Mass singularity and confinement in QED_3

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Abstract

Infrared behaviour of the fermion propagator is determined based on spectral representation and LSZ reduction formula.Our model shows confining property.Mass generation or mass changing effect is shown by superposition of the different mass in dispersion integral.
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1 Problems in QED_3

1 Quenched propagator:massgeneration & condensation
2 effects of vacuum polarization : long range part modified to weak
3 confinement ? Z_2^{-1},high energy behaviour of the propagator
4 Analyticity in Minkowski space

2 Spectral representation of the propagator

$$S_F(p) = \int d^3x \exp(-ip\cdot x) \left\langle \Omega | T\psi(x)\overline{\psi}(0)|\Omega \right\rangle = \int d\omega \frac{\gamma\cdot p\rho_1(\omega) + \omega\rho_2(\omega)}{p^2 - \omega^2 + i\epsilon}, \tag{1}$$

$$\frac{1}{\pi}\Im S(p) = \int \frac{d\omega^2}{\sqrt{\omega^2}}\delta(p^2-\omega^2)[\gamma\cdot p\rho_1(\sqrt{\omega^2}) + \sqrt{\omega^2}\rho_2(\sqrt{\omega^2})] = \gamma\cdot p\rho_1(p) + \rho_2(p), \tag{2}$$

$$S_F(x) = \left\langle \Omega | T\psi(x)\overline{\psi}(0)|\Omega \right\rangle. \tag{3}$$

The field ψ is renormalized and is taken to be a spinor with mass m.Here we introduce intermediate states that contribute the spectral function

$$\rho(p) = \gamma{\cdot}p\rho_1(p){+}\rho_2(p) = (2\pi)^2 \sum_N \delta^3(p{-}p_N) \int d^3x \exp(ip{\cdot}x) \left\langle \Omega|\psi(x)|N\right\rangle \left\langle N|\overline{\psi}(0)|\Omega\right\rangle . \tag{4}$$

Total three-momentum of the state $|N\rangle$ is p^μ_N.The only intermediates N contain one spinor and an arbitrary number of photons.Setting

$$|N\rangle = |r; k_1, ..., k_n\rangle, \tag{5}$$

where r is the momentum of the spinor $r^2 = m^2$,and k_i is the momentum of ith soft photon,we have

$$\rho(p) = \int d^3x \exp(-ip\cdot x) \int \frac{md^2r}{r^0} \sum_{n=0}^{\infty} \frac{1}{n!} \times (\int \frac{d^3k}{(2\pi)^3} \theta(k_0)\delta(k^2) \sum_{\epsilon})_n \delta(p - r - \sum_{i=1}^{n} k_i)$$
$$\times \left\langle \Omega|\psi(x)|r; k_1, .., k_n\right\rangle \left\langle r; k_1, ..k_n|\overline{\psi}(0)|\Omega\right\rangle . \tag{6}$$

Here the notations

$$(f(k))_0 = 1, (f(k))_n = \prod_{i=1}^{n} f(k_i) \tag{7}$$

have been introduced to denote the phase space integral of each photon.The initial sum over ϵ is a sum over polarization of photon.To evaluate the contribution of the soft-photons,we consider when only the nth photon is soft.

2.0.1 LSZ for scalar

$$\begin{aligned}
\phi_{in}(x) &= \int d^2x[a_{in}(k)f_k(x) + a^+(k)f^*_k(x)], \\
f_k(x) &= \frac{1}{\sqrt{(2\pi)^2 2\omega_k}} \exp(-ik\cdot x), \\
\int f_k(x)f^*_{k'}(x)d^2x &= \frac{1}{2\omega_k}\delta^{(2)}(k-k') \\
a_{in}(k) &= -i\int d^2x f^*_k(x)\overleftrightarrow{\partial_0}\phi_{in}(x) \\
a^+_{in}(k) &= i\int d^2x f^*_{-k}(x)\overleftrightarrow{\partial_0}\phi_{in}(x)
\end{aligned}$$

$$\begin{aligned}
\langle\beta_{out}|\alpha, p_{in}\rangle &= \left\langle\beta_{out}|a^+_{in}(p)|\alpha_{in}\right\rangle \\
&= \left\langle\beta_{out}|a^+_{out}(p)|\alpha_{in}\right\rangle + \left\langle\beta_{out}|a^+_{in}(p) - a^+_{out}(p)|\alpha_{in}\right\rangle \\
&= \langle\beta - p_{out}|\alpha_{in}\rangle - i\left\langle\beta_{out}|\int d^2x f_p(x)\overleftrightarrow{\partial_0}[\phi_{in}(x) - \phi_{out}(x)]|\alpha_{in}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
& -i\left\langle \beta_{out}\right| \int d^2x f_p(x)\overleftrightarrow{\partial_0}[\phi_{in}(x)-\phi_{out}(x)]|\alpha_{in}\Big\rangle \\
= & \frac{i}{\sqrt{Z}}(\lim_{t\to-\infty}-\lim_{t\to\infty})\int d^2x f_p(x)\overleftrightarrow{\partial_0}\langle\beta_{out}|\phi(x,t)|\alpha_{in}\rangle \\
= & \frac{i}{\sqrt{Z}}\int_{-\infty}^{\infty} d^3x[f_p(x)\partial_0^2\langle\beta_{out}|\phi(x,t)|\alpha_{in}\rangle-\partial_0^2 f_p(x)\langle\beta_{out}|\phi(x,t)|\alpha_{in}\rangle]
\end{aligned}
$$

$$
\frac{\partial^2 f_p(x)}{\partial t^2}=(\nabla^2-m^2)f_p(x)
$$

Using Partial integral onto $\phi(x,t)$, we get

$$
\langle\beta_{out}|\alpha,p_{in}\rangle=\frac{i}{\sqrt{Z}}\int d^3x f_p(x)(\Box+m^2)\langle\beta_{out}|\phi(x,t)|\alpha_{in}\rangle
$$

2.0.2 Main problem ;detemination of the matrixelement

Here we define the following matrix element

$$
\begin{aligned}
T_n &= \langle\Omega|\psi|r;k_1,..k_n\rangle && (8)\\
&= \left\langle\Omega|\psi a^{+}_{in}(k_n)|r;k_1,..k_{n-1}\right\rangle . && (9)
\end{aligned}
$$

We consider T_n for $k_n^2\neq 0$,we continue off the photon mass-shell by Lehmann-Symanzik-Zimmermann(LSZ)formula:

$$
\begin{aligned}
T_n &= \epsilon_n^\mu T_{n\mu}, \\
\epsilon_n^\mu T_n^\mu &= \frac{i}{\sqrt{Z_3}}\int d^3y\exp(ik_n\cdot y)\Box_y\langle\Omega|T\psi(x)\epsilon^\mu A_\mu(y)|r;k_1,...,k_{n-1}\rangle \\
&= -\frac{i}{\sqrt{Z_3}}\int d^3y\exp(ik_n\cdot x)\langle\Omega|T\psi(x)\epsilon^\mu j_\mu(y)|r;k_1,...,k_{n-1}\rangle . \qquad (10)
\end{aligned}
$$

where the electromagnetic current is

$$
j^\mu(x)=-e\overline{\psi}(x)\gamma_\mu\psi(x). \qquad (11)
$$

$$
\begin{aligned}
& \Box_x T(\psi A_\mu(x)=T\psi\Box_x A_\mu(x)=T\psi(-j_\mu(x)+\frac{d-1}{d}\partial_\mu^x(\partial\cdot A(x)), && (12)\\
& \partial\cdot A^{(+)}|phys>=0. && (13)
\end{aligned}
$$

From the definition (9) T_n^μ is seen to satisfies Ward-Takahashi-identity:

$$
k_{n\mu}T_n^\mu(r,k_1,..k_n)=eT_{n-1}(r,k_1,..k_{n-1}),r^2=m^2, \qquad (14)
$$

provided the equal-time commutation relations

$$
\begin{aligned}
\partial_\mu^x T(\psi j_\mu(x)) &= -e\psi(x), \\
\partial_\mu^x T(\overline{\psi} j_\mu(x)) &= e\overline{\psi}(x). \qquad (15)
\end{aligned}
$$

In the Bloch-Nordsieck approximation we have most singular contributions of photons which are emitted from external lines.In perturbation theory one photon matrix element is given

$$\begin{aligned} T_1 &= \left\langle in|T(\psi_{in}(x), ie\int d^3y\overline{\psi}_{in}(y)\gamma_\mu\psi_{in}(y)A^\mu_{in}(y))|r;k\ in\right\rangle \\ &= ie\int d^3yd^3zS_F(x-y)\gamma_\mu\delta^{(3)}(y-z)\exp(i(k\cdot y+r\cdot z))\epsilon^\mu(k,\lambda)U(r,s) \\ &= -ie\frac{(r+k)\cdot\gamma+m}{(r+k)^2-m^2}\gamma_\mu\epsilon^\mu(k,\lambda)\exp(i(k+r)\cdot x)U(r,s), \end{aligned} \quad (16)$$

where $U(r,s)$ is a four-component free particle spinor with positive energy.$U(r,s)$ satisfies the relations

$$\begin{aligned} (\gamma\cdot r-m)U(r,s) &= 0, \\ \sum_S U(r,s)\overline{U}(r,s) &= \frac{\gamma\cdot r+m}{2m}. \end{aligned} \quad (17)$$

In this case the Ward-Takahasi-identity follows

$$\begin{aligned} k_\mu T_1^\mu &= -ie\frac{1}{\gamma\cdot(r+k)-m}(\gamma\cdot k)U(r,s) \\ &= -ieU(r,s)=eT_0, \end{aligned} \quad (18)$$

provided the lowest-order Ward-identity

$$\gamma\cdot k=(\gamma\cdot(r+k)-m)-(\gamma\cdot r-m). \quad (19)$$

For general T_n low-energy theorem determines the structure of non-singular terms in k_n. Detailed dicussions are given in ref [1] and non-singular terms are irrelevant for the single particle singularity in four-dimension.Under the same assumption in three-dimension we have

$$T_n|_{k_n^2=0}=e\frac{\gamma\cdot\epsilon}{\gamma\cdot(r+k_n)-m}T_{n-1}. \quad (20)$$

From this relation the n-photon matrix element

$$\langle\Omega|\psi(x)|r;k_1,..,k_n\rangle\,\langle r;k_1,..k_n|\overline{\psi}(0)|\Omega\rangle \quad (21)$$

is reduced to the product of lowest-order one-photon matrix element

$$T_n\overline{T_n}=\prod_{j=1}^n T_1(k_j)T_1^+(k_j)\gamma_0. \quad (22)$$

In this case the spectral function ρ in (6) is given by exponentiation of one-photon matrix element,which yields an infinite ladder approximation for the propagator.Thus we obtain the spectral function and the propagator in the followings forms,

$$\begin{aligned} \rho(p) &= \int d^3x \exp(-ip\cdot x) \int \frac{md^2r}{r^0} \exp(ir\cdot x) \exp(\mathsf{F}), \\ F &= \sum_{\text{one photon}} \langle\Omega|\psi(x)|r;k\rangle \langle r;k|\overline{\psi}(0)|\Omega\rangle \\ &= \int \frac{d^2k}{(2\pi)^2} \delta(k^2)\theta(k_0) \exp(ik\cdot x) \sum_{\lambda,S} T_1\overline{T_1}, \end{aligned} \quad (23)$$

$$\sum_{\lambda,S} T_1 T_1 = e^2 \left[\frac{(r+k)\cdot\gamma + m}{(r+k)^2 - m^2} \gamma^\mu \frac{r\cdot\gamma + m}{2m} \gamma^\nu \frac{(r+k)\cdot r + m}{(r+k)^2 - m^2} \Pi_{\mu\nu} \right]. \quad (24)$$

Here $\Pi_{\mu\nu}$ is the polarization sum

$$\Pi_{\mu\nu} = \sum_\lambda \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda) = -g_{\mu\nu} - (d-1)\frac{k_\mu k_\nu}{k^2}, \quad (25)$$

and the free photon propagator is

$$D_0^{\mu\nu} = \frac{1}{k^2 + i\epsilon}[g_{\mu\nu} + (d-1)\frac{k_\mu k_\nu}{k^2}]. \quad (26)$$

We get

$$F = -e^2 \int \frac{d^3k}{(2\pi)^2} \exp(ik\cdot x)\theta(k^0)[\delta(k^2)(\frac{m^2}{(r\cdot k)^2} + \frac{1}{(r\cdot k)}) + (d-1)\frac{\delta(k^2)}{k^2}]. \quad (27)$$

The second term $\delta(k^2)/k^2$ equals to $-\delta'(k^2)$.Our calculation is the same with the imaginary part of the photon propagator.To avoid infrared divergence which arises in the phase space integral we must introduce small photon mass μ as an infrared cut-off.Therefore (22) is modified to

$$\begin{aligned} F &= -e^2 \int \frac{d^3k}{(2\pi)^2} \exp(ik\cdot x)\theta(k^0) \\ &\quad \times[\delta(k^2-\mu^2)(\frac{m^2}{(r\cdot k)^2} + \frac{1}{(r\cdot k)}) - (d-1)\frac{\partial}{\partial k^2}\delta(k^2-\mu^2)]. \end{aligned} \quad (28)$$

Here we assume $\rho_1(\omega) = \rho_2(\omega) = \rho$ which is valid in the infrared(i.e.$\gamma\cdot r = m$).In general case there are two kinds of spectral function which is given in the appendix.It is helpful to use function $D_+(x)$ to determine F

$$\begin{aligned} D_+(x) &= \frac{1}{(2\pi)^2 i} \int \exp(ik\cdot x) d^3k \theta(k^0)\delta(k^2-\mu^2) \\ &= \frac{1}{(2\pi)^2 i} \int_0^\infty J_0(kx) \frac{\pi k dk}{2\sqrt{k^2+\mu^2}} = \frac{\exp(-\mu x)}{8\pi i x}. \end{aligned} \quad (29)$$

If we use parameter trick(exponential cut-off)

$$\lim_{\epsilon\to 0}\int_0^\infty d\alpha \exp(i(k+i\epsilon)\cdot(x+\alpha r)) = i\frac{\exp(ik\cdot x)}{k\cdot r}, \tag{30}$$

$$\lim_{\epsilon\to 0}\int_0^\infty \alpha d\alpha \exp(i(k+i\epsilon)\cdot(x+\alpha r)) = -\frac{\exp(ik\cdot x)}{(k\cdot r)^2}, \tag{31}$$

History

the function F is written in the following form

$$F = ie^2m^2\int_0^\infty \alpha d\alpha D_+(x+\alpha r,\mu) - e^2\int_0^\infty d\alpha D_+(x+\alpha r,\mu) - ie^2(d-1)\frac{\partial}{\partial\mu^2}D_+(x,\mu)$$

$$= F_1 + F_2 + F_g \tag{32}$$

$$= \frac{e^2m^2}{8\pi r^2}(-\frac{\exp(-\mu|x|)}{\mu} + x\,\mathrm{Ei}(\mu|x|)) + \frac{e^2}{8\pi r}\mathrm{Ei}(\mu|x|) + (d-1)\frac{e^2}{8\pi\mu}\exp(-\mu|x|), \tag{33}$$

where the function $\mathrm{Ei}(\mu|x|)$ is defined

$$\mathrm{Ei}(\mu|x|) = \int_1^\infty \frac{\exp(-\mu|x|t)}{t}dt. \tag{34}$$

It is understood that all terms which vanishes with $\mu\to 0$ are ignored.The leading non trivial contributions to F are

$$\mathrm{Ei}(\mu|x|) = -\gamma - \ln(\mu|x|) + O(\mu|x|), \tag{35}$$

$$F_1 = \frac{e^2m^2}{8\pi r^2}\left(-\frac{1}{\mu} + |x|(1-\ln(\mu|x|)-\gamma)\right) + O(\mu),$$

$$F_2 = \frac{e^2}{8\pi\sqrt{r^2}}(\ln(\mu|x|)+\gamma) + O(\mu),$$

$$F_g = \frac{e^2}{8\pi}(\frac{1}{\mu} - |x|)(d-1) + O(\mu), \tag{36}$$

$$F = \frac{e^2}{8\pi\mu}(d-2) + \frac{\gamma e^2}{8\pi m} + \frac{e^2}{8\pi m}\ln(\mu|x|) - \frac{e^2}{8\pi}|x|\ln(\mu|x|) - \frac{e^2}{8\pi}|x|(d-2+\gamma), \tag{37}$$

where γ is Euler's constant.Hereafter we use integrals for intermediate state for on-shell fermion

$$\int d^3x \exp(-ip\cdot x)\int d^3r\delta(r^2-m^2)\exp(ir\cdot x)f(r) = f(m). \tag{38}$$

$$\int d^3x \exp(-ip\cdot x)\int d^3r\frac{\exp(-m|x|)}{4\pi|x|}\exp(ir\cdot x)\delta(r^2-m^2)\frac{1}{m^2+p^2}. \tag{39}$$

And we set $r^2 = m^2$ in the phase space integral;

$$\rho(p) = \int d^3x \exp(-ip\cdot x) \int \frac{md^2r}{\sqrt{r^2+m^2}} \exp(ir\cdot x) \exp(F(m,|x|))$$
$$= \int d^3x \exp(-ip\cdot x) \frac{\exp(-m|x|)}{4\pi|x|} \exp(F(m,|x|)), \tag{40}$$
$$\overline{\rho(x)} = \frac{\exp(-m|x|)}{4\pi|x|} \exp(F(m,|x|)). \tag{41}$$

$$\exp(F(m,|x|) = \exp(-\frac{e^2}{8\pi}|x|(d-2+\gamma))(\mu|x|)^{D-C|x|},$$
$$D = \frac{e^2}{8\pi m}, C = \frac{e^2}{8\pi}. \tag{42}$$

Linear infrared divergences cancell by higher order correction.Here $\exp(-m|x|)/4\pi|x|, |x| = \sqrt{-x^2}$ is a free scalar propagator with physical mass m and $\exp(F)$ denotes the quantum correction for the propagator involving infinite numbers of photons in our approximation.

2.1 Confining property

Here we mention the confining property of the propagator $S_F(x)$ in position space

$$S_F(x) = (\frac{i\gamma\cdot\partial}{m} + 1)[\frac{m\exp(-m|x|)}{4\pi|x|}(\mu|x|)^{D(1-m|x|)}]. \tag{43}$$

The S_F dumps strongly at large x provided

$$\lim_{x\to\infty}(\mu|x|)^{-Dm|x|} = 0. \tag{44}$$

The profiles of the $\overline{\rho(x)}$ for various values of $D \geq 1$ are shown in Fig.1.The effect of $(\mu|x|)^{-Dm|x|}$ in position space is seen to decrease the value of the propagator at low energy and shown in Fig.2.

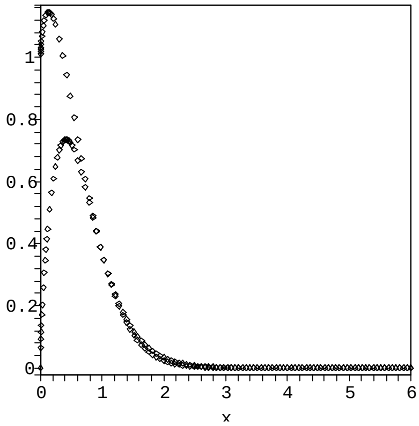

Fig.1 $\overline{\rho(x)}$ for $m = \mu = unit,\ D = 1, 1.5$

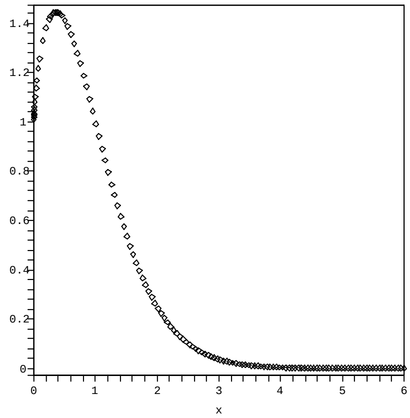

Fig.2 $(\mu\,|x|)^{-Dm^{|x|}}$ for $m = \mu = unit, D = 1$

In section IV and V we discuss dynamical mass,the renormalization constant,and bare mass in connection of each terms in F.

2.2 O(e^2) propagator in Momentum space

After angular integration of (39),we get the propagator

$$S_F(p) = (\frac{\gamma \cdot p}{m} + 1)\rho(p), \tag{45}$$

$$\rho(p) = \frac{m}{2\pi\sqrt{-p^2}} \int_0^\infty d\,|x| \sin(\sqrt{p^2 x^2}) \exp(A - (m_0 + B)\,|x|)(\mu\,|x|)^{-C|x|+D}, \tag{46}$$

where

$$A = \frac{e^2}{8\pi\mu}(d-2) + \frac{\gamma e^2}{8\pi m}, B = \frac{e^2}{8\pi}(d-2+\gamma), C = \frac{e^2}{8\pi}, D = \frac{e^2}{8\pi m}. \tag{47}$$

If we discuss the Euclidean or off-shell propagator we can omitt the linear infrared divergent part in A.In this case m denotes a physical mass

$$m = |m_0 + \frac{e^2}{8\pi}(d-2+\gamma)|. \tag{48}$$

Here we show the propagator $\rho(p)$ up to $O(e^2)$ and the spectral function

$$\rho^{(2)}(p) = \int d^3x \exp(-ip\cdot x) \int \frac{md^2r}{r^0} \exp(ir\cdot x)F(x)$$
$$= \frac{m}{\sqrt{-p^2}} \int_0^\infty d|x| \sin(\sqrt{p^2x^2}) \exp(-m|x|)[1 + A - C|x| \ln(\mu|x|) + D\ln(\mu|x|)]$$
$$= [\frac{m(1+A)}{m^2+p^2} + m(DI_1 - CI_2)], \quad (49)$$
$$\rho^{(2)}(p) = \frac{1}{\pi}\Im\rho^{(2)}(-p^2), \quad (50)$$

where I_1, I_2 are the following integrals

$$I_1 = \int_0^\infty \frac{\sin(\sqrt{p^2x^2})\exp(-m|x|)}{\sqrt{-p^2}} \ln(\mu|x|)d|x|$$
$$= \frac{-\gamma}{m^2+p^2} - \frac{\ln((m^2+p^2)/\mu^2)}{2(m^2+p^2)} - \frac{\ln((m-\sqrt{-p^2})/(m+\sqrt{-p^2}))}{m^2+p^2}, \quad (51)$$

$$I_2 = \int_0^\infty \frac{\sin(\sqrt{p^2x^2})\exp(-m|x|)}{\sqrt{-p^2}} |x|\ln(\mu|x|)d|x|$$
$$= \frac{-m}{(m^2+p^2)^2}[\ln((m-\sqrt{-p^2})/(m+\sqrt{-p^2}))$$
$$+\ln((m^2+p^2)/\mu^2) - 2(1-\gamma)]. \quad (52)$$

From these expressions we see that the gauge dependent terms A, D and B, C are wave function renormalization constant and mass renormalization respectively.In this order the wave function renormalization constant

$$Z_2^{(2)} = 1 + A - \frac{e^2}{8\pi m}(\gamma + \frac{1}{2}\ln((m^2+p^2)/\mu^2) + \ln((m-\sqrt{-p^2})/(m+\sqrt{-p^2})) \quad (53)$$

is divergent at $p^2 = -m^2$ and $p^2 = \infty$.

3 Analysis in momentum space

To search the infrared behaviour we expand the propagator in the powers of the coupling constant e^2 and obtained the Fourier transform of $\rho(x)$ [2].In that case it is not enough to see the structure of infrared behaviour which can be compared to the well-known four dimensional QED[10].Instead we make Laplace transformation of $(\mu|x|)^{-Dm|x|}$,which leads the general spectral representation of the propagator in momentum space.After that we show the roles of Coulomb energy and position dependent mass. The former determines the dimension of the propagator and the latter acts to change mass.Let us begin to study the effect of position dependent mass(Self-energy),Coulomb energy in momentum space

$$\exp(-|x| M(x)) = \exp(-|x| \frac{e^2}{8\pi} \ln(\mu |x|), \tag{54}$$

$$\exp(-Coulomb\ energy/m) = \exp(-\frac{e^2}{8\pi m} \ln(\mu |x|)). \tag{55}$$

Similar discussion was given to study the effects of self-energy and bare potential in the stabilty of massless $e^+ e^-$composite in lattice simulation[11].The position space free propagator

$$S_F(x, m_0) = -(i\gamma \cdot \partial + m_0)\frac{\exp(-m_0 |x|)}{4\pi |x|} \tag{56}$$

is modified by these two terms which are related to dynamical mass and wave function renormalization.To see this let us think about position space propagator

$$\overline{\rho(x)} = \frac{\exp(-m |x|)}{4\pi |x|}(\mu |x|)^{-\frac{e^2}{8\pi}|x|}(\mu |x|)^{\frac{e^2}{8\pi m}}. \tag{57}$$

It is easy to see that the probability of particles which are separated with each other in the long distance is supressed by the factor $(\mu |x|)^{-Dm|x|}$,and the Coulomb energy modifies the short distance behaviour from the bare $1/|x|$ to $1/|x|^{1-D}$.The effect of Coulomb energy for the infrared behaviour of the free particle with mass m can be seen by its fourier transform[2,10]

$$\begin{aligned} & 4\pi \int_0^\infty x^2 \frac{\sin(\sqrt{p^2 x^2})}{\sqrt{p^2 x^2}} \frac{\exp(-m |x|)}{4\pi |x|}(\mu |x|)^D d|x| \\ & = \mu^D \frac{\Gamma(D+1) \sin((D+1)\arctan(\sqrt{-p^2}/m))}{\sqrt{-p^2}(-p^2+m^2)^{(1+D)/2}} \end{aligned} \tag{58}$$

$$\sim \mu^D(\sqrt{-p^2} - m)^{-1-D} \text{ near } p^2 = -m^2. \tag{59}$$

Above formula shows the structure in momentum space is modified for both infrared and ultraviolet regions. Usually constant D represents the coefficent of the leading infrared divergence for fixed mass in four dimension. Therefore Coulomb energy in three dimension has the same effects as in four dimension but change the ultraviolet behaviour since the coupling constant e^2 is not renormalized. Now we consider the role of $M(x)$ as the dynamical mass at low momentum.First we define Fourier transform of the scalar part of the propagator;

$$\rho(p) = F.T(\frac{m \exp(-m |x|)}{4\pi |x|}(\mu |x|)^{-C|x|+D}) \tag{60}$$

$$= \int \exp(-ip \cdot x)\frac{m \exp(-m |x|)}{4\pi |x|}(\mu |x|)^{-C|x|+D} d^3x. \tag{61}$$

For definiteness we show the Fourier transformations of the propagator for ($D = 0, C = 0$) and ($D = 1, C = 0$) cases,

$$\int d^3x \exp(-ip\cdot x)\frac{\exp(-m\,|x|)}{4\pi\,|x|}\,|x|^0 = \frac{1}{p^2+m^2},$$
$$\int d^3x \exp(-ip\cdot x)\frac{\exp(-m\,|x|))}{4\pi\,|x|}\,|x| = \frac{2m}{(p^2+m^2)^2}. \qquad (62)$$

The second case is the same with dynamical mass generation as we have seen before.If we include $\exp(-|x|\,M(x)) = (\mu\,|x|)^{-C|x|}$ term it is easy to see that the value of the proagator $\rho(p = 0)$ decreases which is shown numerically in Fig.3. If we use Laplace transformation

$$F(s) = \int_0^\infty \exp(-s\,|x|)(m\,|x|)^{-C|x|}d\,|x|$$

we easily see that $(m\,|x|)^{-C|x|}$ acts as mass changing operator $m \to m - s$

$$\exp(-m\,|x|)(m\,|x|)^{-C|x|} = \int dsF(s)\exp(-(m-s)\,|x|). \qquad (63)$$

To separate the μ dependence we change by

$$(\mu\,|x|)^{-C|x|} \to (\mu/m)^{-C|x|}(m\,|x|)^{-Dm|x|} \to \exp(-Dm\,|x|\ln(\mu/m))(m\,|x|)^{-Dm|x|}$$

and the former factor absorbs as mass renormalization as $m^* = m(1+D\ln(\mu/m))$.

$$\rho(p) = (\gamma\cdot p+m)F.T.[\frac{\exp(-m\,|x|)}{4\pi\,|x|}(m\,|x|)^D\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\exp(s\,|x|)F(s)ds]. \quad (64)$$

$$\rho(p) = -\frac{\Gamma(D+1)\mu^D}{2\sqrt{-p^2}}\int_0^\infty dsF(s)\frac{\sin((D+1)\arctan(\sqrt{-p^2}/(m^*-s))}{\sqrt{((m^*-s)^2-p^2)^{D+1}}}, D = \frac{e^2}{8\pi m}.$$

$F(s)$ is shown in Fig.4.

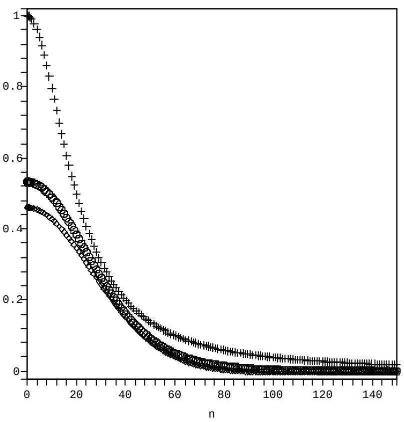

Fig.3 $\rho(p)$ for $m = \mu = unit$, $D = 0(upper), 1(middle), 1.5(upper)$,$p = n/20$.

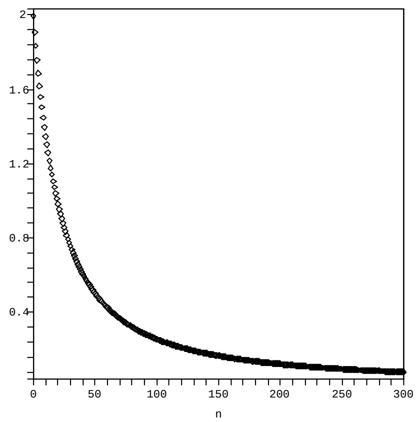

Fig.4 $F(s)$ for $(m\,|x|)^{-Dm|x|}, m = 1, D = 1, s = n/20$.

We have the complete the expression of the propagators based on spectral representation.The spinor propagator in position space is expressed in the following for $\rho_1 = \rho_2 = \rho$ which is the case in our model[7] The equation for the renormalization constant in terms of the spectral functions read

$$\begin{aligned} \psi_0 &= \sqrt{Z_2}\psi_r, \\ \overline{\psi}_0 &= \sqrt{Z_2}\overline{\psi}_r, \qquad (65) \\ \langle 0|T(\psi_0(x)\psi_0(y))|0\rangle &= Z_2 \langle 0|T(\psi_r(x)\psi_r(y))|0\rangle\,, \qquad (66) \\ \frac{Z_2^{-1}}{\gamma\cdot p - m_0} &= S_F(p). \qquad (67) \end{aligned}$$

$$m_0 Z_2^{-1} = m \int \omega \rho_2(\omega) d\omega = \lim_{p\to\infty}[p^2 S_F(p)], \tag{68}$$

$$Z_2^{-1} = \int \rho_1(\omega) d\omega = \lim_{p\to\infty}[\gamma \cdot p S_F(p)]. \tag{69}$$

we obtain

$$m_0 Z_2^{-1} \sim \lim_{p^2\to\infty} \sqrt{-p^2}^{-D} = \begin{bmatrix} 0 & (0 < D) \\ m & (0 = D) \end{bmatrix}, \tag{70}$$

$$Z_2^{-1} = \lim_{p^2\to\infty} \sqrt{-p^2}^{-D} = \begin{bmatrix} 0 & (0 < D) \\ 1 & (0 = D) \end{bmatrix}. \tag{71}$$

This means that propagator in the high energy limit has no part which is proportional to the free one. Usually mass is a parameter which appears in the Lagrangean.For example chiral symmetry is defined for the bare quantity.In ref[9] the relation between bare mass and renormalized mass of the fermion propagator in QED is discussed based on renormalization group equation with the assumption of ultraviolet stagnant point and shown that the bare mass vanishes in the high energy limit even if we start from the finite bare mass in the theory.It suggests that symmetry properties can be discussed in terms of renormalized quantities.In QCD bare mass vanishes in the short distance by asymptotic freedom.And the dynamical mass vanishes too[8].In our apprximation this problem is understood that at short distance propagator in position space tends to

$$\overline{\rho(x)} = \frac{\exp(-m_0 |x|)}{4\pi |x|} \exp(-B|x| + D\ln(\mu|x|) - C|x|\ln(\mu|x|))_{x\to 0} \to |x|^{D-1} (\mu|x|)^{-C|x|}, \tag{72}$$

where we have $\overline{\rho(0)} = finite$ at $D = 1$ case which is independent of the bare mass m_0.Thus we have a same effect as vanishing bare mass in four dimensional model.Of course we have a dynamical mass generation which is $m = |\frac{e^2}{8\pi}(d - 2 + \gamma)|$ for $m_0 = 0$ in our approximation.There is a chiral symmetry at short distance where the bare or dynamical mass vanishes in momentum space but its breaking must be discussed in terms of the values of the order parameter.Therefore it is interesting to study the possibility of pair condensation in our approximation.The vacuum expectation value of pair condensate is evaluated

$$\langle \overline{\psi}\psi \rangle = -tr S_F(x) = -2 \int_0^\infty \frac{p^2 d\sqrt{p^2}}{2\pi^2} \frac{\Gamma(D+1)\mu^D}{(D+1)2\sqrt{p^2}}$$
$$\times \int_0^\infty ds G(s) \frac{\sin((D+1)\arctan(\sqrt{p^2}/(m-s))}{\sqrt{((m-s)^2 + p^2)^{D+1}}}, \tag{73}$$

is finite for $D \geq 1$ for finite cut-off μ.it looks like a wave packet at finite range where long range correlation appears.In the weak coupling limit we obtain $Z_2 =$

$1, m_0 = m$ and $\langle \overline{\psi}\psi \rangle = \infty$.If we introduce chiral symmetry as a global $U(2n)$,it breaks dynamically into $SU(n) \times SU(n) \times U(1) \times U(1)$ as in QCD[8,11] for $D = 1$ for finite infrared cut-off.Our model may be applicable to relativistic model of super fluidity in three dimension.Usually we do not find the critical coupling $D = 1$ in the analysis of the Dyson-Schwinger equation in momentum space where only continuum contributions are taken into account and we do not define physical mass.·

3.0.1 Vaccum polarization

$$\Pi_{\mu\nu}(k) = -\frac{e^2 T_{\mu\nu}}{8\pi}[-4m + (\sqrt{k^2} + \frac{4m}{\sqrt{k^2}})\ln(\frac{2m+\sqrt{k^2}}{2m-\sqrt{k^2}}),$$
$$\to D^{-1} = k^2 + \frac{e^2}{8}\sqrt{k^2}$$

$$D(x) = \int \frac{d^3k}{(2\pi)^3}\exp(ik\cdot x)\frac{1}{k^2 + \frac{e^2}{8}k}$$
$$= \frac{\pi\cos(\frac{e^2}{8}x) + 2\,\mathrm{Ci}(\frac{e^2}{8}x)\sin(\frac{e^2}{8}x) - 2\,\mathrm{Si}(\frac{e^2}{8}x)\cos(\frac{e^2}{8}x)}{4\pi^2 x},$$
$$D(x)_{x\to 0} = \frac{1}{4\pi x} - \frac{e^2(1+\gamma)}{16\pi^2} - \frac{e^2}{16\pi^2}\ln(\frac{e^2}{8}x) + O(x),$$
$$D(x)_{x\to\infty} = \frac{4}{\pi^2 e^2 x^2}$$

We cannot remove infrared cut-off μ.

Short distance:log correction

Long distance Coulomb force is weak and no massgeneration but we have wave renormalization→cut structure

Modified spectral function

$$\rho(p^2) = F.T.[(i\gamma\cdot\partial + m)\frac{\exp(-m|x|)}{4\pi x}[\exp(F(c|x| \ll 1) + ..\exp(F(cx \gg 1)], c = \frac{e^2 N}{8}. \quad (74)$$

For short distance

$$\rho(p) = 2\pi f(\epsilon)\int_0^\infty ds\int_0^{1/c} x^2 d|x|\,\frac{\sin(\sqrt{-p^2}\,|x|)}{\sqrt{-p^2}\,|x|}\frac{\exp(-(m-s)\,|x|)}{4\pi\,|x|}G(s)(c\,|x|)^D$$
$$= \frac{ic^D f(\epsilon)}{4\sqrt{-p^2}}\int_0^\infty dsG(s)[(m-s+i\sqrt{-p^2})^{-(D+1)}\gamma(D+1,(m-s+i\sqrt{-p^2})/c)$$
$$-(m-s-i\sqrt{-p^2})^{-(D+1)}\gamma(D+1,(m-s-i\sqrt{-p^2}/c)],$$
$$= \frac{\Gamma(D+1)ic^D f(\epsilon)}{4\sqrt{-p^2}}\int_0^\infty dsG(s)[\frac{(m-s-i\sqrt{-p^2})^{D+1}-(m-s+i\sqrt{-p^2})^{D+1}}{((m-s)^2-p^2)^{D+1}}]$$
$$= -\frac{\Gamma(D+1)c^D f(\epsilon)}{2\sqrt{-p^2}}\int_0^\infty dsG(s)\frac{\sin((D+1)\arctan(\sqrt{-p^2}/(m-s))}{\sqrt{((m-s)^2-p^2)^{D+1}}}, D=\frac{4c}{N\pi m}. \tag{75}$$

Critical coupling for $\langle\overline{\psi}\psi\rangle \neq 0 : D \geq 1$

Long distance

$$\rho(p) = \epsilon^{\frac{8}{N\pi^2}}\int_{1/c}^\infty x^2 dx\frac{\sin(\sqrt{-p^2}x)}{\sqrt{-p^2}x}\frac{\exp(-mx)}{4\pi x}(cx)^D$$
$$= \frac{\mu^D i}{2\sqrt{-p^2}}[(m+i\sqrt{-p^2})^{-(D+1)}\Gamma(D+1,(m+i\sqrt{-p^2})/c]$$
$$-(m-i\sqrt{-p^2})^{-(D+1)}\Gamma(D+1,(m-i\sqrt{-p^2})/c]. \tag{76}$$

near mass shell

$$\rho(p^2) \simeq (\frac{\mu}{m})^D\frac{\gamma\cdot p+m}{2m^2}(1-\frac{p^2}{m^2})^{-(1+D)}, D=\frac{8}{N\pi^2}. \tag{77}$$

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