Scattering phases and finite volume eigenstates

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Pion–nucleon cross section
Resonance at large quark mass

\[ E_{\pi\pi} = 2\sqrt{m_{\pi}^2 + \vec{p}_{lat}^2} \]
\[ \vec{p}_{lat} = \frac{2\pi}{L} \vec{n} \]

\[ m_{\pi} L = 2\pi \]

CP-PACS(2002)
Baryons: Simple parameterisation

• First “guess” at baryons; use simple (one parameter) form

\[ M_B = M_B^0 \left( \frac{1}{1 + \frac{3}{2} \frac{m_\pi}{M_B^0}} + \frac{3}{2} \frac{m_\pi}{M_B^0} \right) \]
Non-interacting energies

$E$ (GeV)

$m_{\pi}^2$ (GeV$^2$)
Interactions: much more interesting

- Real part of rho meson mass

![Graph showing the relationship between $M_V$ (GeV) and $m_\pi^2$ (GeV$^2$)](image)
Lüscher: Unstable particles / phase shifts

- Map out volume dependence of energy levels

Wiese [Lat88] “free” 2-particle discrete spectrum

- Resonance “plateau” for weakly interacting system ie. small width
Quantum Mechanics: 1D

- 2-Body scattering
  - Consider 2 identical bosons
    \[ \psi(x_1, x_2) = \psi(x_2, x_1) \]
  - Interact by potential dependent on relative coordinate only
    \[ V = V(|x_1 - x_2|) = V(|x|) \]
  - Potential only acts over a finite range
    \[ V(x) = 0 \text{ for all } |x| > R \]
1D: 2-body scattering

• Kinetic energy operator

\[ K = -\frac{1}{2m} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \]

\[ = -\frac{1}{2M} \frac{\partial^2}{\partial X^2} - \frac{1}{2\mu} \frac{\partial^2}{\partial x^2} \]

\[ X \equiv (x_1 + x_2)/2, \quad x \equiv x_1 - x_2 \]

\[ M \equiv 2m, \quad \mu = m/2 \]

• Separable solution

\[ \psi(x, X) = \psi(x) \Psi(X) \]

• Bose symmetry

\[ \psi(x) = \psi(-x) \]

• We’ll consider just this wavefunction in the relative coordinate
1D: 2-body scattering

• General solution
  • Right \( \psi^{(r)}(x) = Ae^{-ikx} + Be^{ikx} \)
  • Left \( \psi^{(l)}(x) = Ae^{ikx} + Be^{-ikx} \)

• Steady-state: conservation of probability \( \Rightarrow |B| = |A| \)
  • Define \( \frac{B}{A} = e^{i2\delta} \)

\( \Rightarrow \psi^{(r)}(x) = A \left( e^{-ikx} + e^{i2\delta(k)}e^{ikx} \right) \)

• For any potential, scattering process is entirely determined by the phase shift \( \delta(k) \)
1D: Delta function well

- **Example** potential
  \[ V(x) = -\alpha \delta(x) \]

- **Bound state**
  \[ E = -\frac{\mu \alpha^2}{2} \equiv -E_b \]

- **Scattering**
  \[ \tan 2\delta = \frac{2k\mu\alpha}{k^2 - \mu^2\alpha^2} \]
1D: Finite “volume”

- Periodic boundary conditions
  - “Box” length $L$
    
    $\psi(x + L) = \psi(x)$

- Boundary $x = \pm L/2$
  - Continuous: Bose symmetry
  
    $\Rightarrow \psi^{(r)}(L/2) = \psi^{(l)}(-L/2)$
  - Smooth
    
    $-ik \left( Ae^{ikL/2} - Be^{-iKL/2} \right) = ik \left( Ae^{ikL/2} - Be^{-iKL/2} \right),$
    
    $\Rightarrow \frac{B}{A} = e^{-ikL}.$

- Potential defines phase shift through interaction region

  $e^{i2\delta(k)} = e^{-ikL}$

- $\Rightarrow$ Eigenvalue equation

  $0 = \delta(k) + \frac{KL}{2} + n\pi \quad (n \in \mathbb{Z})$
1D: Finite “volume”

\[ 0 = \delta(k) + \frac{kL}{2} + n\pi \quad (n \in \mathbb{Z}) \]

- Eigenstates of the finite-volume system are entirely determined if the phase shift is known.

- Lattice QCD (if it were 1D QM with a finite-range potential):
  - Finite volume energies correspond uniquely to the scattering phase shift at particular momenta.
1D: Delta function well in a finite volume

- Eigenvalue equation

- Solutions correspond to eigenvalue of the momentum in the non-interacting region
1D: Delta function well in a finite volume

- Solid lines: Eigenenergies on the finite box
- Dashed lines: Periodic solutions in absence of potential
3D: Spherical potential scattering

- Hamiltonian
  \[ H = \frac{-1}{2\mu} \nabla^2 + V(r), \]
  \[ r = |\vec{r}|, \quad \vec{r} = \vec{x}_1 - \vec{x}_2 \]

- Stationary state
  \[ H\psi = E\psi \]

- Radial wavefunction decomposition
  \[ \psi(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\hat{r}) \psi_{lm}(r) \]
  satisfies radial Schrödinger equation
3D: Spherical potential scattering

- Radial SE independent of $m$, write
  \[ \psi_{lm}(r) = b_{lm} u_l(r; k) \]
- For $r > R$, potential vanishes, solution expressed in terms of spherical Bessel functions
  \[ u_l(r; k) = \alpha_l(k) j_l(kr) + \beta_l(k) n_l(kr) \]
  - Note: integrability requires a unique solution for each $k$ and $l$
- Partial wave scattering phase
  \[ e^{i2\delta_l(k)} = \frac{\alpha_l(k) + i\beta_l(k)}{\alpha_l(k) - i\beta_l(k)} \]
- Scattering determined by phase shifts
3D: Finite volume

- Circles don’t go in squares!
3D: Finite volume

Want solutions in non-interacting region

Smooth periodic solutions of Helmholtz equation in the exterior region

\[(\nabla^2 + k^2)\psi(\vec{r}) = 0\]

For \(R < r < L/2\), these are just superpositions of radial solutions
3D: Finite volume

- Exterior region \((r<L/2)\)
  \[
  \psi_{lm}(\vec{r}) = b_{lm} \{ \alpha_l(k)j_l(kr) + \beta_l(k)n_l(kr) \}
  \]

- Coefficients chosen to squeeze our circles into the box
  - That is, to obtain the solutions that are smooth and periodic
    \((The Toddler Problem: Perhaps another lecture sometime?)\)
3D: Finite volume

- Toddler solution

\[ G_{lm}(\hat{r}; k^2) = \frac{(-1)^l k^{l+1}}{4\pi} \left[ Y_{lm}(\hat{r}) n_l(kr) + \sum_{js} M_{lm;js}(k) Y_{js}(\hat{r}) j_j(kr) \right] \]

\[ M_{l'm';lm}(k^2) = \frac{(-1)^{l'}}{\pi^{3/2}} \sum_{j'=|l'-l|} \sum_{s=-j} \frac{j^i j^j}{q^{j+1}} Z_{js}(1; q^2) C_{l'm';js;lm} \]

\[ C_{l'm';js;lm} = (-1)^m i^{l'-j+l} \sqrt{(2l'+1)(2j+1)(2l+1)} \times \begin{pmatrix} l' & j & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l' & j & l \\ m' & s & -m \end{pmatrix} \]

\[ Z_{lm}(s; q^2) = \sum_{\vec{n} \in \mathbb{Z}^3} \frac{|\vec{n}|^l Y_{lm}(\hat{n})}{(\vec{n}^2 - q^2)^s} \]

Generally requires renormalisation (eg. analytic continuation in \(s\))
3D: Finite volume

- General solution of Helmholtz equation

\[
\psi(\vec{r}; k^2) = \sum_{l,m} v_{lm} G_{lm}(\vec{r}; k^2)
\]

\[
\sum_{l,m} v_{lm} G_{lm}(\vec{r}; k^2) = \sum_{lm} \left[ \frac{(-1)^l k_1^{l+1}}{4\pi} v_{lm} n_l(kr) \right. \\
+ \sum_{l'm'} \frac{(-1)^{l'} k^{l'+1}}{4\pi} v_{l'm'} \mathcal{M}_{l'm';lm}(k) j_l(kr) \left. \right] Y_{lm}(\hat{r})
\]

- Match scattering solution (non-zero \(c\)'s)

\[
\sum_{l'm'} c_{l'm'} \left\{ U_{l'm';lm} - e^{i2\delta_{l'} \delta_{l'm'}} \right\} = 0
\]

\[
U_{l'm';lm} = \frac{\mathcal{M}_{l'm';lm} + i\delta_{l'l} \delta_{mm'}}{\mathcal{M}_{l'm';lm} - i\delta_{l'l} \delta_{mm'}}
\]
Extracting phase shifts

- Finite-volume spectra give phase shifts

Wiese [Lat88] “free” 2-particle discrete spectrum

Resonance “plateau”
Finite-momentum trick

Lattice volume: $m_\pi L \gtrsim 2\pi$

Lowest non-zero momentum:

$$p_{\text{min}} = \frac{2\pi}{L} \sim m_\pi$$

Rest system: rho couples to 2 back-to-back pions

2-pion energy

$$2\sqrt{m_\pi^2 + p_{\text{min}}^2} \sim 2\sqrt{2}m_\pi$$

Rho “decay” requires

$$\frac{m_\pi}{m_\rho} \lesssim 0.35$$

**BOOST rho to** $p_{\text{min}}$  
Rummukainen & Gottlieb, NPB(1995)

Rho energy

$$\sqrt{m_\rho^2 + p_{\text{min}}^2}$$

2-pion energy

$$m_\pi + \sqrt{m_\pi^2 + p_{\text{min}}^2}$$

“decay” $\frac{m_\pi}{m_\rho} \lesssim 0.45$
Finite-momentum

• Using boosted frames, can extract the scattering phases at many more eigenenergies (momenta)

Dudek et al. arXiv:1212.0830
Coupled-channel resonances
Multi-channel S-matrix on finite volume

- Extension of Lüscher by He, Feng & Liu JHEP(2005)

\[ S^{(l)}(E) = \begin{pmatrix} \eta l e^{2i \delta_1} & i \sqrt{1 - \eta^2} e^{i(\delta_1 + \delta_2)} \\ i \sqrt{1 - \eta^2} e^{i(\delta_1 + \delta_2)} & \eta l e^{2i \delta_2} \end{pmatrix} \]

- Scattering parameters as a function of E
  \[ \delta_1, \delta_2, \eta \]

- Finite-volume eigenvalue equation (s-wave)

\[ \cos(\Delta_1 + \Delta_2 - \delta_1^0 - \delta_2^0) = \eta \cos(\Delta_1 - \Delta_2 - \delta_1^0 + \delta_2^0) \]

\[ \Delta_i = M_{00;00}(q_i^2) = \frac{Z_{00}(1; q_i^2)}{\pi^{3/2} q_i} \]
Toy model for the “Roper”

Two-channel extraction of physical P11 scattering parameters
The “Roper” on a finite volume
Outlook

• In elastic 2-body scattering, we have a straightforward relation that maps finite-volume spectra to scattering parameters

• Beyond these simplest of systems, more work remains to be done
  • Exploration of the inelastic threshold and multi-channel systems
  • Many-body channels
  • Exponential corrections to energies
    • Potentials are not necessarily vanishing at the boundary