A Perspective on Dyson-Schwinger equation

toy model of Pion

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Outline

• Introduction
• Pion Distribution Amplitude
• Pion gamma Transition Form Factor
• Conclusions
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Introduction
The Challenge of QCD

- QCD is the only known example in nature of a fundamental quantum field theory that is innately nonperturbative; Solving QCD will have profound implications for our understanding of the natural world.

- QCD is characterized by two emergent phenomena:

  confinement and dynamical chiral symmetry breaking (DCSB);
  a world without DCSB would be profoundly different.

- Must discover the origin of confinement, its relationship to DCSB and understand how these phenomenon influence hadronic observables.
Nature's strong messenger--Pion

- 1947-Pion discovered by Cecil Frank Powell; almost massless, Goldstone mode associated with dynamical chiral symmetry breaking in QCD; strong couplings to baryons.
- Measure $\pi \gamma$ transition form factor in space like region

**FIG. 1:** A Feynman diagram for $\gamma\gamma^* \rightarrow q\bar{q} \rightarrow \pi^0$ in $e^+e^-$ collisions

**FIG. 24:** Comparison of the results for the product $Q^2|F(Q^2)|$ for the $\pi^0$ from different experiments. The error bars are a quadratic sum of statistical and systematic uncertainties. For the Belle and BaBar results, only a $Q^2$-dependent systematic-error component is included. The two curves denoted fit(A) use the BaBar parameterization while the curve denoted fit(B) uses Eq. (23) (see the text). The dashed line shows the asymptotic prediction from pQCD ($\sim 0.185$ GeV).

**CELLO** (1991) 0.7-2.2 GeV^2
**CLEO** (1998) 1.6-8.0 GeV^2
**BaBar** (2009) 4-40 GeV^2
**Belle** (2010) 4-40 GeV^2
Introduction

Nature's strong messenger--Pion

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Pion gamma transition form factor at lowest-order pQCD and leading twist 2

\[ Q^2 F(Q^2) = 2f_\pi \frac{1}{3} \int_0^1 dx \frac{\varphi^{(2)}_\pi(x)}{1 - x} \]

FIG. 24: Comparison of the results for the product \( Q^2|F(Q^2)| \) for the \( \pi^0 \) from different experiments. The error bars are a quadratic sum of statistical and systematic uncertainties. For the Belle and BaBar results, only a \( Q^2 \)-dependent systematic-error component is included. The two curves denoted fit(A) use the BaBar parameterization while the curve denoted fit(B) uses Eq. [23] (see the text). The dashed line shows the asymptotic prediction from pQCD (\( \sim 0.185 \text{ GeV} \)).
Introduction

Nature's strong messenger--Pion

• 1947-Pion discovered by Cecil Frank Powell; almost massless, Goldstone mode associated with dynamical chiral symmetry breaking in QCD; strong couplings to baryons.

\[
\langle 0 | \bar{q}(-x) \gamma_5 \gamma_\mu L[-x, x] q(x) | \pi(P) \rangle = f_\pi P_\mu \int_0^1 du e^{-i(2u-1)x\cdot P} \varphi^{(2)}_\pi(u) + \text{high twist}
\]

★ Pion gamma transition form factor at lowest-order pQCD and leading twist 2

\[
Q^2 F(Q^2) = 2f_\pi \frac{1}{3} \int_0^1 dx \frac{\varphi^{(2)}_\pi(x)}{1-x}
\]

Distribution amplitudes are nonperturbative quantities

QCD Sum Rules (Chernyak, Zhitnitsky 1982)
Nonlocal QCD Sum Rules (Bakulev, Mikhailov, BMS...)
Instanton-vacuum (Dorokhov, Polyakov...)
NJL (Arriola et al...)
AdS/QCD (Brodsky et al...)
Lattice QCD (Braun et al; Donnellan et al; Arthur et al...)
Dyson-Schwinger equations (Chang et al...)

......
Meeting Challenge—QCDs Dyson-Schwinger Equations

- The equations of motion of QCD
  - Each Green function satisfies integral equation involving other functions...an infinite tower of coupled integral equations
  - The symmetric preserving truncation is necessary

- The most important DSE is QCDs gap equation---quark propagator

\[
[S]^{-1} = [S_0]^{-1} + [\gamma]^{-1} [S]^{-1} [D]
\]

Modeling gluon propagator
Si-xue Qin, et al
Phys. Rev. C. 84 (2011) 042202

Landau Gauge;
Infrared constant;
Massive-like infrared and Massless in ultraviolet.
Introduction
Meeting Challenge—QCDs Dyson-Schwinger Equations

Model quark-gluon vertex

J. S. Ball and Ting-Wai Chiu, Phys. Rev. D 22 (1980) 2542

Nonperturbative truncation game


See Sixue Qin's talk
Ayse Kizilersu's talk

\[
P_\mu \Lambda^{us}_{5\mu\beta}(k, q; P) = \Gamma^u_\beta(q_+, k_+)i\gamma_5 + i\gamma_5 \Gamma^s_\beta(q_-, k_-) - i(m_u + m_s)\Lambda^{us}_{5\beta}(k, q; P)
\]

Necessary and sufficient to ensure axial-vector identity satisfied.

Kernel’s Ward-Takahashi identity provides means by which to construct a symmetry preserving kernel of the Bethe-Salpeter equation that is matched to any reasonable Ansatz for the dressed-quark-gluon vertex

Solution is not unique, the separate ansatz is useful.
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Pion's valence quark Distribution Amplitude

Quark and gluon fields are distributed along the light-cone $x^-$ direction.

Valence quark picture
Definitive of a hadron – it’s how we tell a proton from a neutron
Expresses charge; flavour; baryon number; and other Poincaré-invariant macroscopic quantum numbers
Parton physics involves time-dependent dynamics

Exact expression in QCD for the pion’s valence-quark PDA

$$\varphi_\pi(x) = Z_2 \text{tr}_{CD} \int \frac{d^4k}{(2\pi)^4} \delta(n \cdot k - xn \cdot P) \gamma_5 \gamma \cdot n S(k) \Gamma_\pi(k; P) S(k - P)$$

Fact:
Solving quark and amplitude in Euclidean Space;
Limited information of Quark propagator at $k^2=0$;
Limited information of pion Amplitude at $k^2>0$ and $-1<\frac{k \cdot P}{\sqrt{k^2 P^2}}<1$

Physical require:
$|\frac{k \cdot P}{\sqrt{k^2 P^2}}|<\text{Infinity}$
For more elusive we need to know of amplitude with one quark leg moment square fixed but another goes to infinity
Analytical representation of BS amplitude

General Nakanishi integral representation

$$\Gamma_\pi(k; P) = \gamma_5 \left\{ \gamma \cdot PF(k; P) + \gamma \cdot kG(k; P) + iE(k; P) + \sigma_{\mu\nu}k_\mu P_\nu H(k; P) \right\}$$

$$= \gamma_5 \left\{ \int_{-1}^{1} dz \int_0^\infty d\gamma \frac{\gamma \cdot Pf(\gamma, z) + \gamma \cdot kg(\gamma, z) + ie(\gamma, z) + \sigma_{\mu\nu}k_\mu P_\nu h(\gamma, z)}{(k^2 + zk \cdot P + M^2 + \gamma)^2} \right\}$$

Approximation-I: reduce to 1 dimension

$$e(z, \gamma) \rightarrow \rho(z)\delta(\gamma)$$

Lose $k_{\text{perp}}$ nontrivial dependence;
Integral properties appropriate correct

Approximation-II: numerical fighting

In practice we fit weight function $\rho(z)$ by the limited knowledge of Bethe-Salpeter amplitude;
There are some uncertainties in the approach of extracting $\rho(z)$
Uncertainties would provide error estimates for the PDA and transition form factor
Relate weight function to PDA

An example to illustrate the relation

Quark propagator:

\[ S^{-1}(k) = Z_2 (i \gamma \cdot k + M) \]

Bethe-Salpeter Amplitude:

\[ \Gamma_\pi(k; P) = i \gamma_5 \alpha Z_2 \frac{M}{f_\pi} \int_{-1}^{1} dz \rho(z) \frac{M^2 k \cdot P}{(k^2 + z k \cdot P + M^2)^2} + \text{regularization at bound points} \]

\( \rho(z) \) is weight function accounting relative motion between the partons. 
\( \rho(z) \) is kind of odd function; 
\( \rho(z) \) take no any singularity;

Twist-2 PDA:

\[ f_\pi \phi(x) = Z_2 \text{tr}_{\text{DC}} \int \frac{d^4 k}{(2\pi)^4} \delta(n \cdot k - x n \cdot P) \gamma_5 \gamma \cdot n S(k) \Gamma_\pi \left( k - \frac{P}{2}; P \right) S(k - P) \]

Corresponding relation between \( \rho(z) \) and PDA

Chiral limit would be considered for simplicity

Z2 wave function renormalization constant
Method-I: local moment

\[ \int_0^1 dx x^m \varphi(x) \sim \int dk (n \cdot k)^m \text{tr}[ ... ] \]

Limit number of moment can be calculated on Lattice which can not give enough information about PDA;

Within DSEs approach the moments can be calculated to any large value if one performed analytical representation of quark propagator and BS amplitude; PDA can be extracted with enough information of moments.

Method-II: quasi-PDA


Consider space correlation in a large momentum P in the z-direction

Quark and gluon fields distributed in the z-direction; The matrix element depends on the momentum P; One reaches the stand distribution as P \to infinity;

It is possible to perform finite P calculation on Lattice(matching condition between finite P and stand case);
Relate weight function to PDA

\[ f_{\pi^{(2)}}(x) = Z_2 \text{tr}_{\text{DC}} \int \frac{d^4k}{(2\pi)^4} \delta(n \cdot k - x n \cdot P) \gamma_5 \gamma \cdot n S(k) \Gamma_{\pi}(k - \frac{P}{2}; P) S(k - P) \]

**Blue line**

\[ \frac{d\rho^{\text{up limit}}(z)}{dz} = \frac{3}{4}(1 - z^2) \]

\[ \varphi(x) = 6x(1 - x) \]

QCD's conformal limit

**Black line**

\[ \frac{d\rho^{\text{down limit}}(z)}{dz} = \frac{1}{2}(\delta(1 + z) + \delta(1 - z)) \]

\[ \varphi(x) = 1 \]

Point-particle-like characteristics:

One assigns equal probability to two distinct configurations: valence-quark with all the pion's momentum and valence-antiquark with none or antiquark with all the momentum and quark with none.
Relate weight function to PDA

\[ f_\pi \phi^{(2)}(x) = Z_2 \text{tr}_{DC} \int \frac{d^4 k}{(2\pi)^4} \delta(n \cdot k - x n \cdot P) \gamma_5 \gamma \cdot n S(k) \Gamma_\pi(k - \frac{P}{2}; P) S(k - P) \]

\[ \rho(z) \rightarrow \rho(z; \mu) \]

\( \rho(z) \) should be scale dependent

**Down limit** produce a very low hadronic scale PDA;

**Up limit** provide an appropriate conformal limit;

We estimate the true weight function would lay between these two limits.
Modeling weight function

Default scale is 1GeV!

Model-I: A smooth approximation to the step function

\[ \rho_I(z) = \frac{1}{2} \left( \frac{1}{1 + e^{-\frac{z+z_0}{\ell}}} - \frac{1}{1 + e^{\frac{z+z_0}{\ell}}} \right) \]

Red Line

Model-II: An integrable singularity form at boundary

Green Line \[ \rho_{II}(z) = \text{ArcSin}[z] \]

Model-I: the Bethe-Salpeter amplitude of Pion decreases as

\[ \Gamma_\pi(k_1^2, k_2^2) \to \frac{1}{k_1^2} \]

as \( k_1^2 \to \infty \) with \( k_2^2 \) fixed finite, or \( k_1^2 \to \infty \) with \( k_1^2/k_2^2 \) fixed finite.

Model-II: the Bethe-Salpeter amplitude of Pion decreases as

\[ \Gamma_\pi(k_1^2, k_2^2) \to \frac{1}{\sqrt{k_1^2}}, \frac{1}{k_1^2} \]

as \( k_1^2 \to \infty \) with \( k_2^2 \) fixed finite and as \( k_1^2 \to \infty \) with \( k_1^2/k_2^2 \) fixed finite.
A Euclidean quasi-distribution
Model-I illustration

Quasi-PDA at finite P3

Blue Line: P3/M=2
Green Line: P3/M=10
Red Line: P3/M=100
Black Line: P3/M=Infinity

Open Question: could we read stand PDA based on finite P3 quasi-PDA?
Perturbative one-loop matching condition?
Nonperturbative matching condition?
M = 0.306 GeV to produce F\_pion = 0.0924 GeV

1, Second moment
\[ \int dx \varphi(x)(2x - 1)^2 \]

<table>
<thead>
<tr>
<th>NJL-like</th>
<th>Model-I</th>
<th>Model-II</th>
<th>Asymptotic</th>
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<td>1/4</td>
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<td>1/5</td>
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2, x \to 1 limit
\[ \sqrt{1-x} \]
\[ 1-x \]

3, Inverse moment
\[ \int dx \frac{\varphi(x)}{x} \]

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</thead>
<tbody>
<tr>
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</table>
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Dressed triangle diagram

- Nonperturbative expression of pion gamma transition form factor

\[
\frac{1}{4\pi^2 f_\pi}\epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma G(Q_1^2, \omega^2) = \\
-2\text{tr}_D \int \Gamma_\mu(k + k_2, k - k_1 + k_2) S(k - k_1 + k_2) \Gamma_\nu(k - k_1 + k_2) \Lambda_\pi(k - \frac{k_1}{2} + \frac{k_2}{2}; -k_1 - k_2)
\]

with \( \omega^2 = \frac{Q_1^2 - Q_2^2}{Q_1^2 + Q_2^2} \). Photon asymmetry

- Chiral anomaly (Saturated with a triangle diagram)

\[
\begin{align*}
P_\mu i \Gamma_\mu(k; P) &= S^{-1}(k + \frac{P}{2}) - S^{-1}(k - \frac{P}{2}) \\
P_\mu i \chi_{5\mu}(k; P) &= S(k + \frac{P}{2}) \gamma_5 + \gamma_5 S(k - \frac{P}{2}) \\
S(k) &= \frac{1}{i\gamma \cdot k A(k^2) + B(k^2)} = -i\gamma \cdot k \sigma_A(k^2) + \sigma_B(k^2)
\end{align*}
\]

\[
\begin{align*}
\Gamma_\mu(k; P) &= \gamma_\mu \frac{A(k_+) + A(k_-)}{2} + 2\mu \gamma \cdot t \frac{A(k_+) - A(k_-)}{k_+^2 - k_-^2} + ... \\
F(k; P) &\rightarrow \sigma_A(k^2) \\
G(k; P) &\rightarrow 2k \cdot P \sigma_A'(k^2)
\end{align*}
\]

\[
G(0, \omega) = 1
\]
Asymptotic Pion Transition Form Factor

- Nonperturbative expression of pion gamma transition form factor

\[
\frac{1}{4\pi^2 f_\pi} \epsilon_{\mu \nu \rho \sigma} k_1^\rho k_2^\sigma G(Q_1^2, \omega^2) =
-2 \text{tr}_D \int \Gamma_\mu(k + k_2, k - k_1 + k_2) S(k - k_1 + k_2) \Gamma_\nu(k - k_1 + k_2) \Lambda_\pi(k - \frac{k_1}{2} + \frac{k_2}{2}; -k_1 - k_2)
\]

with \( \omega^2 = \frac{Q_1^2 - Q_2^2}{Q_1^2 + Q_2^2} \).

- Photon asymmetry

- UV leading order contribution (bare quark-photon vertex)

\[
\Gamma_\mu = Z_2 \gamma_\mu
\]

\[
\Gamma_\nu = Z_2 \gamma_\nu
\]

\[
\frac{1}{4\pi^2 f_\pi} G(Q_1 \to \infty, \omega^2 = 1) \to 8 f_\pi \int_{-1}^{1} dz \frac{\log \left[ \frac{1+z}{2} \right]}{3(z-1)} \frac{d\rho(z)}{dz} = 2 f_\pi \int_{0}^{1} dx \frac{\varphi(x)}{3(1-x)}
\]

\[ \text{rho}(z) \text{ and PDA fixed at } 2\text{GeV} \]
Transition Form Factor

Triangular diagram calculation

Large $O$ limit

\[
\frac{1}{4\pi^2 f_\pi} Q^2 G(Q^2 \to \infty) = 2 f_\pi \int_0^1 dx \frac{\varphi(x)}{3(1-x)}
\]
Evolution of Bethe-Salpeter Amplitude

Model-I illustration

With the input of weight function

\[ \rho_I(z) = \frac{1}{2} \left( \frac{1}{1 + e^{-\frac{z + z_0}{t}}} - \frac{1}{1 + e^{\frac{z + z_0}{t}}} \right) \]

the PDA can be well expressed by

\[ \varphi(x) = 6x(1-x) \left( 1 + a_2 C_2^{3/2} (2x - 1) + a_4 C_4^{3/2} (2x - 1) + a_6 C_6^{3/2} (2x - 1) + a_8 C_8^{3/2} (2x - 1) \right) \]

We find the following modified weight function

\[ \rho_{I,m}(z) = \frac{3}{4} (1 - z^2) \left( 1 + 6a_2 C_2^{3/2} (z) + 15a_4 C_4^{3/2} (z) + 28a_6 C_6^{3/2} (z) + 45a_8 C_8^{3/2} (z) \right) \]

can produce the above PDA exactly.

The coefficients in PDA can be evolved to any scale as

\[ a_n(\mu) = a_n(\mu_0) \left( \frac{\alpha(\mu)}{\alpha(\mu_0)} \right)^{-\frac{2n}{\beta_0}} \]

with \( \beta_0 = 11 - \frac{2}{3} N_f \), \( \gamma_j = \frac{4}{3} \left( 3 + \frac{2}{(j+1)(j+2)} - 4 \sum_{k=1}^{j+1} \frac{1}{k} \right) \) and \( \alpha[\mu] \) is the running coupling
Evolution of Transition Form Factor
Model-I illustration

- Red Line: Before evolution
- Black Line: After evolution

- At $Q^2=60\text{GeV}^2$ the evolution suppresses the form factor about 6%.

- Without evolution, the form factor reaches its limit (1 GeV scale PDA) from below.

- Including evolution, the form factor reaches an asymptotic limit (6x(1-x)PDA) from above very slowly.
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- We study the relation between pion BS amplitude and its twist-2 PDA;
- The relation of PDA and pion gamma transition form factor can be read from the BS amplitude;
- A practical method to include QCD evolution in BS amplitude;
Thank you