009-0.13)\times10^{-9}$ | 0 |
| $\mu$ muon | 0.106 | –1 |
| $\nu_H$ heaviest neutrino* | $(0.04-0.14)\times10^{-9}$ | 0 |
| $\tau$ tau | 1.777 | –1 |

**Quarks** spin =1/2

| Flavor | Approx. Mass GeV/c$^2$ | Electric charge |
|---|---|---|
| u up | 0.002 | 2/3 |
| d down | 0.005 | –1/3 |
| c charm | 1.3 | 2/3 |
| s strange | 0.1 | –1/3 |
| t top | 173 | 2/3 |
| b bottom | 4.2 | –1/3 |

**BOSONS** force carriers
spin = 0, 1, 2, ...

**Unified Electroweak** spin = 1

| Name | Mass GeV/c$^2$ | Electric charge |
|---|---|---|
| $\gamma$ photon | 0 | 0 |
| $W^-$ W bosons | 80.39 | –1 |
| $W^+$ W bosons | 80.39 | +1 |
| $Z^0$ Z boson | 91.188 | 0 |

**Strong (color)** spin =1

| Name | Mass GeV/c$^2$ | Electric charge |
|---|---|---|
| g gluon | 0 | 0 |


- Fundamental constituents of the Standard Model (SM) of particle physics
  - ✦ Quantum Chromodynamics (QCD) & Electroweak (EW) theories
- Local non-abelian gauge field theories
  - ✦ a special type of relativistic quantum field theory
- SM Lagrangian has gauge symmetries: $SU(3)_c \otimes SU(2)_L \otimes U_Y(1)$
  - ✦ SM has 19 parameters which need to be determined by experiment
  - ✦ however only 2 parameters are intrinsic to QCD: $\Lambda_{QCD}$ & $\theta_{QCD} \leqslant 10^{-9}$

# *Motivation of Lectures*

- *Explore non-perturbative structure of QCD, through the interplay of theory and experiment, as it relates to hadron and nuclear structure*
- The tools available are:
  - lattice QCD
  - chiral perturbation theory
  - *QCD inspired models*
- We will review the model of Nambu and Jona-Lasinio (NJL model)
  - first proposed in 1961 as a theory of elementary nucleons
  - with advent of QCD reinterpreted as a quark effective theory
- Some of the advantages of models over lattice and $\chi$PT are
  - can explore a wider array of physics problems
  - may provide better insight into important physics mechanisms
  - facilitate a dynamic interplay between experiment and theory

# *Plan of Lectures*

- Lecture 1 – Introduction to QCD and the non-perturbative framework provided by the Nambu–Jona-Lasinio (NJL) model
- Lecture 2 – Relativistic Faddeev (3-body) equation & electromagnetic form factors
- Lecture 3 – Deep inelastic scattering and parton distribution functions
- Lecture 4 – Quark degrees of freedom in nuclei and nuclear matter

# Recommended References


- Y. Nambu and G. Jona-Lasinio, "*Dynamical model of elementary particles based on an analogy with superconductivity I*", Phys. Rev. **122**, 345 (1961).
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- W. Bentz and A. W. Thomas, "*The Stability of nuclear matter in the Nambu-Jona-Lasinio model*", Nucl. Phys. A **696**, 138 (2001).
- I. C. Cloët, W. Bentz and A. W. Thomas, "*EMC and polarized EMC effects in nuclei*", Phys. Lett. B **642**, 210 (2006).
- I. C. Cloët, W. Bentz and A. W. Thomas, "*Isovector EMC effect explains the NuTeV anomaly*", Phys. Rev. Lett. **102**, 252301 (2009).

# Quantum Chromodynamics (QCD)

- QCD is the fundamental theory of the strong interaction, where the *quarks* and *gluons* are the basic degrees of freedom

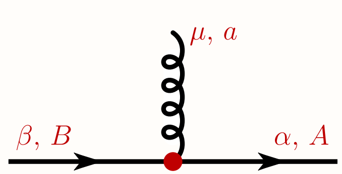
$$(q_\alpha)^A_f \quad \begin{cases} \text{colour} & A = 1,\, 2,\, 3 \\ \text{spin} & \alpha = \uparrow, \downarrow \\ \text{flavour} & f = u,\, d,\, s,\, c,\, b,\, t \end{cases} \qquad\qquad A^a_\mu \quad \begin{cases} \text{colour} & a = 1, \ldots, 8 \\ \text{spin} & \varepsilon^\pm_\mu \end{cases}$$

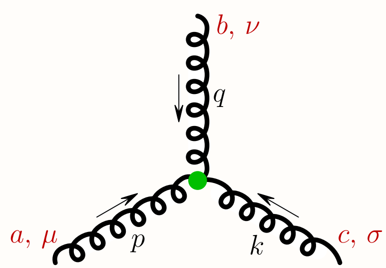
- QCD is a non-abelian gauge theory whose dynamics are governed by the Lagrangian

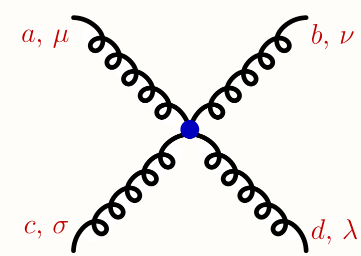
$$\mathcal{L} = \bar{q}_f \left( i\not{D} + m_f \right) q_f - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a ;$$

$$i\not{D} = \gamma^\mu \left( i\partial_\mu + g_s\, A^a_\mu\, T^a \right)$$

$$F^a_{\mu\nu} = \partial_\mu\, A^a_\nu - \partial_\nu\, A^a_\mu + g_s\, f_{abc}\, A^b_\mu\, A^c_\nu$$



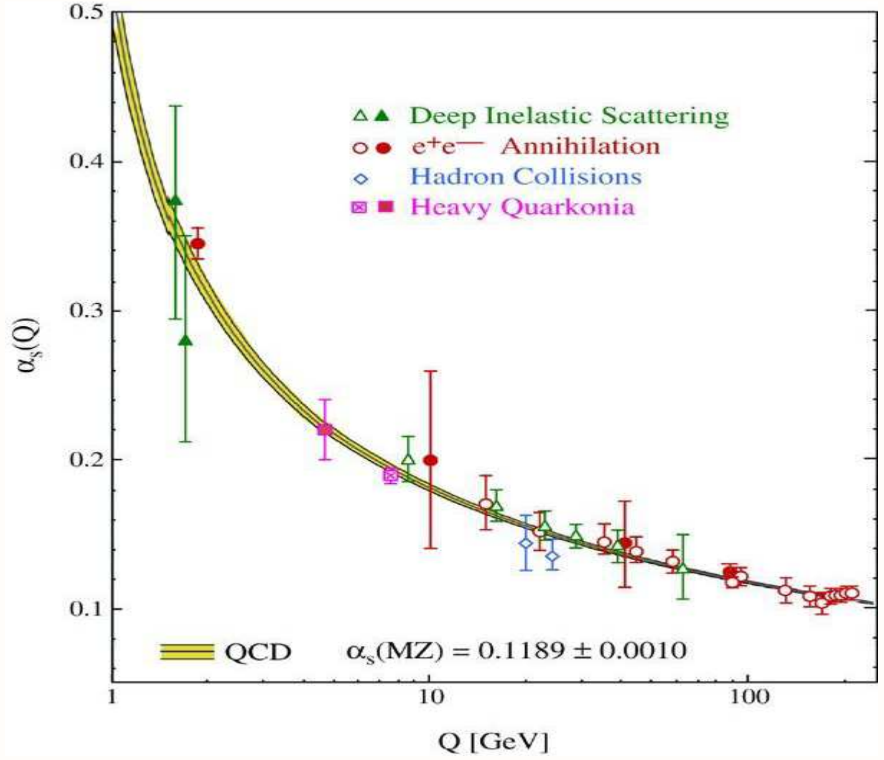




- Gluon self-interactions have many profound consequences

# *Asymptotic Freedom*

- At large $Q^2$ or short distances interaction strength becomes logarithmically small
  - a striking features of QCD
  - QED has opposite behaviour: $\alpha_e \simeq \frac{1}{137}$

$$\alpha_s^{LO}(Q^2) = \frac{4\,\pi}{\left(11 - \frac{2}{3}\,N_f\right)\ln\left(Q^2/\Lambda^2_{QCD}\right)}$$




- Asymptotic Freedom – 2004 Nobel Prize – Gross, Politzer and Wilczek
- $\Lambda_{QCD}$ most important parameter in QCD – [dimenional transmutation of $g_s$]
  - $\Lambda_{QCD} \simeq 200$ MeV $\simeq 1$ fm$^{-1}$ – sets scale, QCDs “standard kilogram”
- Momentum-dependent coupling $\Longleftrightarrow$ coupling depends on separation
  - interaction strength between quarks and gluons grows with separation

# Confinement

- Hadron structure & QCD is characterized by two emergent phenomena
  - confinement and dynamical chiral symmetry breaking (DCSB)
- Both of these phenomena are not evident from the QCD Lagrangian
- All known hadrons are colour singlets, even though they are composed of coloured quarks and gluons: baryons $(qqq)$ & mesons $(\bar{q}q)$
- Confinement conjecture: *particles that carry the colour charge cannot be isolated and can therefore not be directly observed*
- Related to $1 million Millennium Prize:

**Yang-Mills Existence And Mass Gap:** *Prove that for any compact simple gauge group $G$, quantum Yang-Mills theory on $\mathbf{R}^4$ exists and has a mass gap $\Delta > 0$.*

  - for $SU(3)_c$ must prove that glueballs have a lower bound on their mass
  - partial explanation as to why strong force is short ranged

# Chiral Symmetry

- Define left- and right-handed fields: $\psi_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\,\psi$
- The QCD Lagrangian then takes the form $[\mathbf{m} = \mathrm{diag}\,(m_u, m_d, m_s, \ldots)]$

$$\mathcal{L} = \bar{\psi}_L \, i\not{D}\, \psi_L + \bar{\psi}_R \, i\not{D}\, \psi_R - \bar{\psi}_R \, \mathbf{m}\, \psi_L - \bar{\psi}_L \, \mathbf{m}\, \psi_R - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$$

- Therefore for $\mathbf{m} = 0$ QCD Lagrangian is chirally symmetric

$$SU(N_f)_L \otimes SU(N_f)_R \implies \psi_{L,R} \to e^{-i\,\omega^a_{L,R}\,T^a}\,\psi_{L,R}$$

- $SU(N_f)_L \otimes SU(N_f)_R$ chiral symmetry is equivalent to

$$SU(N_f)_V \otimes SU(N_f)_A \implies \psi \to e^{-i\,\omega^a_V\,T^a}\psi, \;\; \psi \to e^{-i\,\omega^a_A\,T^a\,\gamma_5}\psi$$

- Global symmetries: Wigner-Weyl or Nambu-Goldstone modes
  - ✦ Wigner-Weyl mode: vacuum is also invariant
  - ✦ Nambu-Goldstone mode: vacuum breaks symmetry

# *Dynamical Chiral Symmetry Breaking*

- Recall for $\mathbf{m} = 0$ QCD Lagrangian is invariant under

$$SU(N_f)_L \otimes SU(N_f)_R \quad \Longleftrightarrow \quad SU(N_f)_V \otimes SU(N_f)_A$$

- $N_f = 2$ corresponds to the isospin subgroup of $SU(N_f)_V$ transformations
  - ✦ hadronic mass spectrum tells us nature largely respects isospin symmetry
  - ✦ $m_{\pi^-} \simeq m_{\pi^0} \simeq m_{\pi^+}, \quad m_p \simeq m_n, \quad m_{\Sigma^-} \simeq m_{\Sigma^0} \simeq m_{\Sigma^+}$
  - ✦ therefore $SU(N_f)_V$ is realized in the Wigner-Weyl mode

- $SU(N_f)_A$ transformations mix states of opposite parity
  - ✦ expect hadronic mass spectrum to exhibit parity degeneracy
  - ✦ $m_\sigma - m_\pi \sim 300$ MeV, $m_{a_1} - m_\rho \sim 490$ MeV, $m_N - m_{N^*} \sim 600$ MeV, etc
  - ✦ recall: $m_u \simeq m_d \simeq 5$ MeV $\Longrightarrow$ cannot produce large mass splittings
  - ✦ therefore $SU(N_f)_A$ must be realized in the Nambu-Goldstone mode

- Chiral symmetry broken dynamically: $SU(N_f)_L \otimes SU(N_f)_R \Longrightarrow SU(N_f)_V$

# Goldstone's Theorem

- Goldstone's theorem: *if a continuous global symmetry is broken dynamically, then for each broken group generator there must appear in the theory a massless spinless particle (Goldstone boson)*

- QCDs chiral symmetry is explicitly broken by small current quark masses

$$m_u = 1.5 - 3.5\,\text{MeV} \quad \& \quad m_d = 3.5 - 6.0\,\text{MeV} \qquad (\ll \Lambda_{QCD})$$

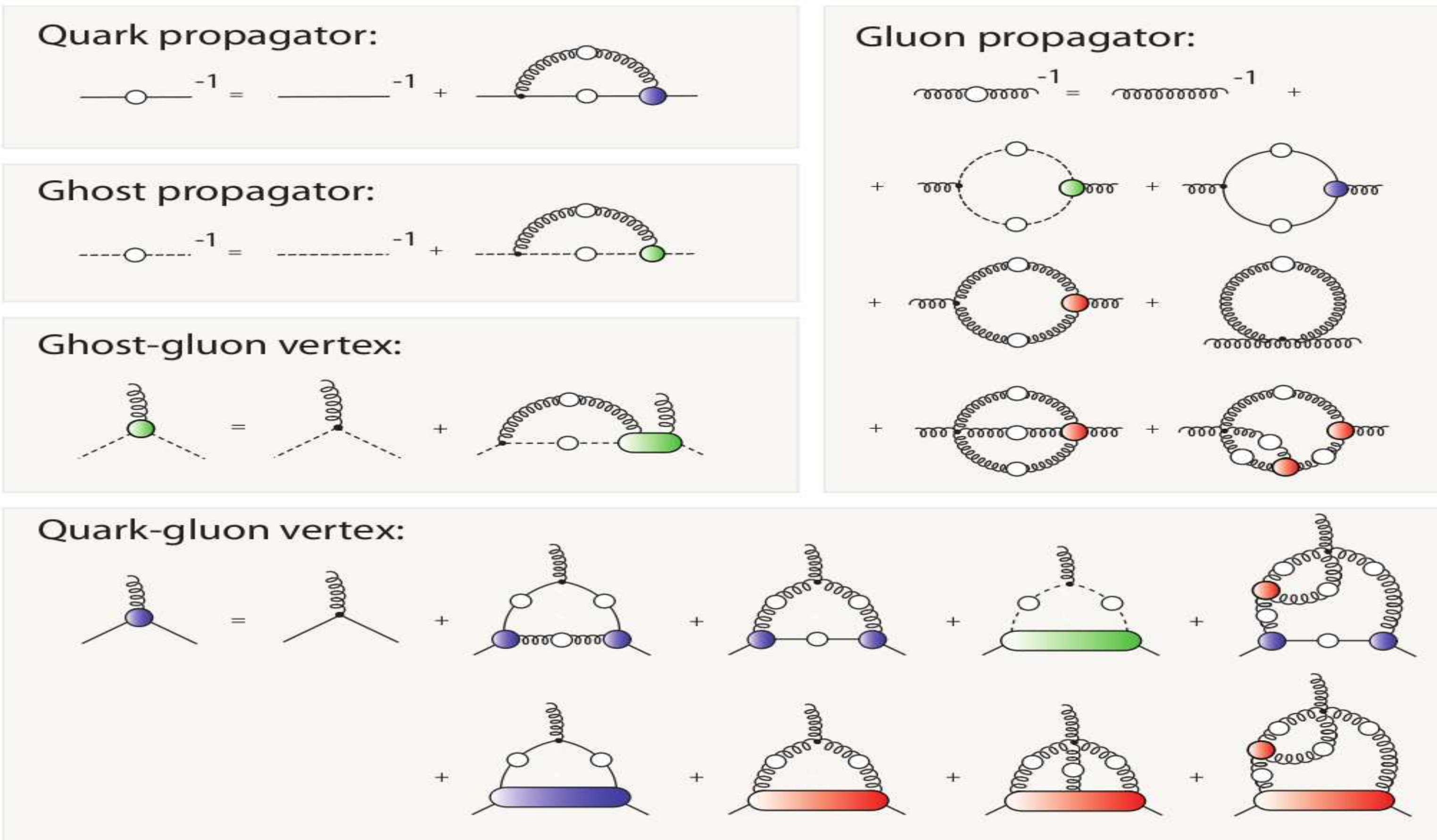
- For $N_f = 2$ expect $N_f^2 - 1 = 3$ Goldstone bosons: $\pi^+,\ \pi^0,\ \pi^-$
  - ✦ physical particle masses are not zero – $m_\pi \sim 140\,\text{MeV}$ – because of explicit chiral symmetry breaking: $m_{u,d} \neq 0$

- Chiral symmetry and its dynamical breaking has profound consequences for the QCD mass spectrum and hadron structure
  - ✦ this is not apparent from the QCD Lagrangian and is an innately non-perturbative phenomena

- Need non-perturbative methods to understand all consequences of QCD

# QCDs Dyson–Schwinger Equations

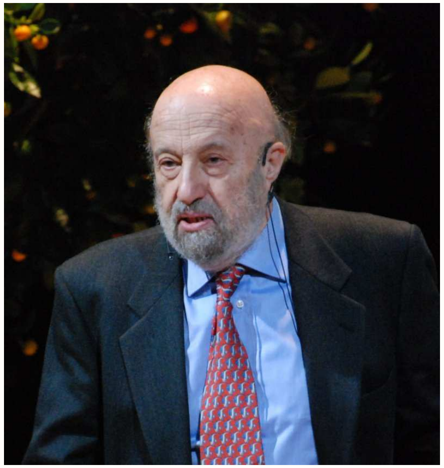
- DSEs are the equations of motion for a quantum field theory
  - ✦ must truncate infinite tower of coupled integral equations




*ETC!*

- Truncation: gluon propagator becomes constant – $D^{\mu\nu}(k) \to g^{\mu\nu}$
- *Largely equivalent to the Nambu–Jona Lasinio (NJL) model*

# *The Nambu–Jona-Lasinio Model*

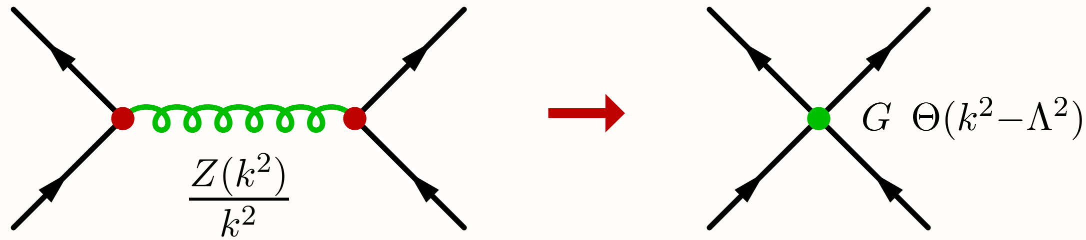


- The Nambu–Jona-Lasinio (NJL) Model was invented in 1961 by *Yoichiro Nambu* and *Giovanni Jona-Lasinio* while at The University of Chicago
  - ✦ was inspired by the BCS theory of superconductivity
  - ✦ was originally a theory of elementary nucleons
  - ✦ rediscovered in the 80s as an effective quark theory
- It is a relativistic quantum field theory, that is relatively easy to work with, and is very successful in the description of hadrons, nuclear matter, etc
- Nambu won half the 2008 Nobel prize in physics in part for the NJL model:

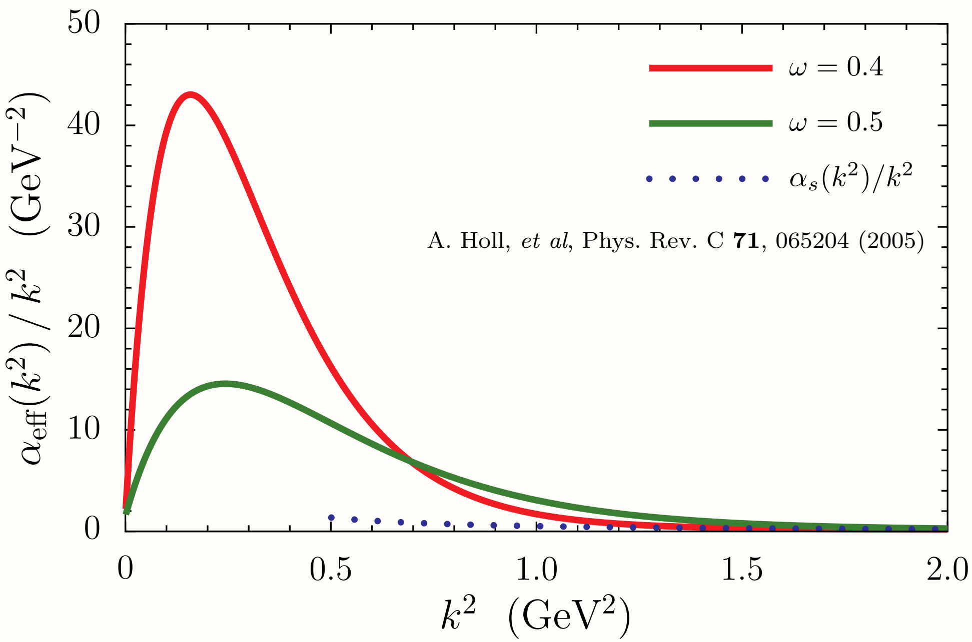
  "*for the discovery of the mechanism of spontaneous broken symmetry in subatomic physics*" [Nobel Committee]

# *NJL Model*

- NJL model is interpreted as low energy chiral effective theory of QCD



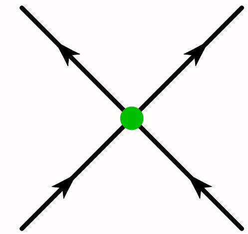

- Can be motivated by infrared enhancement of quark–gluon interaction
  e.g. DSEs and Lattice QCD




- Investigate the role of quark degrees of freedom
- NJL has same flavour symmetries as QCD
- NJL is non-renormalizable $\Longrightarrow$ cannot remove regularization parameter

# NJL Lagrangian

- In general the NJL Lagrangian has the form



$$\mathcal{L}_{NJL} = \mathcal{L}_0 + \mathcal{L}_I = \overline{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi + \sum\nolimits_\alpha G_\alpha\,\left(\overline{\psi}\,\Gamma_\alpha\,\psi\right)^2$$

  - ✦ $\Gamma_\alpha$ represents a product of Dirac, colour and flavour matrices

- What about $\mathcal{L}_I$? – effective theories should maintain symmetries of QCD
- In chiral limit QCD Lagrangian has symmetries

$$\mathcal{S}_{QCD} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes U(1)_A \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

  - ✦ $SU(N_f)_A$ is broken dynamically – DCSB
  - ✦ $U(1)_A$ is broken in the anomalous mode – $U(1)$ problem – massive $\eta'$

- NJL interaction Lagrangian must respect the symmetries

$$\mathcal{S}_{NJL} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

  - ✦ in NJL $SU(3)_c$ will be considered a global gauge symmetry
  - ✦ $U(1)_A$ is often broken explicitly $\Longrightarrow m_{\eta'} \neq 0$

# *NJL Lagrangian (2)*

$$\mathcal{S}_{NJL} = SU(3)_c \otimes SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes \mathcal{C} \otimes \mathcal{P} \otimes \mathcal{T}$$

- The NJL Lagrangian should be symmetric under the transformations

$$\begin{aligned}
SU(N_f)_V: &\quad \psi \longrightarrow e^{-i\,\boldsymbol{t}\cdot\boldsymbol{\theta_V}}\,\psi & \bar\psi \longrightarrow \bar\psi\, e^{i\,\boldsymbol{t}\cdot\boldsymbol{\theta_V}} \\
SU(N_f)_A: &\quad \psi \longrightarrow e^{-i\,\gamma_5\,\boldsymbol{t}\cdot\boldsymbol{\theta_A}}\,\psi & \bar\psi \longrightarrow \bar\psi\, e^{-i\,\gamma_5\,\boldsymbol{t}\cdot\boldsymbol{\theta_A}} \\
U(1)_V: &\quad \psi \longrightarrow e^{-i\,\theta}\,\psi & \bar\psi \longrightarrow \bar\psi\, e^{i\,\theta} \\
U(1)_A: &\quad \psi \longrightarrow e^{-i\,\gamma_5\,\theta}\,\psi & \bar\psi \longrightarrow \bar\psi\, e^{-i\,\gamma_5\,\theta}
\end{aligned}$$

- Nambu and Jona-Lasinio choose the Lagrangian

$$\mathcal{L} = \bar\psi\,(i\partial\!\!\!/ - m)\,\psi + G_\pi\left[(\bar\psi\psi)^2 - (\bar\psi\,\gamma_5\boldsymbol{\tau}\,\psi)^2\right]$$

- Can choose any combination of these $4-$fermion interactions

$$\begin{aligned}
&(\bar\psi\psi)^2, \quad (\bar\psi\,\gamma_5\,\psi)^2, \quad (\bar\psi\,\gamma^\mu\,\psi)^2 \quad (\bar\psi\,\gamma^\mu\gamma_5\,\psi)^2, \quad (\bar\psi\,i\sigma^{\mu\nu}\,\psi)^2, \\
&(\bar\psi\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,\gamma_5\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,\gamma^\mu\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,\gamma^\mu\gamma_5\,\boldsymbol{t}\,\psi)^2, \quad (\bar\psi\,i\sigma^{\mu\nu}\,\boldsymbol{t}\,\psi)^2.
\end{aligned}$$

# NJL Lagrangian (3)

- The most general $N_f = 2$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + G_\pi\left[\left(\bar{\psi}\psi\right)^2 - \left(\bar{\psi}\,\gamma_5\boldsymbol{\tau}\,\psi\right)^2\right] + G_\omega\left(\bar{\psi}\,\gamma^\mu\,\psi\right)^2 + G_\rho\left[\left(\bar{\psi}\,\gamma^\mu\boldsymbol{\tau}\,\psi\right)^2 + \left(\bar{\psi}\,\gamma^\mu\gamma_5\boldsymbol{\tau}\,\psi\right)^2\right]$$
$$+G_h\left(\bar{\psi}\,\gamma^\mu\gamma_5\,\psi\right)^2 + G_\eta\left[\left(\bar{\psi}\,\gamma_5\,\psi\right)^2 - \left(\bar{\psi}\,\boldsymbol{\tau}\,\psi\right)^2\right] + G_T\left[\left(\bar{\psi}\,i\sigma^{\mu\nu}\psi\right)^2 - \left(\bar{\psi}\,i\sigma^{\mu\nu}\boldsymbol{\tau}\,\psi\right)^2\right]$$

✦ $\mathcal{L}_I$ is $U(1)_A$ invariant if: $G_\pi = -G_\eta$ & $G_T = 0$

$$\bar{\psi}\psi \longleftrightarrow \sigma \longleftrightarrow (J^P, T) = (0^+, 0)$$
$$\bar{\psi}\,\gamma_5\boldsymbol{\tau}\,\psi \longleftrightarrow \pi \longleftrightarrow (J^P, T) = (0^-, 1)$$
$$\bar{\psi}\,\gamma^\mu\,\psi \longleftrightarrow \omega \longleftrightarrow (J^P, T) = (1^-, 0)$$
$$\bar{\psi}\,\gamma^\mu\boldsymbol{\tau}\,\psi \longleftrightarrow \rho \longleftrightarrow (J^P, T) = (1^-, 1)$$
$$\bar{\psi}\,\gamma^\mu\gamma_5\,\psi \longleftrightarrow f_1, h_1 \longleftrightarrow (J^P, T) = (1^+, 0)$$
$$\bar{\psi}\,\gamma^\mu\gamma_5\boldsymbol{\tau}\,\psi \longleftrightarrow a_1 \longleftrightarrow (J^P, T) = (1^+, 1)$$
$$\bar{\psi}\boldsymbol{\tau}\psi \longleftrightarrow a_0 \longleftrightarrow (J^P, T) = (0^+, 1)$$
$$\bar{\psi}\,\gamma_5\,\psi \longleftrightarrow \eta, \eta' \longleftrightarrow (J^P, T) = (0^-, 0)$$

# NJL Lagrangian (4)

- The most general $N_f = 2$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L}_I = \tfrac{1}{2}\, G_\pi \left[ (\bar\psi\psi)^2 - (\bar\psi\,\gamma_5 \boldsymbol{\tau}\,\psi)^2 \right] - \tfrac{1}{2}\, G_\omega\, (\bar\psi\,\gamma^\mu\,\psi)^2 - \tfrac{1}{2}\, G_\rho \left[ (\bar\psi\,\gamma^\mu \boldsymbol{\tau}\,\psi)^2 - (\bar\psi\,\gamma^\mu\gamma_5 \boldsymbol{\tau}\,\psi)^2 \right]$$
$$+ \tfrac{1}{2}\, G_f\, (\bar\psi\,\gamma^\mu\gamma_5\,\psi)^2 - \tfrac{1}{2}\, G_\eta \left[ (\bar\psi\,\gamma_5\,\psi)^2 - (\bar\psi\,\boldsymbol{\tau}\,\psi)^2 \right] - \tfrac{1}{2}\, G_T \left[ (\bar\psi\, i\sigma^{\mu\nu}\psi)^2 - (\bar\psi\, i\sigma^{\mu\nu} \boldsymbol{\tau}\,\psi)^2 \right]$$

  - ✦ $\mathcal{L}_I$ is $U(1)_A$ invariant if: $G_\pi = -G_\eta$ & $G_T = 0$

- The most general $N_f = 3$ NJL Lagrangian that respects the symmetries is

$$\mathcal{L}_I = G_\pi \left[ \tfrac{1}{6}\, (\bar\psi\psi)^2 + (\bar\psi\, \boldsymbol{t}\,\psi)^2 - \tfrac{1}{6}\, (\bar\psi\,\gamma_5\,\psi)^2 - (\bar\psi\,\gamma_5\, \boldsymbol{t}\,\psi)^2 \right]$$
$$-\frac{1}{2}\, G_\rho \left[ (\bar\psi\,\gamma^\mu\, \boldsymbol{t}\,\psi)^2 + (\bar\psi\,\gamma^\mu\gamma_5\, \boldsymbol{t}\,\psi)^2 \right] - \frac{1}{2}\, G_\omega\, (\bar\psi\,\gamma^\mu\,\psi)^2 - \frac{1}{2}\, G_f\, (\bar\psi\,\gamma^\mu\gamma_5\,\psi)^2$$

- Enlarging the $SU(N_f)_V \otimes SU(N_f)_A$ chiral group from $N_f = 2$ to $N_f = 3$ reduces the number of coupling from six to four

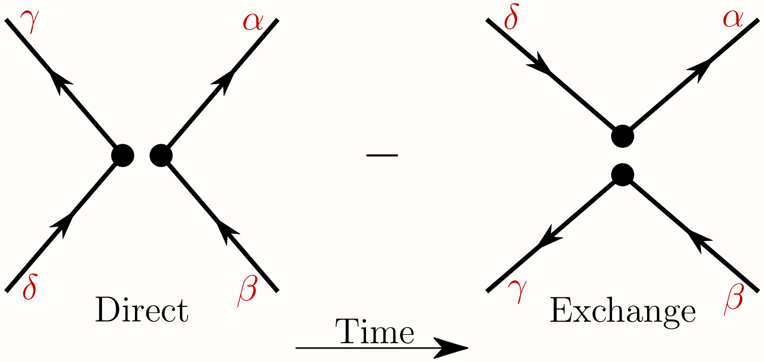
- The $N_f = 3$ Lagrangian is automatically $U(1)_A$ invariant

  - ✦ $U(1)_A$ is then often broken by the 't Hooft term – a 6-quark interaction

$$\mathcal{L}_I^{(6)} = K \left[ \det\left(\bar\psi(1+\gamma_5)\psi\right) + \det\left(\bar\psi(1-\gamma_5)\psi\right) \right]$$

# NJL Interaction Kernel

- Using Wick's theorem and the NJL Lagrangian their are 2 diagrams for the interaction between a quark and an anti-quark




$$2i\,G\left[\Omega^i_{\alpha\beta}\overline{\Omega}^i_{\gamma\delta} - \Omega^i_{\alpha\delta}\overline{\Omega}^i_{\gamma\beta}\right]$$

- Using Fierz transformations can express each *exchange term* as a sum of *direct terms*
- The $SU(2)$ NJL interaction kernel then takes the form

$$K_{\alpha\beta,\gamma\delta} = 2i\,G_\pi\left[(\mathbb{1})_{\alpha\beta}(\mathbb{1})_{\gamma\delta} - (\gamma_5\boldsymbol{\tau})_{\alpha\beta}(\gamma_5\boldsymbol{\tau})_{\gamma\delta}\right] - 2i\,G_\omega\,(\gamma_\mu)_{\alpha\beta}(\gamma^\mu)_{\gamma\delta}$$
$$- 2i\,G_\rho\left[(\gamma_\mu\boldsymbol{\tau})_{\alpha\beta}(\gamma^\mu\boldsymbol{\tau})_{\gamma\delta} + (\gamma_\mu\gamma_5\boldsymbol{\tau})_{\alpha\beta}(\gamma^\mu\gamma_5\boldsymbol{\tau})_{\gamma\delta}\right] + \ldots$$

- This kernel enters the NJL gap and meson Bethe-Salpeter equations

# Regularization Schemes

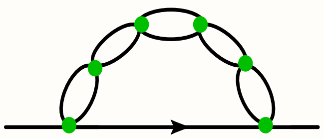
- The NJL model is non-renormalizable $\Longrightarrow$ cannot remove regularization
  - regularization parameter(s) play a dynamical role
- Popular choices are:
  - 3-momentum cutoff: $\vec{p}^2 < \Lambda^2$
  - 4-momentum cutoff $p_E^2 < \Lambda^2$
  - Pauli-Villars
- We will use the proper-time regularization scheme

$$\frac{1}{X^n} = \frac{1}{(n-1)!}\int_0^\infty d\tau\; \tau^{n-1}\, e^{-\tau\, X} \;\rightarrow\; \frac{1}{(n-1)!}\int_{1/\Lambda_{UV}^2}^{1/\Lambda_{IR}^2} d\tau\; \tau^{n-1}\, e^{-\tau\, X}$$

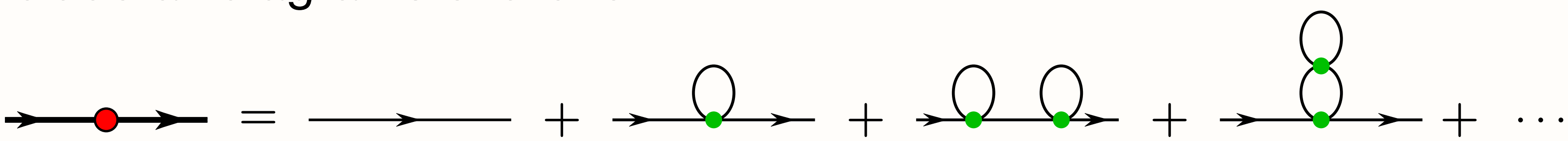
  - only $\Lambda_{UV}$ is need to render the theory finite
  - however, as we shall see, $\Lambda_{IR}$ plays a very important role; it prevents quarks going on their mass shell and hence simulates confinement

# NJL Quark Propagator

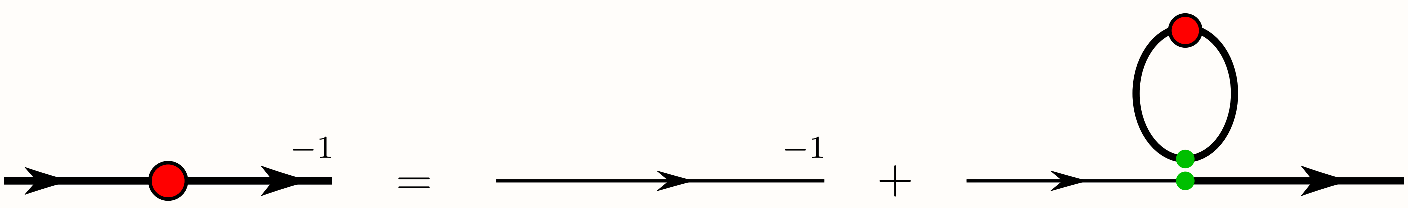
- Complete expression for the quark propagator cannot be obtained
  - ✦ need a truncation
- Do not in include diagrams like:



  - ✦ would give a momentum dependent mass function
- Include all diagrams of the form:



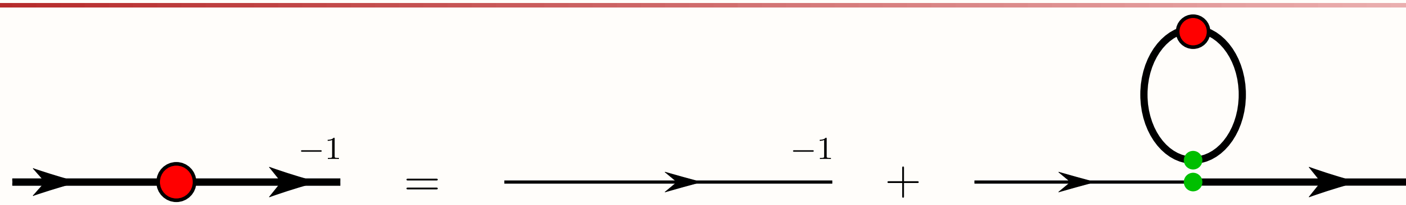
- All these diagrams can be summed via an integral equation




- The most general quark propagator has the form

$$S(p) = \frac{1}{\not{p} - m - \Sigma(p)} = \frac{Z(p^2)}{\not{p} - M(p^2)}$$

# NJL Gap Equation




- The NJL gap equation has the form

$$S^{-1}(k) = S_0^{-1}(k) - \Sigma(k) = [\not{k} - m] - \sum_j \int \frac{d^4\ell}{(2\pi)^4} \mathrm{Tr}\left[S(\ell)\,\overline{\Omega}^j\right]\Omega^j$$

- The only piece of the interaction kernel that contributes is:

$$K^{\sigma}_{\alpha\beta,\gamma\delta} = 2i\,G_\pi\,(\mathbb{1})_{\gamma\delta}\,(\mathbb{1})_{\alpha\beta}$$

- Solving this equation give a quark propagator of the form

$$S^{-1}(k) = \not{k} - M + i\varepsilon$$

- The constituent quark mass satisfies the equation

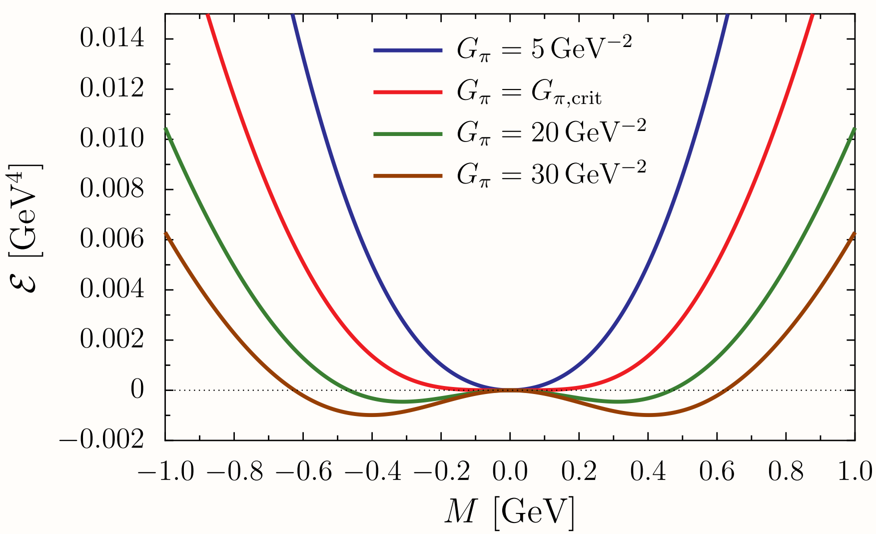
$$M = m + 48i\,G_\pi\,M \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2 - M^2 + i\varepsilon} = m + M\,\frac{3\,G_\pi}{\pi^2}\int d\tau\,\frac{e^{-\tau\,M^2}}{\tau^2}$$

# NJL Gap Equation (2)

$$M = m + M\,\frac{3\,G_\pi}{\pi^2}\int d\tau\,\frac{e^{-\tau\,M^2}}{\tau^2}$$

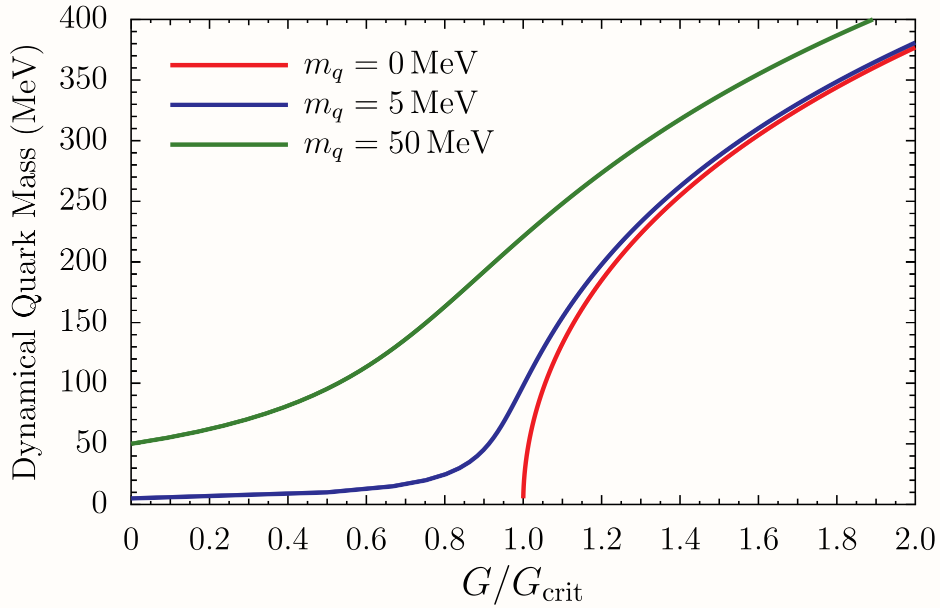
- For the case $m = 0$ the gap equation has two solutions:
  - trivial solution: $M = 0$ & non-trivial solution: $M \neq 0$
- Which solution does nature choose, that is, which solution minimizes the energy. Compare vacuum energy density, $\mathcal{E}$, for each case

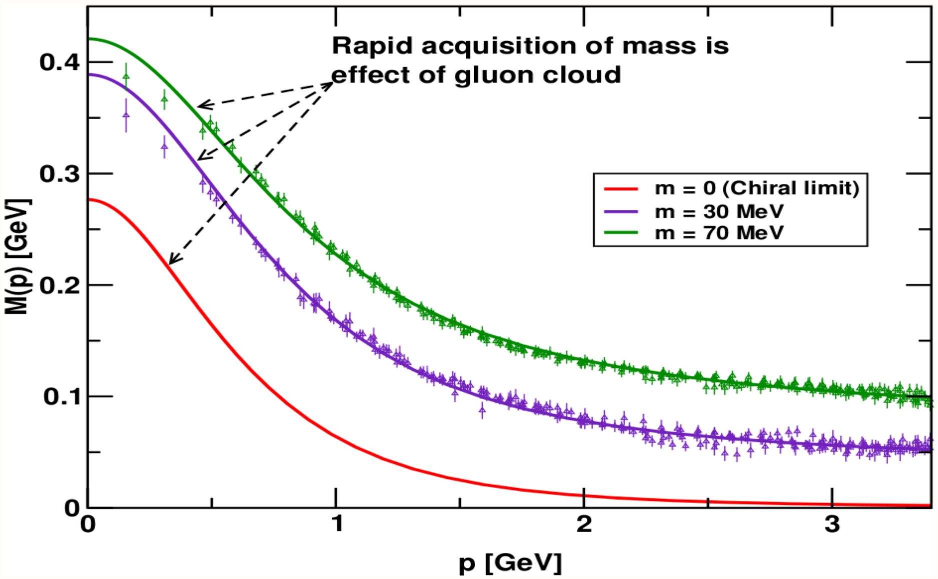
$$\mathcal{E}(M) - \mathcal{E}(M=0) = -\frac{3}{4\pi^2}\int d\tau\,\frac{1}{\tau^3}\left(e^{-\tau M^2} - 1\right) + \frac{M^2}{4\,G_\pi}$$




- For $G_\pi > G_{\pi,\text{crit}}$ the lowest energy solution has $M \neq 0$
- Therefore for $G_\pi > G_{\pi,\text{crit}}$ NJL has DCSB
- DCSB $\Longleftrightarrow$ generates mass from nothing

# NJL & DSE gap equations




- NJL constituent mass is given by: $M = m - 2\,G_\pi\,\langle\bar{\psi}\psi\rangle$
- Chiral condensate is defined by

$$\langle\bar{\psi}\psi\rangle \equiv \lim_{x\to y} \mathrm{Tr}\,[-iS(x-y)] = -\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,[i\,S(k)]$$

- Mass is generated via interaction with vacuum
- Dynamically generated quark masses $\iff$ $\langle\bar{\psi}\psi\rangle \neq 0$
- $\langle\bar{\psi}\psi\rangle = \langle\bar{u}u + \bar{d}d\rangle$ is an order parameter for DSCB

# Confinement in NJL model

- In general the NJL model is not confining; quark propagator is simply

$$S(k) = \frac{1}{\not{k} - M + i\varepsilon} = \frac{\not{k} + M}{k^2 - M^2 + i\varepsilon}$$

  - ✦ quark propagator has a pole $\Longrightarrow$ quarks are part of physical spectrum

- However the proper-time scheme is unique

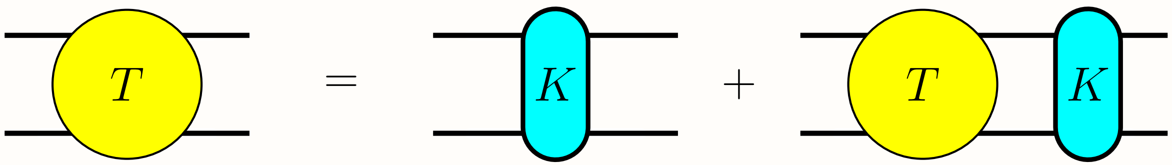
$$S(k) = \int_0^\infty d\tau \, (\not{k} + M) \, e^{-\tau\left(k^2 - M^2\right)} \to \underbrace{\frac{\left[e^{-(k^2-M^2)/\Lambda_{UV}^2} - e^{-(k^2-M^2)/\Lambda_{IR}^2}\right]}{k^2 - M^2}}_{\equiv Z(k^2)} [\not{k} + M]$$

- quark propagator does not have a pole: $Z(k^2) \overset{k^2 \to M^2}{=} \frac{1}{\Lambda_{IR}^2} - \frac{1}{\Lambda_{UV}^2} \neq \infty$

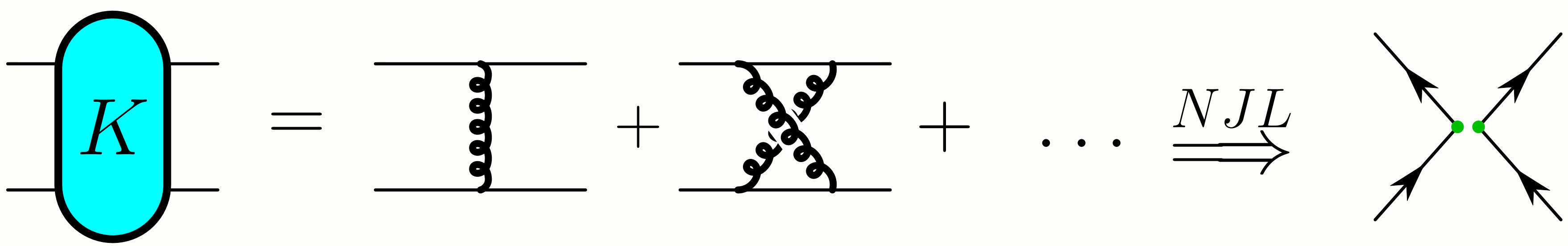
- Are confinement and DCSB related?
  - ✦ NJL model is proof that DCSB can exist without confinement
  - ✦ however commonly believed cannot have confinement without DCSB

# Hadron Spectrum

- In QFT physical states appear as poles in $n$-point Green Functions
- For example, the full quark–antiquark scattering matrix or $T$-matrix, contains poles for all $\bar{q}q$ bound states, that is, the physical mesons
- The quark–antiquark $T$-matrix is obtained by solving the Bethe-Salpeter equation




- In principle kernel, $K$, contains all possible 2PI diagrams



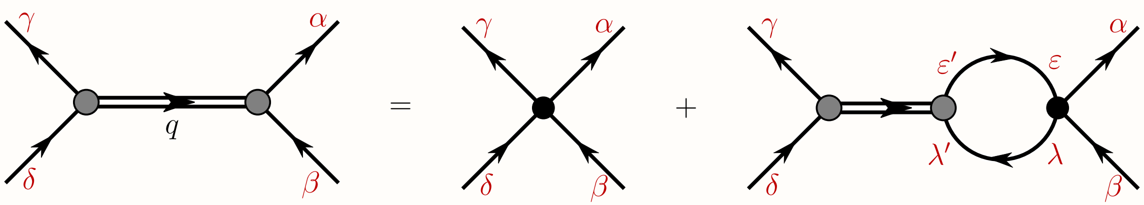

- Kernel of gap and BSEs are intimately related

$$q_\mu \, \Gamma_5^{\mu,i}(p',p) = S^{-1}(p')\,\gamma_5\,\tfrac{1}{2}\tau_i + \tfrac{1}{2}\tau_i\,\gamma_5\,S^{-1}(p) + 2\,m\,\Gamma_\pi^i(p',p)$$

# Bethe-Salpeter Equation for the Pion

- How does the pion become (almost) massless when it is composed of two massive constituents
- The pion is realized as the lowest lying pole in the quark anti-quark $T$-matrix in the pseudo-scalar channel
- In the NJL model this $T$-matrix is given by

$$\mathcal{T}(q)_{\alpha\beta,\gamma\delta} = \mathcal{K}_{\alpha\beta,\gamma\delta} + \int \frac{d^4k}{(2\pi)^4}\, \mathcal{K}_{\alpha\beta,\lambda\epsilon}\, S(q+k)_{\varepsilon\varepsilon'}\, S(k)_{\lambda'\lambda}\, \mathcal{T}(q)_{\varepsilon'\lambda',\gamma\delta},$$




$$\mathcal{K}_\pi = -2i\, G_\pi\, (\gamma_5 \boldsymbol{\tau})_{\alpha\beta} (\gamma_5 \boldsymbol{\tau})_{\lambda\epsilon}$$

- The NJL pion $t$-matrix is

$$\mathcal{T}(q)^i_{\alpha\beta,\gamma\delta} = (\gamma_5 \tau_i)_{\alpha\beta}\, \frac{-2i\, G_\pi}{1 + 2\, G_\pi\, \Pi_\pi(q^2)}\, (\gamma_5 \tau_i)_{\gamma\delta}$$

- The pion mass is then given by: $1 + 2\, G_\pi\, \Pi_\pi(q^2 = m_\pi^2) = 0$

# *The Pion as a Goldstone Boson*

- Recall the pion pole condition – $-1 + 2\,G_\pi\,\Pi_\pi(q^2 = m_\pi^2) = 0$ – where

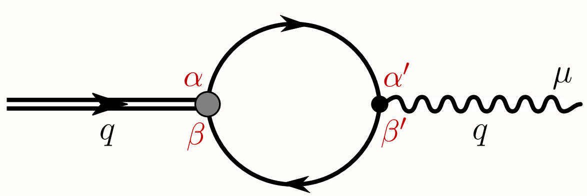
$$\Pi_\pi(q^2) = \frac{m}{2\,G_\pi\,M} - \frac{1}{2\,G_\pi} - q^2\,I(q^2)$$

  - ✦ have used the gap equation to obtain this expression

- The pion pole condition therefore implies

$$m_\pi^2 = \frac{m}{2\,G_\pi\,M\,I(m_\pi^2)}$$

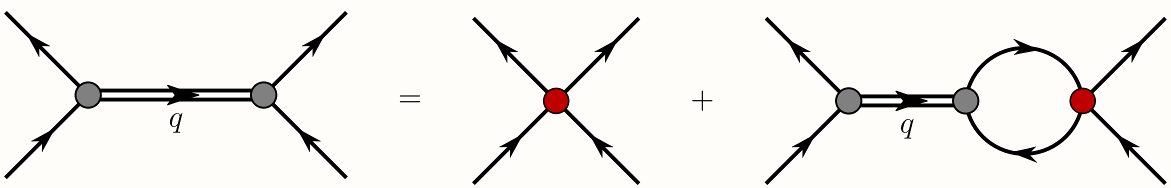
- Therefore in the chiral limit – $m \to 0 \quad (M \neq 0)$ – pion is massless
- Can show other chiral symmetry relations are also satisfied:
  - ✦ $f_\pi\,g_{\pi qq} = M\,g_{Aqq}$ Goldberger–Treiman (GT) relation
  - ✦ $f_\pi^2\,m_\pi^2 = \frac{1}{2}\,(m_u + m_d)\,\langle \bar{u}u + \bar{d}d \rangle$ Gell-Mann–Oakes–Renner (GMOR)




$$\langle 0\,|A_a^\mu|\,\pi_b(p)\rangle = i\,f_\pi\,p^\mu\,\delta_{ab}$$

# Chiral Partners

- If chiral symmetry was *NOT* dynamically broken in nature expect mass degenerate chiral partners, e.g., $m_\sigma \simeq m_\pi \quad \& \quad m_{a_1} \simeq m_\rho$
- The $\rho$ and $a_1$ are the lowest lying vector ($J^P = 1^-$) and axial-vector ($J^P = 1^+$) $\bar{q}q$ bound states: $m_\rho^{\text{exp't}} \simeq 770\,\text{MeV}$ & $m_{a_1}^{\text{exp't}} \simeq 1230\,\text{MeV}$
- Masses given by $T$-matrix poles in the vector and axial-vector $\bar{q}q$ channels



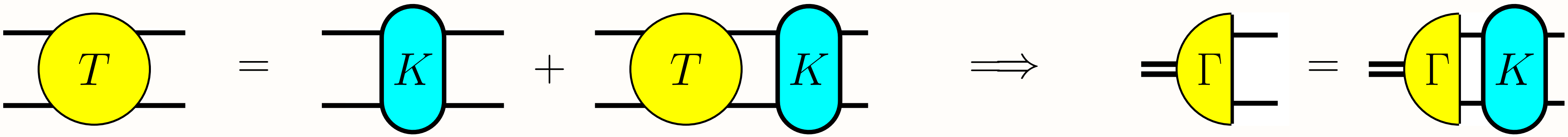

$$\mathcal{K} = -2i\,G_\rho\left[(\gamma_\mu \boldsymbol{\tau})(\gamma^\mu \boldsymbol{\tau}) + (\gamma_\mu\gamma_5 \boldsymbol{\tau})(\gamma^\mu\gamma_5 \boldsymbol{\tau})\right]$$

- Pole conditions: $1 + 2\,G_\rho\,\Pi_\rho(m_\rho^2) = 0$ & $1 + 2\,G_\rho\,\Pi_{a_1}(m_{a_1}^2) = 0$

$$\Pi_{a_1}(q^2) = M^2\,I(q^2) + \Pi_\rho(q^2)$$

- DCSB splits masses; NJL gives: $m_\rho \equiv 770\,\text{MeV} \quad \& \quad m_{a_1} \simeq 1098\,\text{MeV}$
  - ✦ good agreement with the Weinberg sum rule result: $m_{a_1} \simeq \sqrt{2}\,m_\rho$
- Pion conditions for $\pi$ and $\sigma$ $\implies$ $m_\sigma^2 \simeq m_\pi^2 + 4\,M^2$

# Homogeneous Bethe-Salpeter vertex functions




- Near a bound state pole of mass $m$ a two-body $t$-matrix behaves as

$$\mathcal{T}(p,k) \to \frac{\Gamma(p,k)\,\bar{\Gamma}(p,k)}{p^2 - m^2} \qquad \text{where} \qquad p = p_1 + p_2, \quad k = p_1 - p_2$$

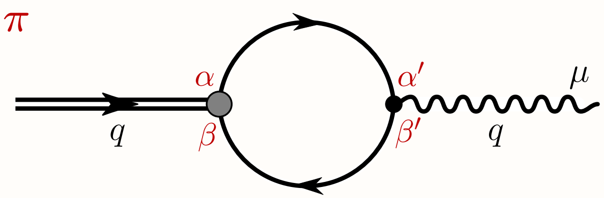
- $\Gamma(p,k)$ is the homogeneous Bethe-Salpeter vertex and describes the relative motion of the quark and anti-quark while they form the bound state
- Expanding the pion $T$-matrix about the pole gives

$$\mathcal{T} = \gamma_5\tau_i\, \frac{-2i\,G_\pi}{1+2\,G_\pi\,\Pi_\pi(q^2)}\,\gamma_5\tau_i \to \frac{i\,g_{\pi qq}^2}{q^2-m_\pi^2}(\gamma_5\tau_i)(\gamma_5\tau_i) \quad \Longrightarrow \quad \Gamma_\pi = \sqrt{g_\pi}\,\gamma_5\tau_i$$

  - ✦ $g_{\pi qq}$ is effective pion-quark coupling constant

- Bethe-Salpeter vertex needed for calculations e.g. $f_\pi$

$$i\,f_\pi\,q^\mu\,\delta_{ij} = \int \frac{d^4k}{(2\pi)^4}\,\text{Tr}\left[\tfrac{1}{2}\,\gamma^\mu\gamma_5\tau_j\,S(k)\,\Gamma_\pi^i\,S(k-q)\right]$$



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