Wave functions of the Nucleon

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Introduction

- What do we mean by 'Wave function'?
- As a state in quantum field theory, the nucleon doesn't have a simple wave function in the naive quantum mechanic sense.
- Lattice QCD interpolating fields naturally correspond to a fixed number of quarks
- This leads to a description using the Nambu-Bethe-Saltpeter wavefunction of three quarks

Introduction

- Consider the standard nucleon interpolating fields, but with a displaced d quark

$$\chi_1(x,z;t) = \epsilon_{abc}[u_a^T(x;t)C\gamma_5 d_b(x+z;t)]u_c(x;t)$$
$$\chi_2(x,z;t) = \epsilon_{abc}[u_a^T(x;t)Cd_b(x+z;t)]\gamma_5 u_c(x;t)$$

- Could also consider displacement of the u quarks as long as we consider an interchange $u(x+z) \leftrightarrow u(x)$

Introduction

- Define the wave function given by the two-point correlator of this displaced operator with a standard source operator:

$$W(p,z,t) = \sum_{x \in V} e^{ipx} \langle \chi(x,z;t)\bar{\chi}(x_0,0;t_0) \rangle$$

- Not gauge invariant - fix to Landau gauge

Lattice Parameters

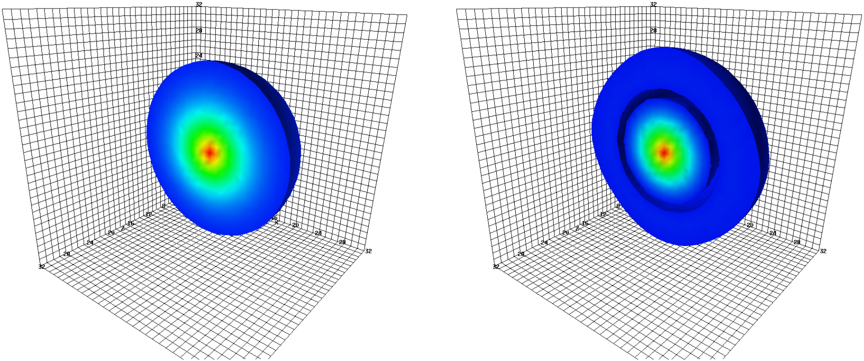
- (2+1) flavour PACS-CS collaboration (Aoki *et al* [arXiv:0807.1661]) via ILDG
- a=0.0907 fm, aL_x=2.9 fm $\approx (68\mathrm{MeV})^{-1}$

κ_{ud}	m_π	N
0.13754	413 MeV	260
0.13770	293 MeV	250×2
0.13781	156 MeV	200×4

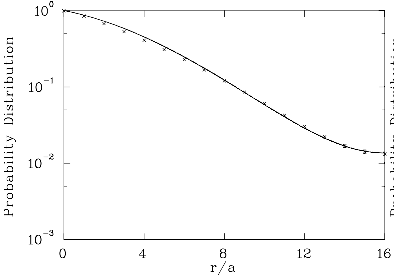
- Gaussian smeared fermion sources, α=0.7

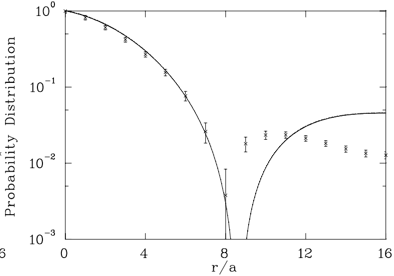
Previous Work

(Roberts *et al* [hep-lat:1311.6626], [hep-lat:1312.2314])



Previous Work





Spherical Harmonics

- $z \neq 0 \rightarrow$ not restricted to zero angular momentum

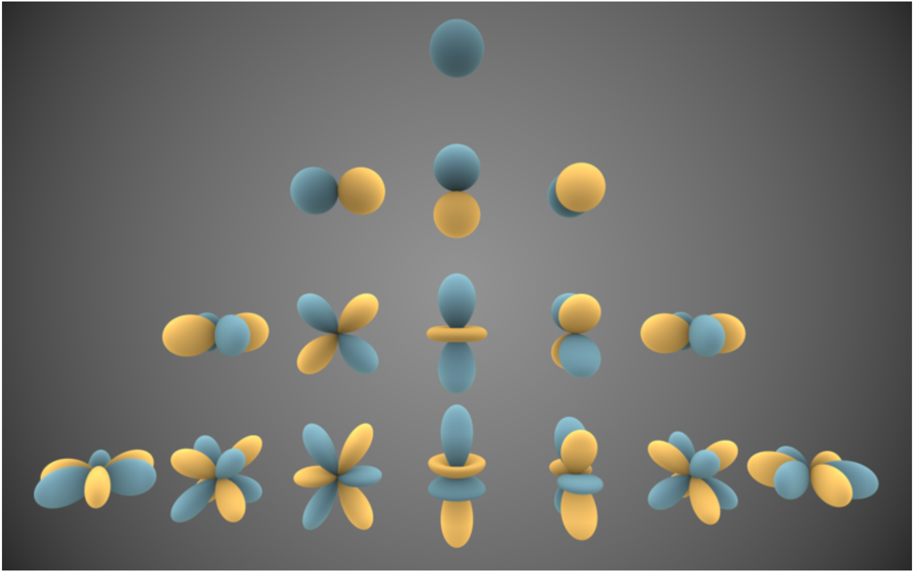
Recall the spherical harmonics $Y_{l,m}$:

$$Y_{0,0} = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

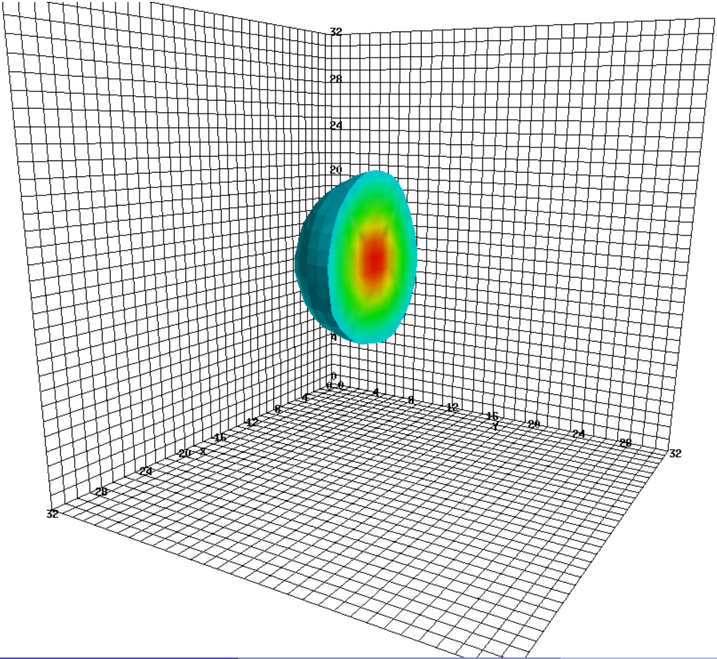
$$Y_{1,0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\mathrm{cos}\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{\mathrm{z}}{\mathrm{r}}$$

$$Y_{1,\pm 1} = \frac{\mp 1}{2}\sqrt{\frac{3}{2\pi}}\mathrm{sin}\theta \mathrm{e}^{\pm \mathrm{i}\phi} = \frac{\mp 1}{2}\sqrt{\frac{3}{\pi}}\frac{\mathrm{x} \pm \mathrm{iy}}{\mathrm{r}}$$

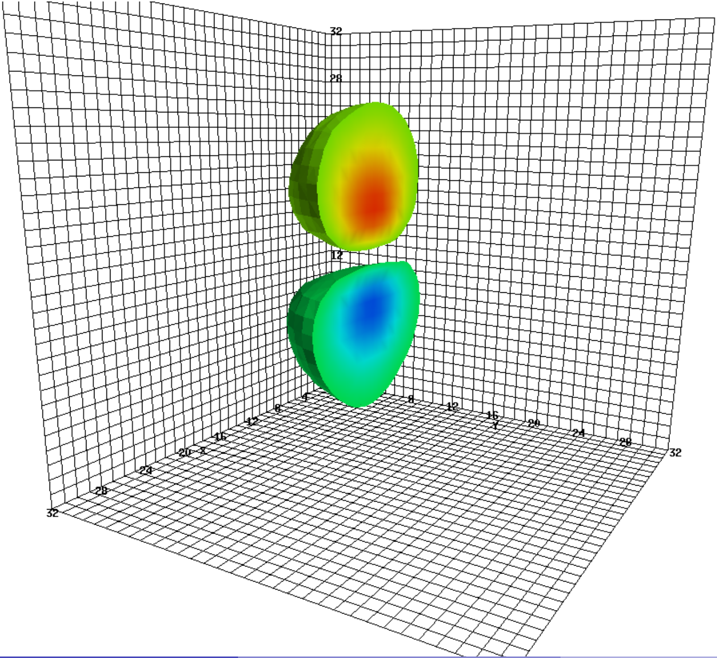
Spherical Harmonics



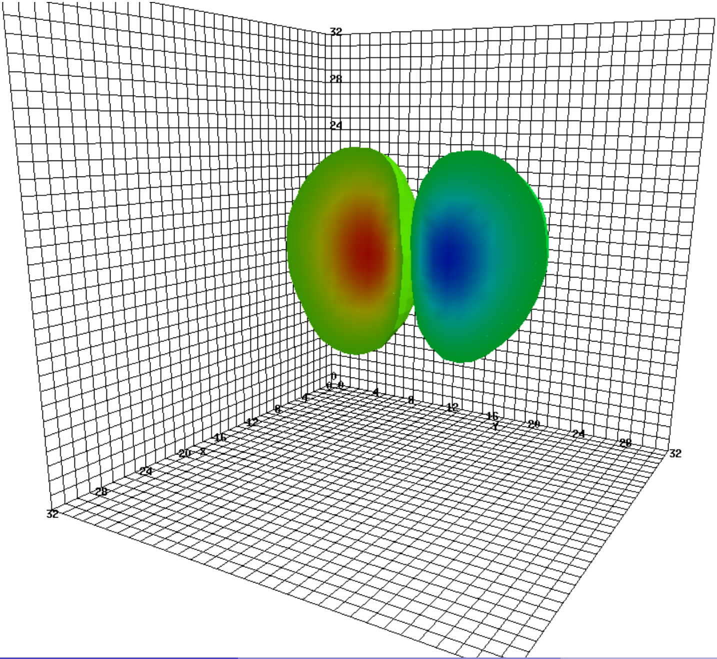
$Y_{0,0} : (i_s = 1, j_s = 1; i_s = 2, j_s = 2)$



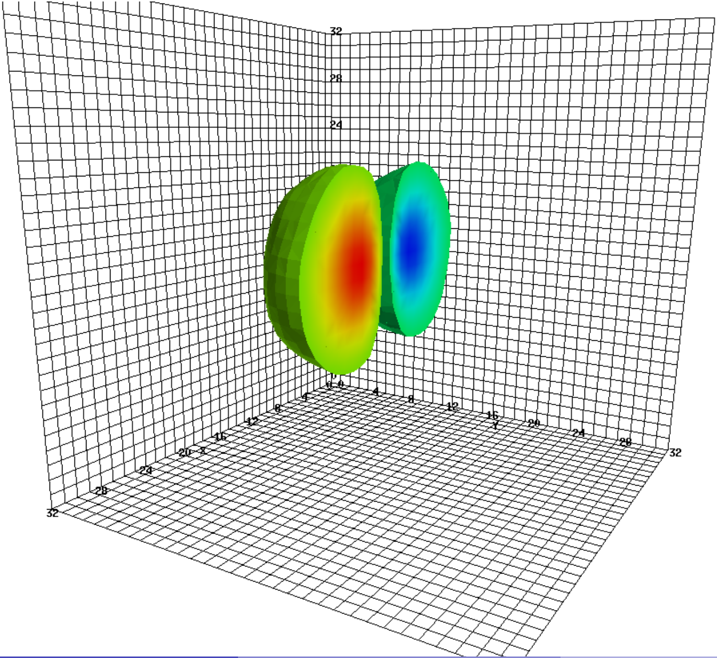
$Y_{1,0} : (i_s = 1, j_s = 3; i_s = 2, j_s = 4)$

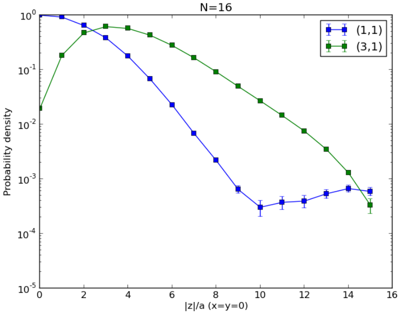


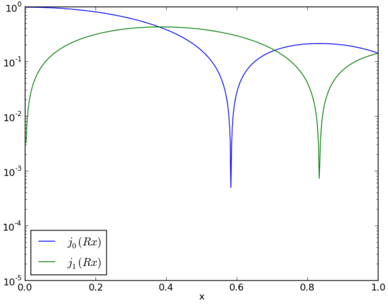
$Y_{1,-1} : (i_s = 1, j_s = 4)$ (real part)



$Y_{1,-1} : (i_s = 1, j_s = 4)$ (imaginary part)




N=16
(1,1)
(3,1)
Probability density
|z|/a (x=y=0)


$j_0(Rx)$
$j_1(Rx)$
x

Variational Analysis

From the standard 2-point correlation function, projected to definite parity:

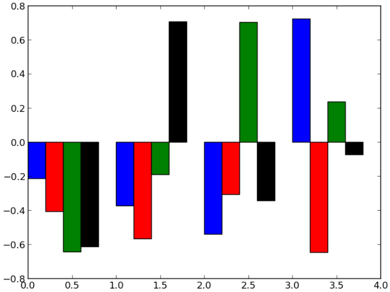
$$G_{ij}(\vec{p}, t) = \sum_{x} e^{-i\vec{p}\cdot\vec{x}} \mathrm{Tr_{sp}}\{\frac{1}{2}(\gamma_0 + \mathrm{I})\langle\chi_\mathrm{i}(\mathrm{x})\bar{\chi}_\mathrm{j}(0)\rangle \tag{1}$$

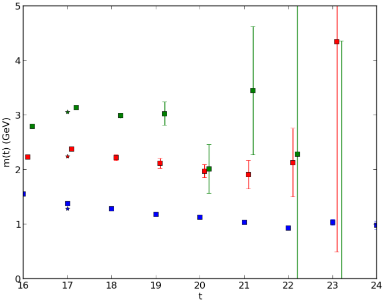
The right eigenvector is defined by the generalized eigenvalue problem:

$$G_{ij}(t_0 + \delta t)v_j = e^{-m\delta t_0} G_{ij}(t)v_j$$

Variational Analysis

t_0=17, dt=3 ; basis N_{sm}=[16, 35, 100, 200]





Normalization

- These eigenvectors are calculated with a normalized correlation matrix

$$G_{ij}(t) \to \frac{G_{ij}(t)}{\sqrt{G_{ii}(0)\,G_{jj}(0)}}$$

- This makes the components of G similar in magnitude
- Otherwise the generalized eigenvalue is badly-behaved
- Then normalize the eigenvectors u, v by their vector norm

Normalization

- In this case, we can't create a full correlation matrix for the wave function (sink smearing would destroy the spatial information we care about)
- However, the sum $v_j \bar{\chi}_j$ has been determined to be the best linear combination of the operators to create a single state from the vacuum.
- The standard normalization for v is then $v_j \rightarrow v_j / \sqrt{G_{jj}(0)}$.

$$\rightarrow W_{proj}(z,t) = \sum_j W_j(z,t) v_j$$

Normalization

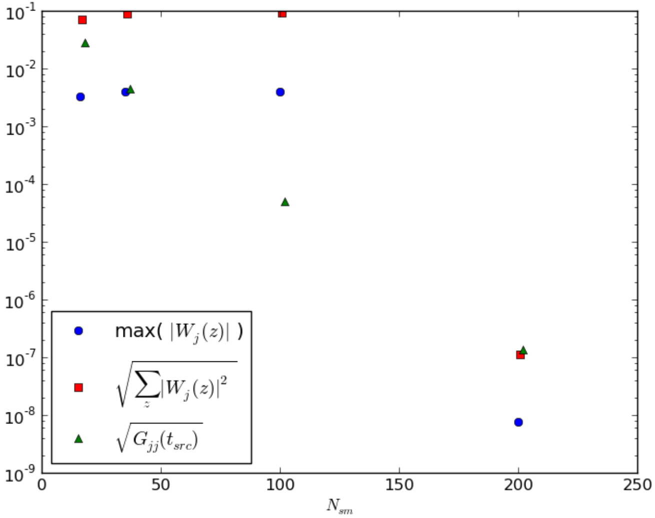
- Consider the probability density for positive parity $P_j(z,t) = |\mathrm{Tr_{Sp}}\Gamma_+ \mathrm{W_j(z,t)}|^2$.
- An alternative normalization could be to scale $W_j(z,t)$ such that

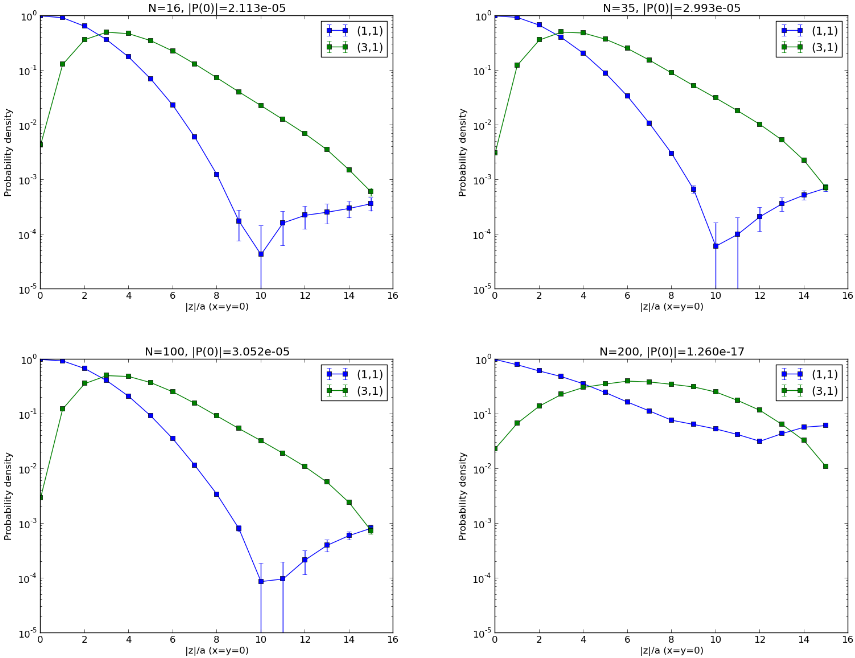
$$\sum_z P_j(z,t) = 1$$

for each individual time value

- How would this change the eigenstate projection?

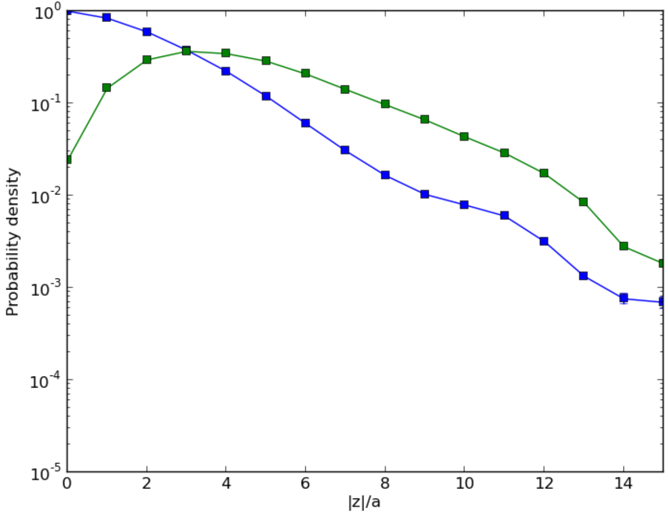
Normalization




N=16, |P(0)|=2.113e-05
N=35, |P(0)|=2.993e-05
N=100, |P(0)|=3.052e-05
N=200, |P(0)|=1.260e-17
(1,1)
(3,1)
Probability density
|z|/a (x=y=0)

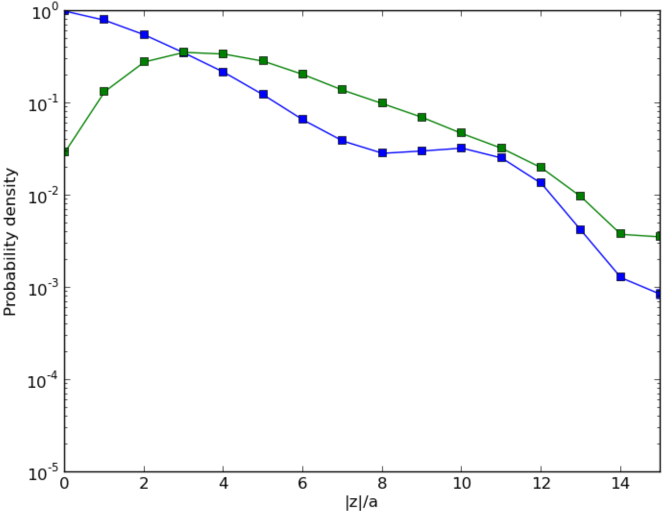
Projected results

Ground state (lightest pion mass):



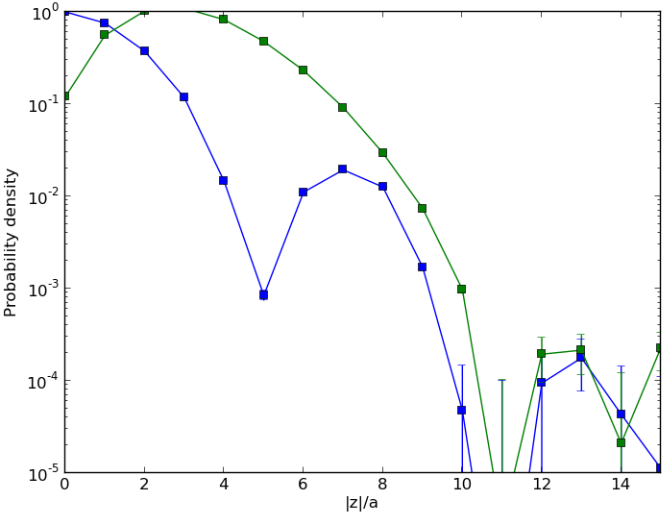
Projected results

First posity parity excited state:



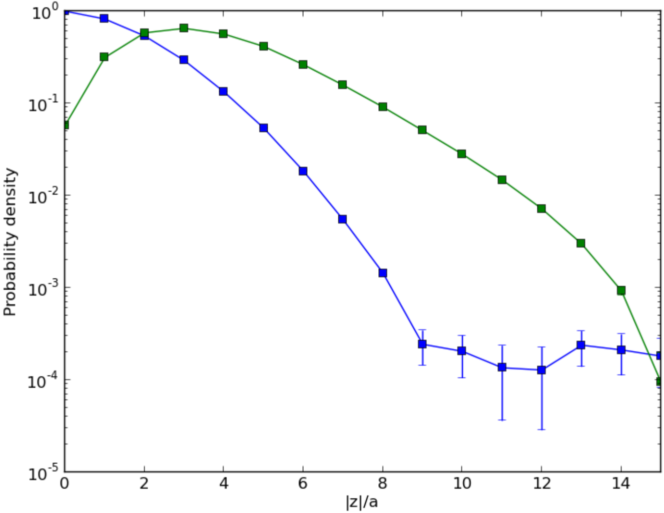
Projected results

Second posity parity excited state:



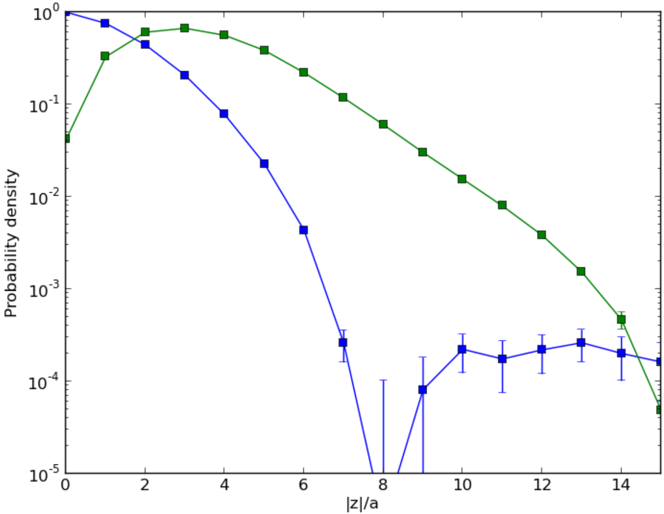
Projected results

Ground state (heavier pion mass):



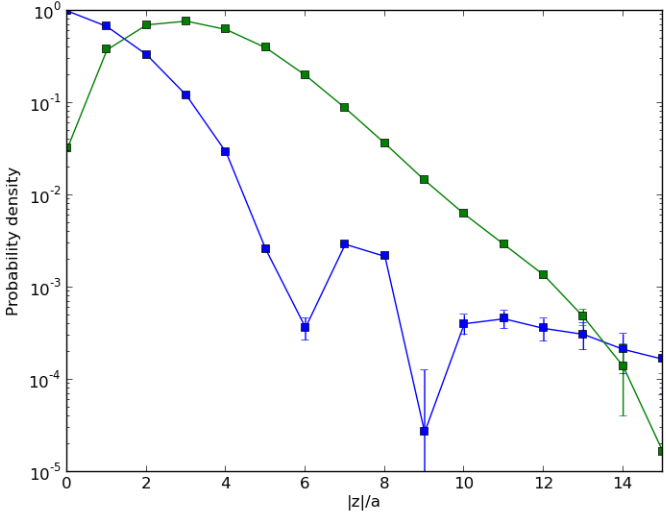
Projected results

First posity parity excited state:

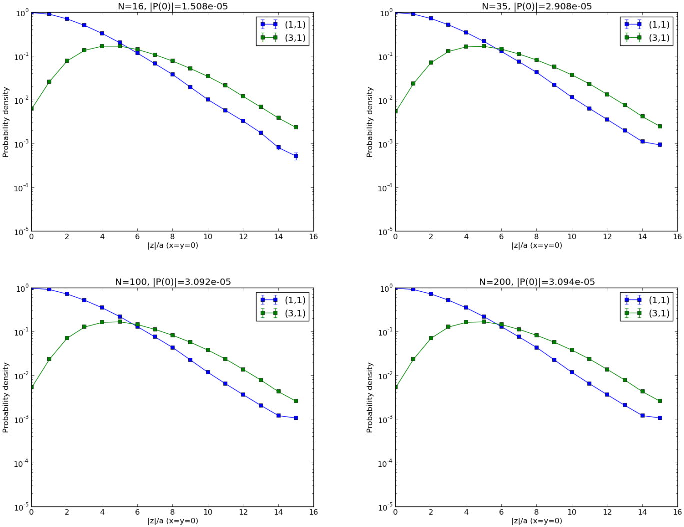


Projected results

Second posity parity excited state:



$\langle \chi_2(z)\bar{\chi_1}\rangle$



Higher spin

- Examining off-diagonal Dirac elements of the wave function has let us obtain angular momenta of $m_l = \pm 1$.
- To access higher values we need a interpolating field for the spin-32 nucleon.
- There is one obvious choice - the interpolator χ_μ:

$$\chi_\mu(x,z) = \epsilon_{abc}[u_a^T(x) C\gamma_5\gamma_\mu d_b(x+z)]\gamma_5 u_c(x+z)$$

- The local version of this operator (z=0) has both spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ contributions.

Higher spin

- The spin-$\frac{3}{2}$ components must obey the Rarita-Schwinger equations ($p_\mu u_\mu = 0, \gamma_\mu u_\mu = 0$) as well as the Dirac equation
- Also, P_{munu} must be idempotent ($P^2 = P$)
- This gives a simple spin-$\frac{3}{2}$ projection operator (at $\vec{p} = 0$):

$$P_{\mu\nu} = \delta_{ij} I - \frac{1}{3}\gamma_i\gamma_j \ (\mu, \nu \neq 0)$$

Spin projection

- The spin-$\frac{1}{2}$ projection operators are :

$$P_{\mu\nu} = \delta_{\mu 0}\delta_{\nu 0}$$

- Equivalent to using $\chi = \epsilon_{abc}[u_a^T C\gamma_0\gamma_5 d_b]\gamma_5 u_c$ at both the source and sink
- and

$$P_{\mu\nu} = \frac{1}{3}\gamma_i\gamma_j \ (\mu, \nu \neq 0)$$

Spin projection

- The explicit form of the projection operator depends on the basis for the gamma matrices
- At zero momentum, the projection operator is block-diagonal in the dirac indices (it does not mix upper and lower components).
- Considering only the upper components and lorentz indices 1,2,3; this gives a 4-dimension eigenspace with eigenvalue 1 (spin-3/2), and a 2-dimension eigenspace with eigenvalue 0 (spin-1/2).

Spin projection

This 4-dimensional space is spanned by the Clebsch-Gordan addition of a spin-1 vector and dirac spinor:

$$\psi(+3/2) = \frac{1}{\sqrt{2}}\{-\psi_1(\uparrow) + i\psi_2(\uparrow)\}$$
$$\psi(+1/2) = \frac{1}{\sqrt{6}}\{-\psi_1(\downarrow) + i\psi_2(\downarrow) + 2\psi_3(\uparrow)\}$$
$$\psi(-1/2) = \frac{1}{\sqrt{6}}\{\psi_1(\uparrow) + i\psi_2(\uparrow) + 2\psi_3(\downarrow)\}$$
$$\psi(-3/2) = \frac{1}{\sqrt{2}}\{\psi_1(\downarrow) + i\psi_2(\downarrow)\}$$

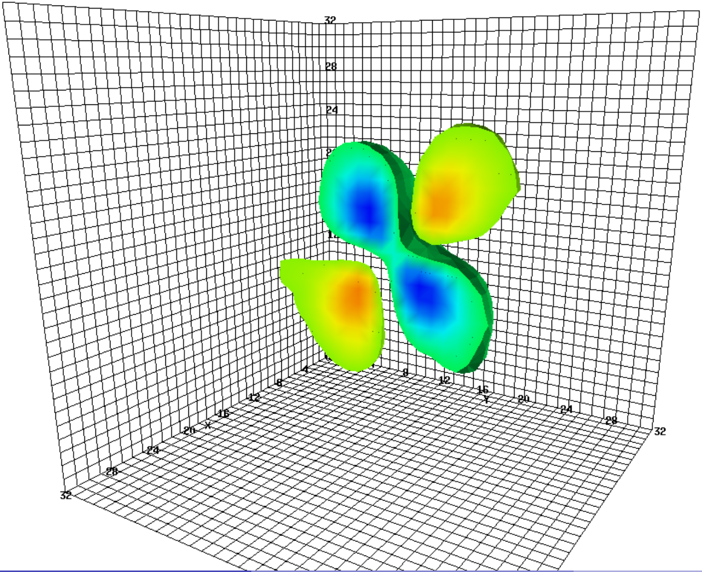
Spin projection

Similarly, the two-dimensional space of the spin-$\frac{1}{2}$ component has basis:

$$\psi(+1/2) = \frac{1}{\sqrt{3}}\{\psi_1(\downarrow) - i\psi_2(\downarrow) + \psi_3(\uparrow)\}$$
$$\psi(-1/2) = \frac{1}{\sqrt{3}}\{-\psi_1(\uparrow) - i\psi_2(\uparrow) + \psi_3(\downarrow)\}$$

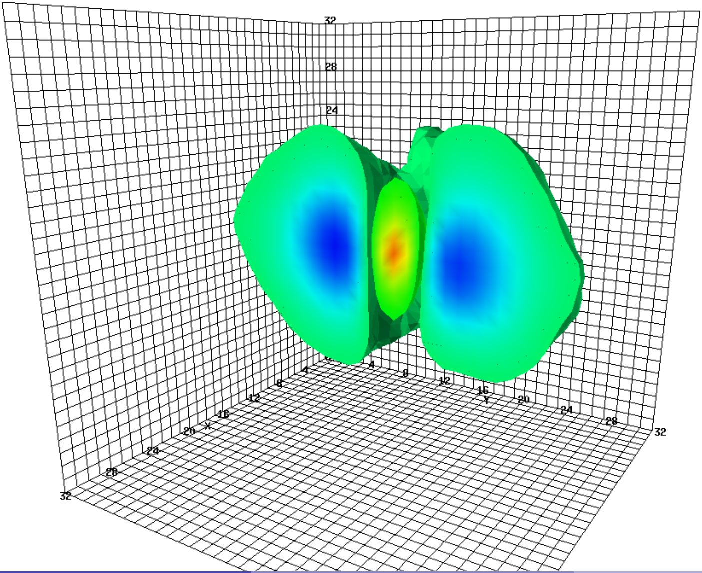
$\psi_1(\uparrow)$ (real part)

Corresponds to $s = \frac{3}{2}$, $m_s = \frac{+3}{2}, \frac{-1}{2}$ and to $s = \frac{1}{2}$, $m_s = \frac{-1}{2}$



$\psi_1(\downarrow)$ (imaginary part)

Corresponds to $s = \frac{3}{2}$, $m_s = \frac{+1}{2}, \frac{-3}{2}$ and to $s = \frac{1}{2}$, $m_s = \frac{+1}{2}$



$$P\psi_1(\downarrow) = P_{1\nu}\psi_\nu$$