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Solving Bethe-Salpeter equation in Minkowski space

V.A. Karmanov[‡] and J. Carbonell[†]

[‡] *Lebedev Physical Institute, Moscow, Russia*

[†] *Laboratoire de Physique Subatomique et Cosmologie, Grenoble, France*

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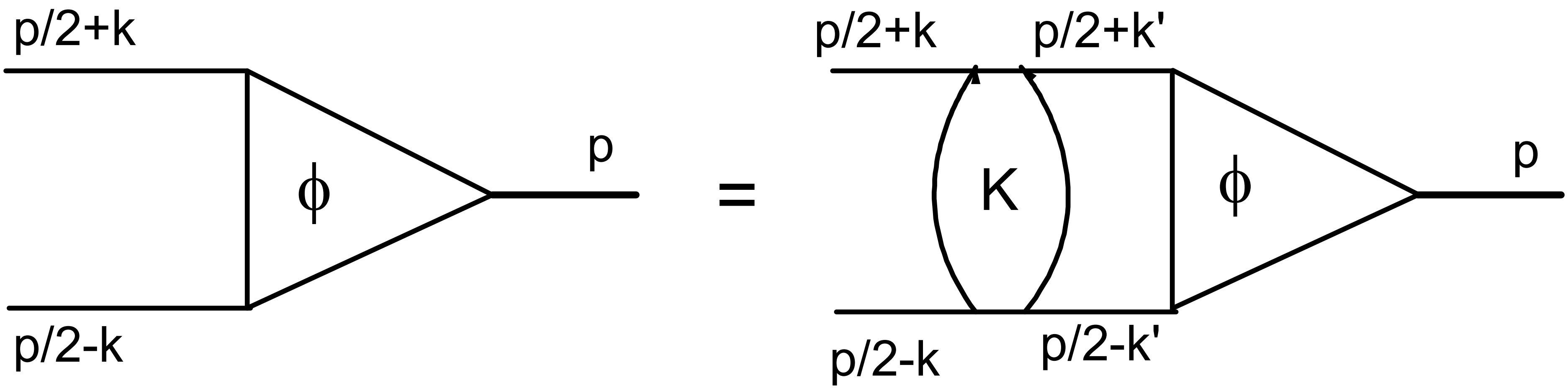
• Plan

- Introduction
- New form of the Bethe-Salpeter equation.
- Test in ladder model.
  - Binding energy.
  - BS amplitude in Minkowski space.
  - LF wave function.
- Solution of BS equation for ladder +cross-ladder kernel.
- Solution of LF equation for ladder +cross-ladder kernel.
- Comparison of BS & LF results.
- Comparison Minkowski & Euclidean space results.
- Conclusion.

• BS equation



$$\Phi(k,p) = \frac{-i}{\left((\frac{p}{2}+k)^2 - m^2 + i\epsilon\right)\left((\frac{p}{2}-k)^2 - m^2 + i\epsilon\right)}$$

$$\times \int \frac{d^4k'}{(2\pi)^4} K(k,k',p)\Phi(k',p)$$

BS equation is singular!

It is not a problem in principle (it is normal).

But it is a problem for numerical solution.

• Wick rotation ⇒ Euclidean space

Propagators ($\vec{p}=0$, $p^2=M^2$, $p\cdot k = Mk_0$):

$$k_0 \longrightarrow ik_4$$

$$\Rightarrow \quad \frac{1}{\left((\frac{p}{2}+k)^2-m^2+i\epsilon\right)} \times \frac{1}{\left((\frac{p}{2}-k)^2-m^2+i\epsilon\right)}$$

$$\frac{1}{\left(m^2-\frac{M^2}{4}+\vec{k}^2+k_4^2\right)^2+M^2k_4^2}$$

Kernel (OBE):

$$K=\frac{-g^2}{(k-k')^2-\mu^2} \rightarrow K_E=\frac{g^2}{(\vec{k}-\vec{k'})^2+(k_4-k_4')^2+\mu^2}$$

BS amplitude:

$$\Phi(\vec{k}, k_0) \rightarrow \Phi_E(\vec{k}, k_4) = \Phi(\vec{k}, ik_0)$$

Equation:

$$\begin{aligned} \Phi_E(\vec{k}, k_4) &= \frac{1}{\left(m^2 - \frac{M^2}{4} + \vec{k}^2 + k_4^2\right)^2 + M^2 k_4^2} \\ &\times \int \frac{d^3k' dk_4}{(2\pi)^4} K_E(k, k') \Phi_E(\vec{k'}, k_4) \end{aligned}$$

Finding solution in Euclidean space is a trivial numerical task.

We find $\Phi_E(\vec{k}, k_4)$ and M.

But we cannot extrapolate $\Phi_E(\vec{k}, k_4)$ back in Minkowski space numerically.

Unstable extrapolation: $\Phi_E(\vec{k}, k_4) \rightarrow \Phi_E(\vec{k}, -ik_4) = \Phi(\vec{k}, k_0)$

Therefore we cannot use $\Phi_E(\vec{k}, k_4)$.

• Minkowski space solution

(first solution, for the ladder kernel only)

K. Kusaka, A.G. Williams, Phys.Rev. **D51** *(1995) 7026;*

K. Kusaka, K. Simpson, A.G. Williams, Phys.Rev. **D56** *(1997) 5071.*

It is based on Nakanishi integral representation for BS amplitude:

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{-ig(\gamma', z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\; z' - i\epsilon\right]^3}.$$

For ladder kernel an equation for $g(\gamma, z)$ is found and solved. Then the BS amplitude $\Phi(k,p)$ is easily found in Minkowski space.

• Another method: projecting on LF

E.g.: *J.H.O. Sales, T. Frederico, B.V. Carlson and P.U. Sauer, Phys. Rev.* **C 61** *(2000) 044003*.

$$\psi(\vec{k}_\perp, x) = \int \Phi(k,p) dk_-$$

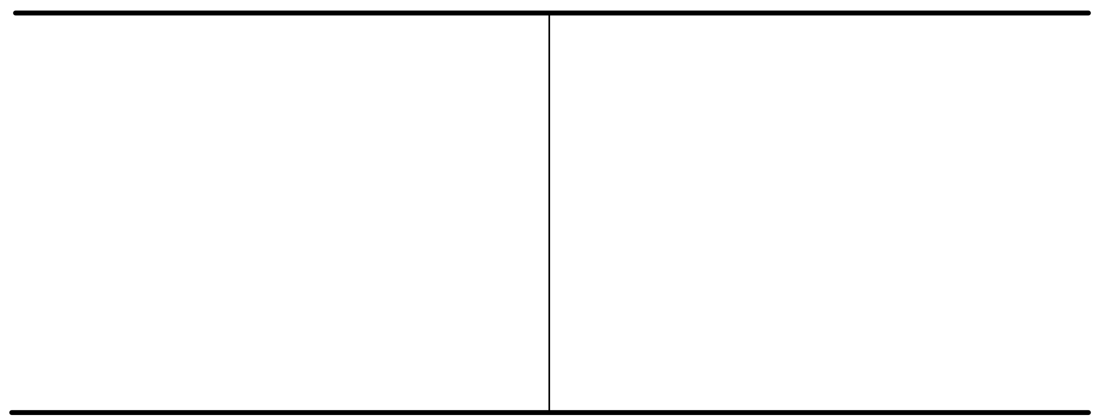
Advantage: Light-front wave function $\psi(\vec{k}_\perp, x)$ is non-singular.

Equation for $\psi(\vec{k}_\perp, x)$:

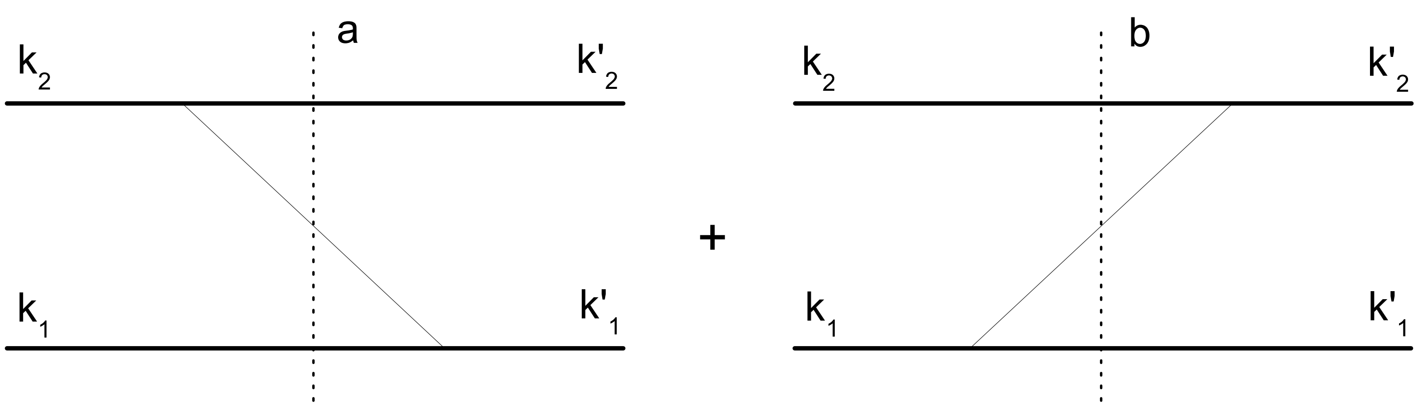
$$\left(\frac{\vec{k}^2_\perp + m^2}{x(1-x)} - M^2\right)\psi(\vec{k}_\perp, x) =$$

$$-\frac{m^2}{2\pi^3}\int \psi(\vec{k}'_\perp, x') V_{LF}(\vec{k}'_\perp, x'; \vec{k}_\perp, x, M^2)\frac{d^2k'_\perp dx'}{2x'(1-x')}$$

• Feynman & LF ladders



Feynman ladder kernel

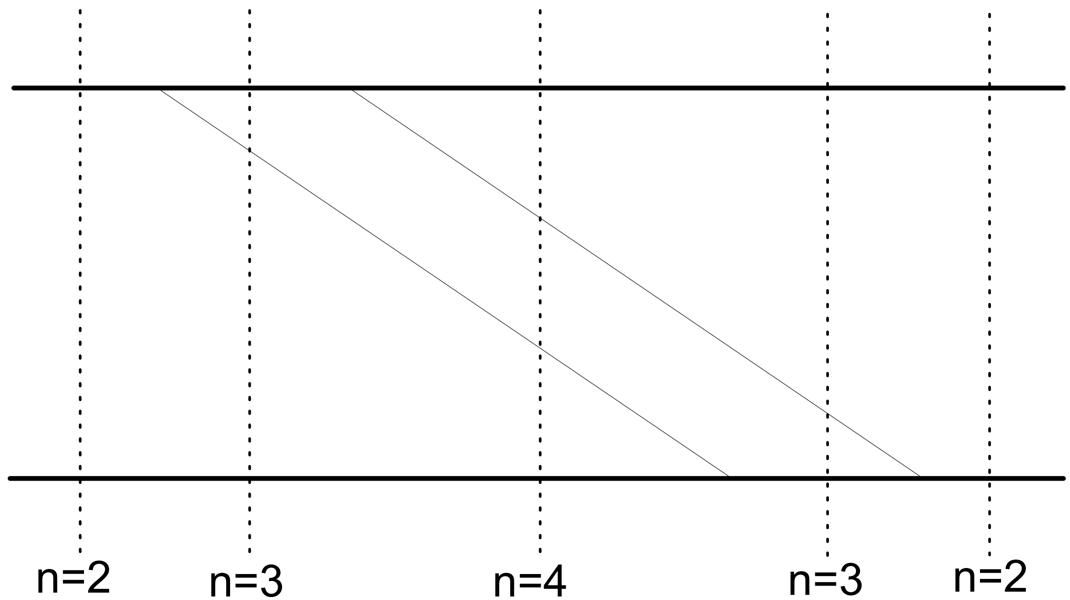


LFD (time ordered) ladder kernel

• Second iteration (Feynman)

Feynman ladder graph with two exchanges
(included in the ladder BS equation)

• Stretched box (LFD)



One of six time-ordered ladder graphs, generated by the Feynman graph (absent in the ladder LFD equation)

- LFD ladder equation is an approximation of the BS one.

• Our (exact) method

Combination of the two:

1. Nakanishi representation
2. Light-front projection

To present method, we consider:

- Spinless particles.
- Zero angular momentum $J = 0$.

• Derivation

- Take BS amplitude in the Nakanishi form:

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_0^\infty d\gamma' \frac{-ig(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\ z' - i\epsilon\right]^3}$$

- Substitute it in the BS equation.
- Apply to both sides of the BS equation the LF projection:

$$\psi(\vec{k}_\perp, x) = \int \Phi(k,p) dk_-$$

- Obtain equation for $g(\gamma, z)$.

• Explicitly covariant LFD

V.A. Karmanov, JETP, ***44*** *(1976) 201.*

J. Carbonell, B. Desplanques, V.A. Karmanov and J.-F.Mathiot, Phys. Reports, ***300*** *(1998) 215*

- All the coordinates x, y, z are equivalent.
- All the LF planes $t+z=0$, $t+x=0$, $t+y=0$ are equivalent.
- All the orientations of the LF plane are equivalent.
- Take LF of general orientation:
  $\omega{\cdot}x=\omega_0 t-\vec{\omega}{\cdot}\vec{x}=0$, $\omega=(\omega_0,\vec{\omega})$ such that $\omega^2=0$.
- Construct LFD, using this general LF plane.

Standard approach: $\omega=(1,0,0,-1)$.
In LF coordinates: $\omega_-=2$, $\omega_+=0$, $\vec{\omega}_\perp=0$.

• Advantages

- Explicit covariance
- Wave function depends on LF plane $\Rightarrow$ its explicit dependence on ω (angular momentum construction, etc.).
- Observables must not depend on LF plane $\Rightarrow$ analysis of their dependence on ω due to approximations.
- Etc.

LF projection

$$
\begin{aligned}
\psi(\vec{k}_\perp, x) &= \int_{-\infty}^{\infty} \Phi(k,p)dk_- \\
&\Rightarrow \int_{-\infty}^{\infty} \Phi(k+\beta\omega,p)d\beta \\
&= \int_{0}^{\infty} \frac{g(\gamma', 1-2x)d\gamma'}{\left[\gamma' + \vec{k}_\perp^2 + m^2 - x(1-x)M^2\right]^2}
\end{aligned}
$$

• Equation for $g(\gamma, z)$

$$\int_0^\infty \frac{g(\gamma', z)d\gamma'}{\left[\gamma' + \gamma + z^2m^2 + (1 - z^2)\kappa^2\right]^2}$$

$$= \int_0^\infty d\gamma' \int_{-1}^1 dz' \, V(\gamma, z; \gamma', z')g(\gamma', z')$$

where $z = 1 - 2x$, $\kappa^2 = m^2 - \frac{1}{4}M^2$.

• This equation is equivalent to the initial BS equation.

Matrix form:

$$\lambda Bx = Ax$$

It is just standard form for well known fortran subroutines.

Kernel

Calculate:

$$I(k,p) = \int \frac{d^4k'}{(2\pi)^4} \frac{iK(k,k',p)}{\left[k'^2 + p\cdot k' z' - \gamma' - \kappa^2 + i\epsilon\right]^3}.$$

Substitute in:

$$\begin{aligned} V(\gamma, z; \gamma', z') &= \frac{\omega\cdot p}{\pi} \int_{-\infty}^{\infty} \frac{-iI(k+\beta\omega, p)d\beta}{\left[(\frac{p}{2}+k+\beta\omega)^2 - m^2 + i\epsilon\right]} \\ &\times \frac{1}{\left[(\frac{p}{2}-k-\beta\omega)^2 - m^2 + i\epsilon\right]}, \end{aligned}$$

- Why LF projection?

To eliminate singularities of the BS amplitude.

We obtain wave function. It has no singularities in physical domain.

• Why Nakanishi representation?

To obtain a self-contained equation.

Namely. Take BS equation:

$$\Phi = \Pi_1 \Pi_2 \int K \Phi d^4 k'$$

Project it on LF plane:

$$\int dk_- \Phi = \int dk_- \Pi_1 \Pi_2 \int K \Phi d^4 k' \Rightarrow \psi = \int dk_- \Pi_1 \Pi_2 \int K \Phi d^4 k'$$

L.h.-side contains ψ, but r.h.-side still contains Φ.
Expressing both through $g(\gamma, z)$, we obtain equation for $g(\gamma, z)$.

• OBE kernel

One-boson exchange (ladder) kernel: $K(k,k',p)=\frac{-g^2}{(k-k')^2-\mu^2+i\epsilon}$

$g^2=16\pi m^2\alpha$. Equation:

$\int_0^\infty \frac{g(\gamma',z)d\gamma'}{\left[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2\right]^2}=\int_0^\infty d\gamma' \int_{-1}^1 dz'\; V(\gamma,z;\gamma',z')g(\gamma',z')$

Kernel $V(\gamma,z;\gamma',z')$:

$$V(\gamma,z;\gamma',z')=\frac{\alpha m^2(1-z)^2}{2\pi\left[\gamma+z^2m^2+(1-z^2)\kappa^2\right]}\int_0^1\frac{v^2dv}{B_1^2}$$

$B_1=B_1(\gamma,z;\gamma',z';v)$ is a polynomial. Integral $\int_0^1\frac{v^2dv}{B_1^2}$ is calculated analytically. Equation is solved numerically.

• Wick-Cutkosky model ($\mu = 0$)

For $\mu = 0$, we are looking the solution in the form:

$$g(\gamma, z) = \delta(\gamma) g(z)$$

The kernel is very simplified. We get:

$$g(z) = \frac{\alpha}{2\pi} \int_{-1}^{1} dz' \, \tilde{K}(z, z') g(z')$$

with

$$\tilde{K}(z', z') = \frac{m^2}{m^2 - \frac{1}{4}(1 - z'^2) M^2} \begin{cases} \frac{(1-z)}{(1-z')}, & \text{if } -1 \leq z' \leq z \\ \frac{(1+z)}{(1+z')}, & \text{if } \quad z \leq z' \leq 1 \end{cases}$$

This is the Wick-Cutkosky equation (1954).

• Numerical solution ($\mu \neq 0$)

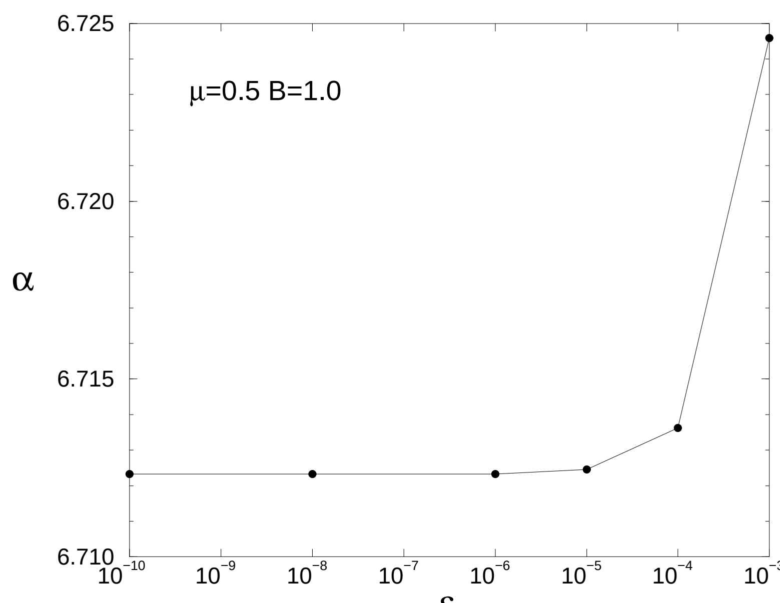
Eigenvalue matrix equation

$$\lambda\, B(M)g = A(M)g$$

Regularization of l.h.-side kernel:

$$B_{ij} \rightarrow B_{ij} + \varepsilon\delta_{ij}$$

Sensitivity to ε:



• Numerical results ($\mu \neq 0$)

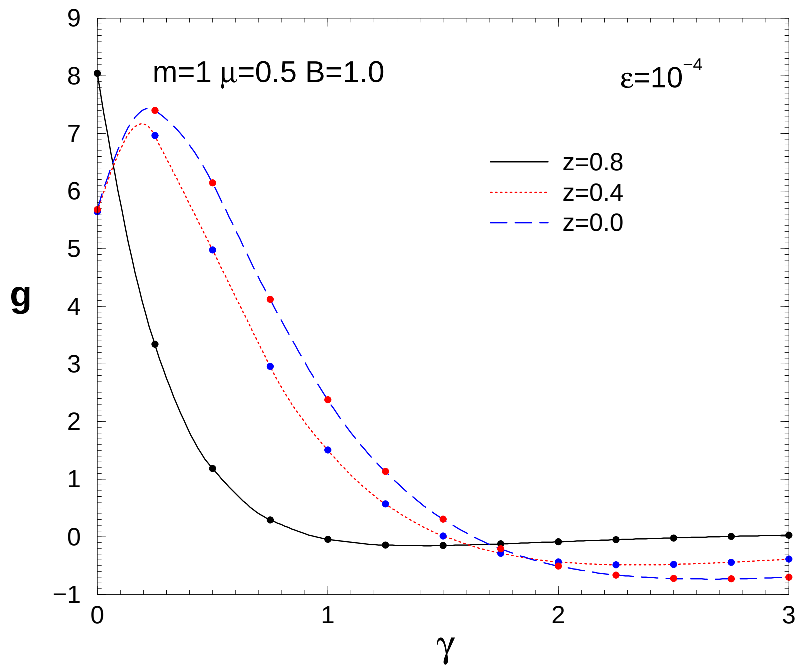
Coupling constant $\alpha = \frac{g^2}{16\pi m^2}$ as a function of the binding energy for $\mu = 0.15$ and $\mu = 0.5$

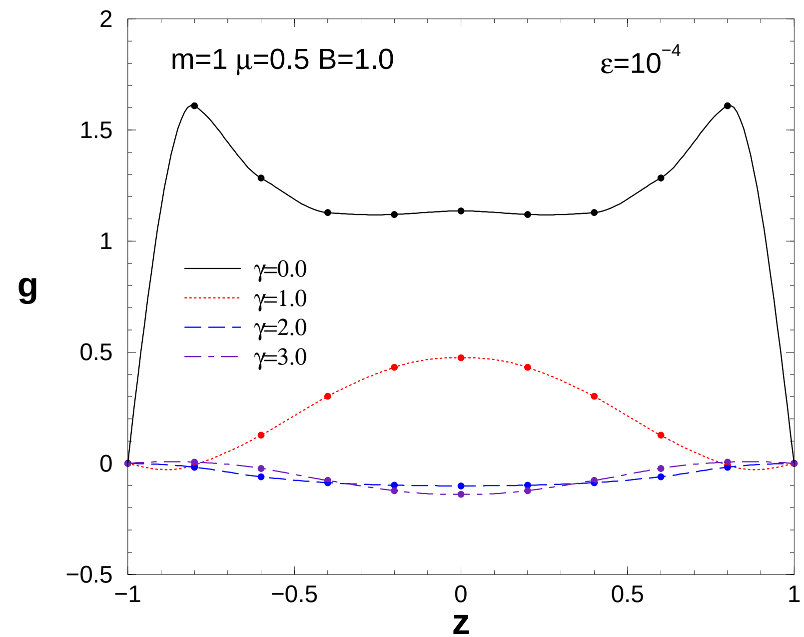
B	$\alpha(\mu = 0.15)$	$\alpha(\mu = 0.50)$
0.01	0.5716	1.440
0.10	1.437	2.498
0.20	2.100	3.251
0.50	3.611	4.901
1.00	5.315	6.712

These results, with all shown digits, exactly coincide with ones obtained in Euclidean space (by Wick rotation).

• This is a test of the method.

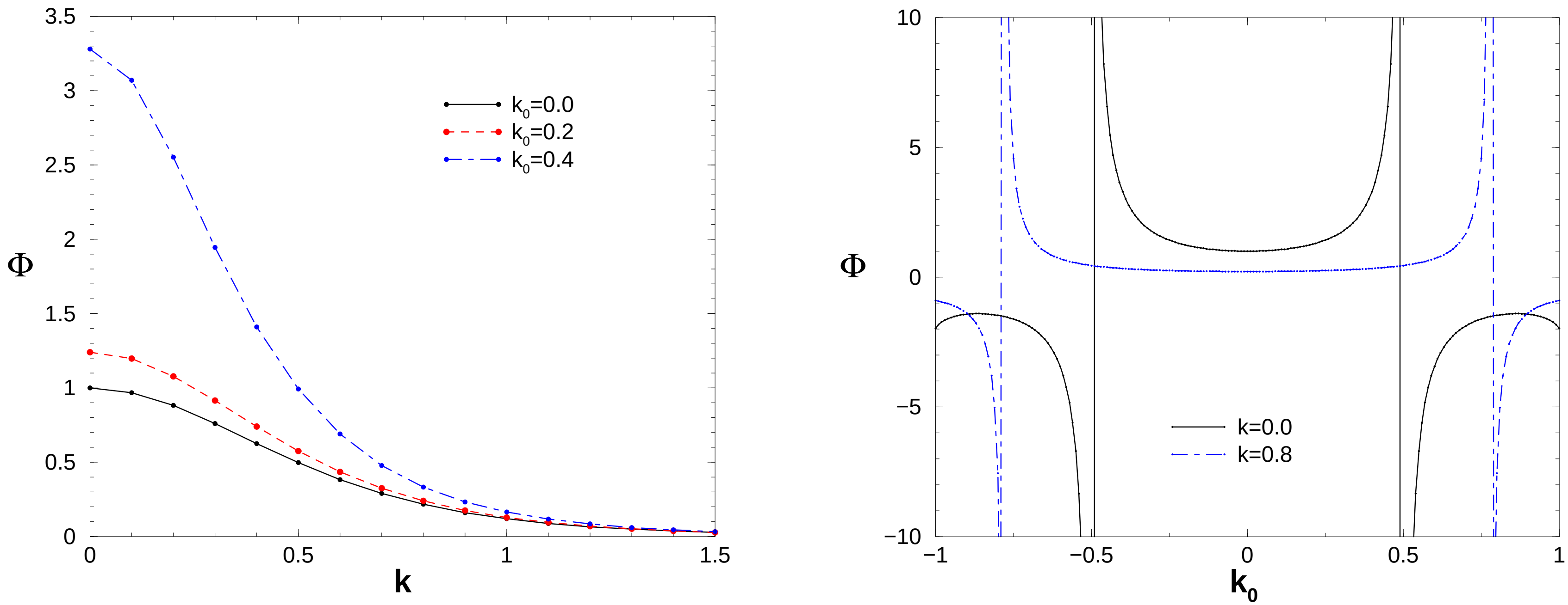
Function $g(\gamma, z)$





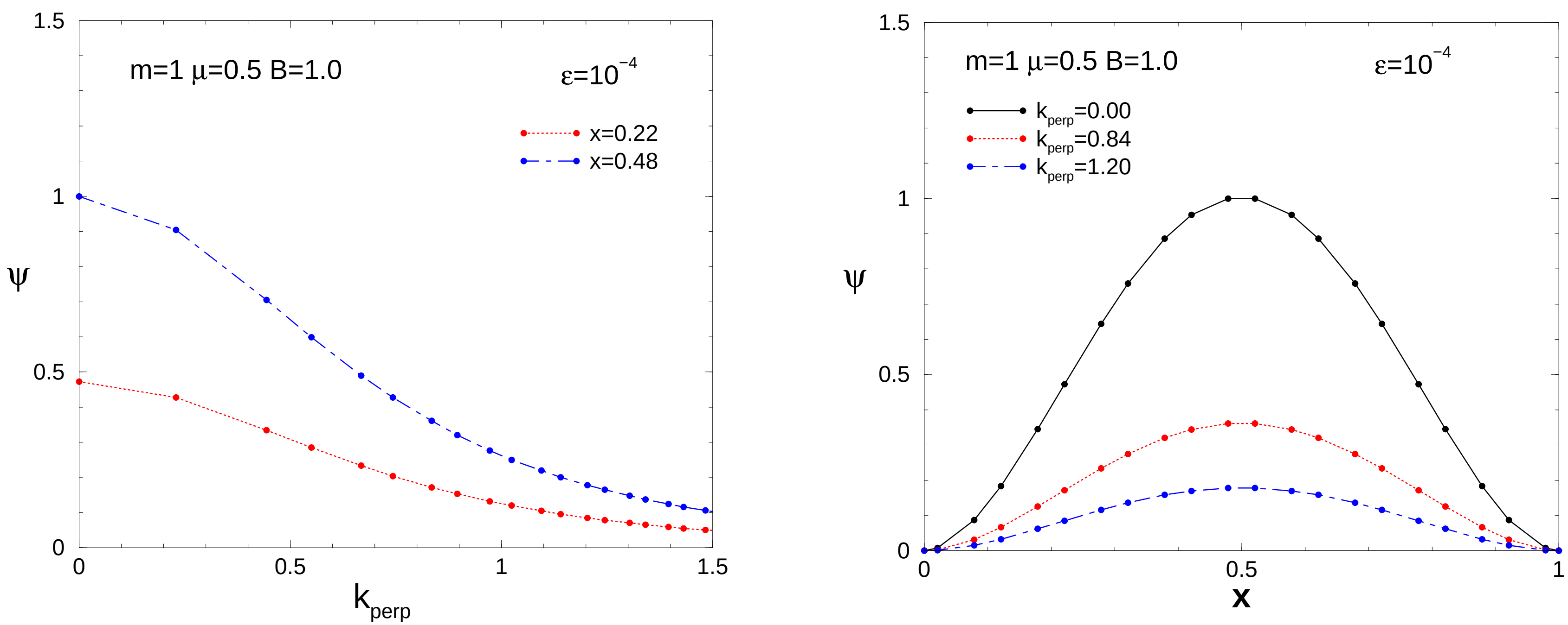
Function $g(\gamma, z)$ for $\mu = 0.5$ and $B = 1.0$. On left – versus γ for fixed values of z and on right – versus z for a few fixed values of γ.

BS amplitude $\Phi(k_0, k)$ $(\vec{p} = 0)$



BS amplitude $\Phi(k_0, k)$ for $\mu = 0.5$ and $B = 1.0$. On left versus k for fixed values of k_0 and on right versus k_0 for a few fixed values of k.

LF wave function $\psi(k_\perp, x)$



Wave function $\psi(k_\perp, x)$ for $\mu = 0.5$ and $B = 1.0$. On left versus $k_\perp$ for fixed values of x and on right versus x for a few fixed values of $k_\perp$.

If we know $g(\gamma, z)$, we can obtain:

- BS amplitude in Minkowski space
  (and in Euclidean space too)
- LF wave function
- Observables

Function $g(\gamma, z)$ is better than BS amplitude!

Example: e.m. form factor

$$(p+p')^\mu F(Q^2) = -i \int \frac{d^4k}{(2\pi)^4} (p+p'-2k)^\mu (m^2-k^2)$$
$$\times \Phi\left(\frac{1}{2}p - k, p\right) \Phi\left(\frac{1}{2}p' - k, p'\right)$$

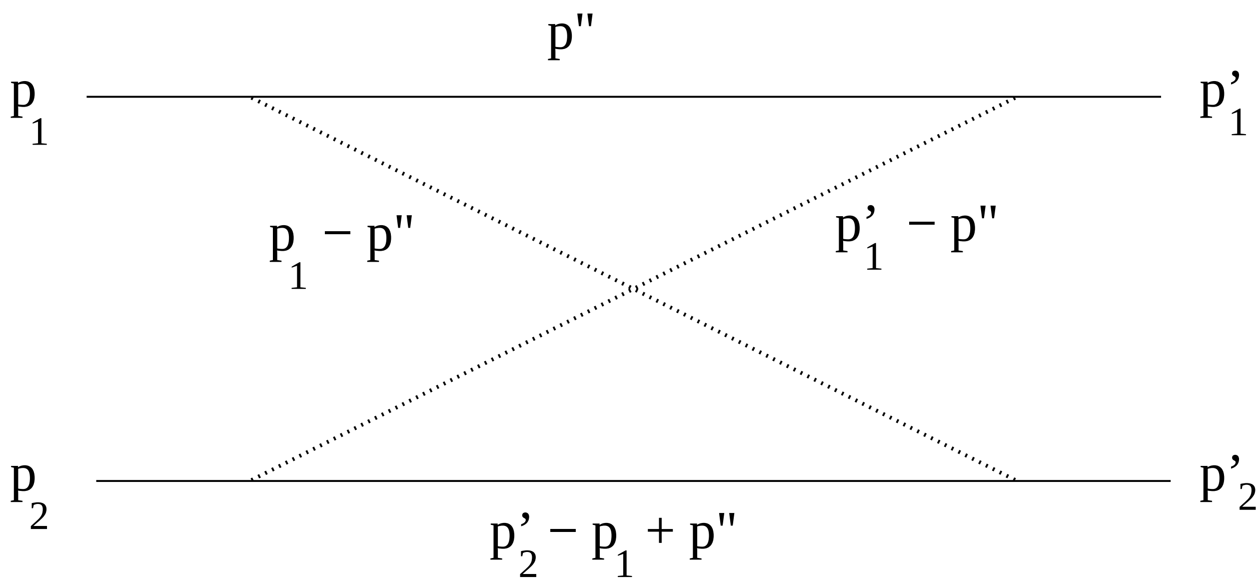
$F(0)$ in Wick-Cutkosky model:

$$\begin{aligned} F(0) &= -\frac{1}{2^5\pi^3} \int_0^1 g(2x-1)dx \int_0^1 g(2x'-1)dx' \\ &\times \int_0^1 u^2\,(1-u)^2 du \frac{\left((6\xi-5)m^2 + 2\xi(1-\xi)M^2\right)}{\left(m^2 - \xi(1-\xi)M^2\right)^4}, \end{aligned}$$

where $\xi = xu + x'(1-u)$.
Form factor F is expressed through $g(\gamma, z)$.

• Cross-box kernel

(first solution)



Feynman cross ladder

Kernel $V(\gamma, z; \gamma', z')$

Calculate:

$$I(k,p) = \int \frac{d^4k'}{(2\pi)^4} \frac{iK(k,k',p)}{\left[k'^2 + p\cdot k' z' - \gamma' - \kappa^2 + i\epsilon\right]^3}.$$

Substitute in:

$$\begin{aligned} V(\gamma, z; \gamma', z') &= \frac{\omega\cdot p}{\pi}\int_{-\infty}^{\infty} \frac{-iI(k+\beta\omega, p)d\beta}{[(\frac{p}{2}+k+\beta\omega)^2 - m^2 + i\epsilon]} \\ &\times \frac{1}{[(\frac{p}{2}-k-\beta\omega)^2 - m^2 + i\epsilon]}, \end{aligned}$$

The integrals for the kernel over four-momenta are calculated analytically. The kernel is represented in the form of 4-dim. integral over the Feynman parameters:

$$V(\gamma, z; \gamma', z') = \int_0^1 y_4(1-y_4)^2 dy_4 \int_0^1 dy_3 \int_0^{1-y_3} dy_2 \int_0^{1-y_2-y_3} dy_1 \frac{c_2}{D^3}$$

where

$$c_2 = 1 - y_4[1 - (1 - y_1 - y_3)(y_1 + y_3)]$$

D is a polynomial.

$$V(\gamma, z; \gamma', z') = V^{(ladder)} + V^{(cr.box)}$$

4-dim. integral for the kernel $V^{(cr.box)}$ is calculated numerically.

• LFD approach

For comparison, we also solved LFD equation for ladder +cross ladders +stretched boxes.

$$\left(\frac{\vec{k}_\perp^2+m^2}{x(1-x)}-M^2\right)\psi(\vec{k}_\perp,x)$$

$$=-\frac{m^2}{2\pi^3}\int\psi(\vec{k}'_\perp,x')V(\vec{k}'_\perp,x';\vec{k}_\perp,x,M^2)\frac{d^2k'_\perp dx'}{2x'(1-x')}$$

$$V(\vec{k}'_\perp,x';\vec{k}_\perp,x,M^2)=V^{(ladder)}+V^{(cr.ladder)}+V^{(str.box)}$$

$$V^{(cr.ladder)} = \sum_{i=1,\dots,6} V_i$$

$$V^{(str.box)} = \sum_{i=7,8} V_i$$

LFD cross ladders

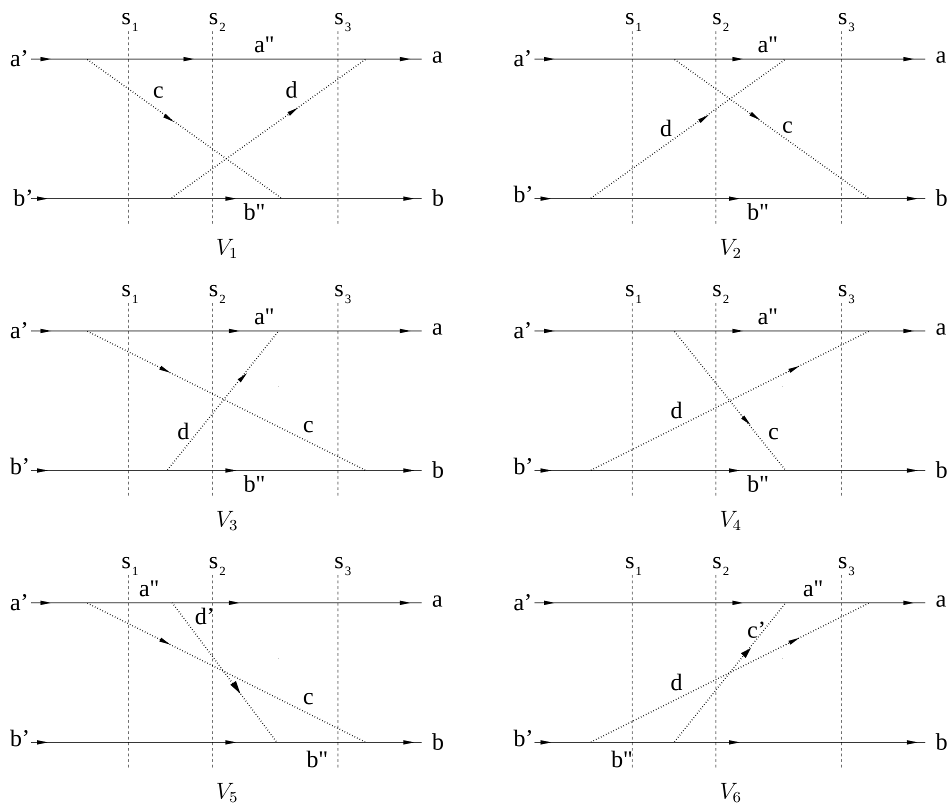
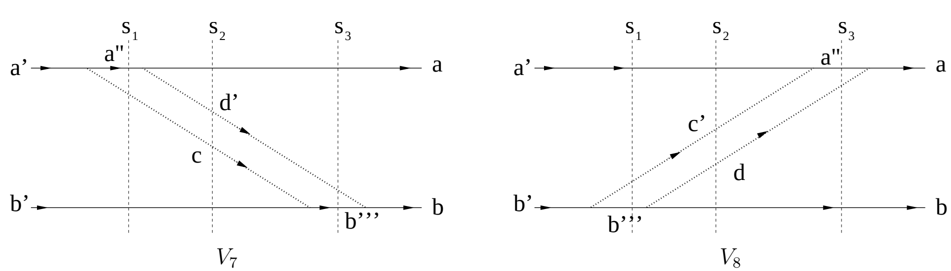


Figure 1: Cross LFD graphs.

LFD stretched boxes



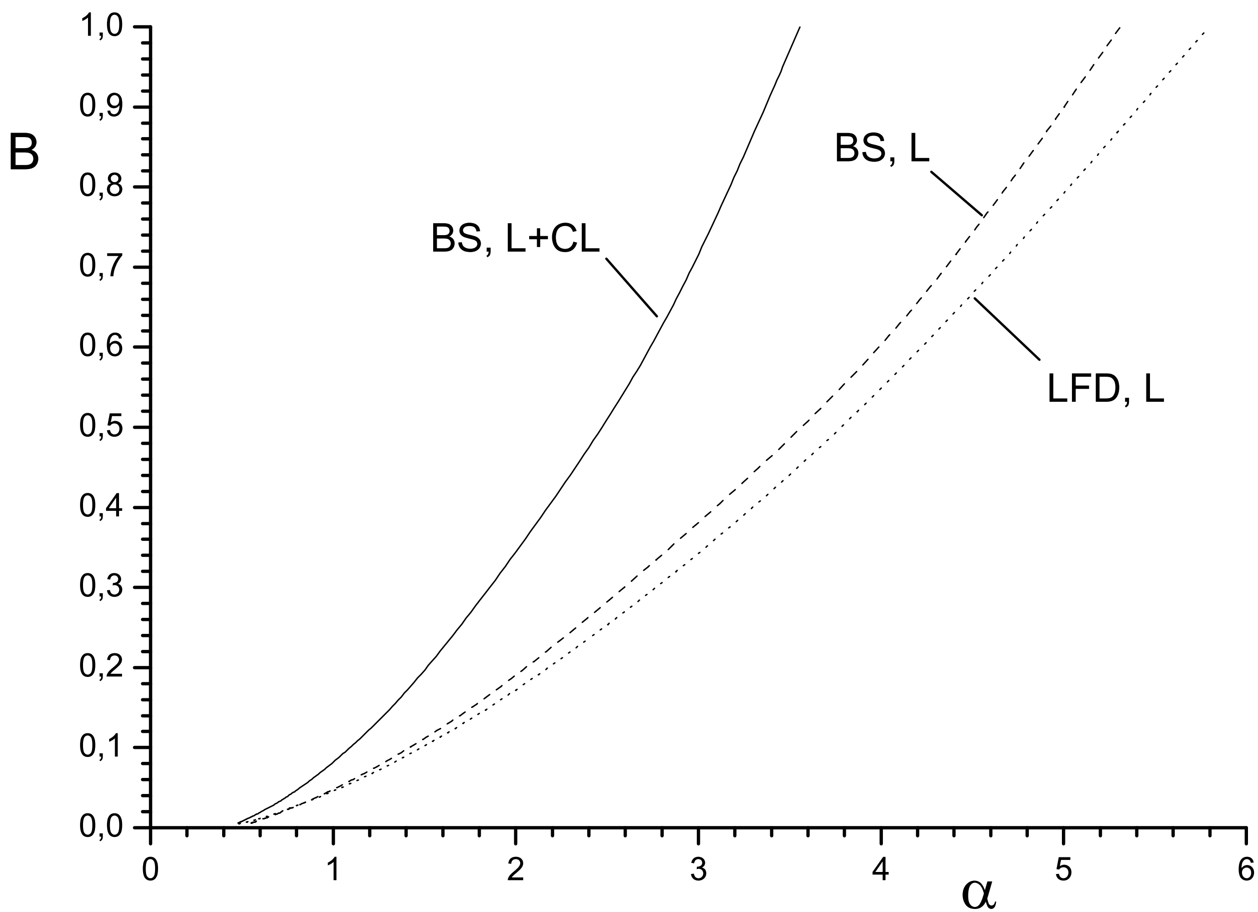
• Numerical results

Coupling constant α for given values of the binding energy B and exchanged mass $\mu = 0.5$ calculated with BS and LF equations for the ladder (L), ladder +cross-ladder (L+CL) and (in LFD) for the ladder +cross-ladder +stretched-box (L+CL+SB) kernels.

B	BS(L)	BS (L+CL)	LF (L)	LFD (L+CL)	LFD (L+CL+SB)
0.01	1.44	1.21	1.46	1.23	1.21
0.05	2.01	1.62	2.06	1.65	1.62
0.10	2.50	1.93	2.57	2.01	1.97
0.20	3.25	2.42	3.37	2.53	2.47
0.50	4.90	3.47	5.16	3.67	3.61
1.00	6.71	4.56	7.17	4.97	4.91

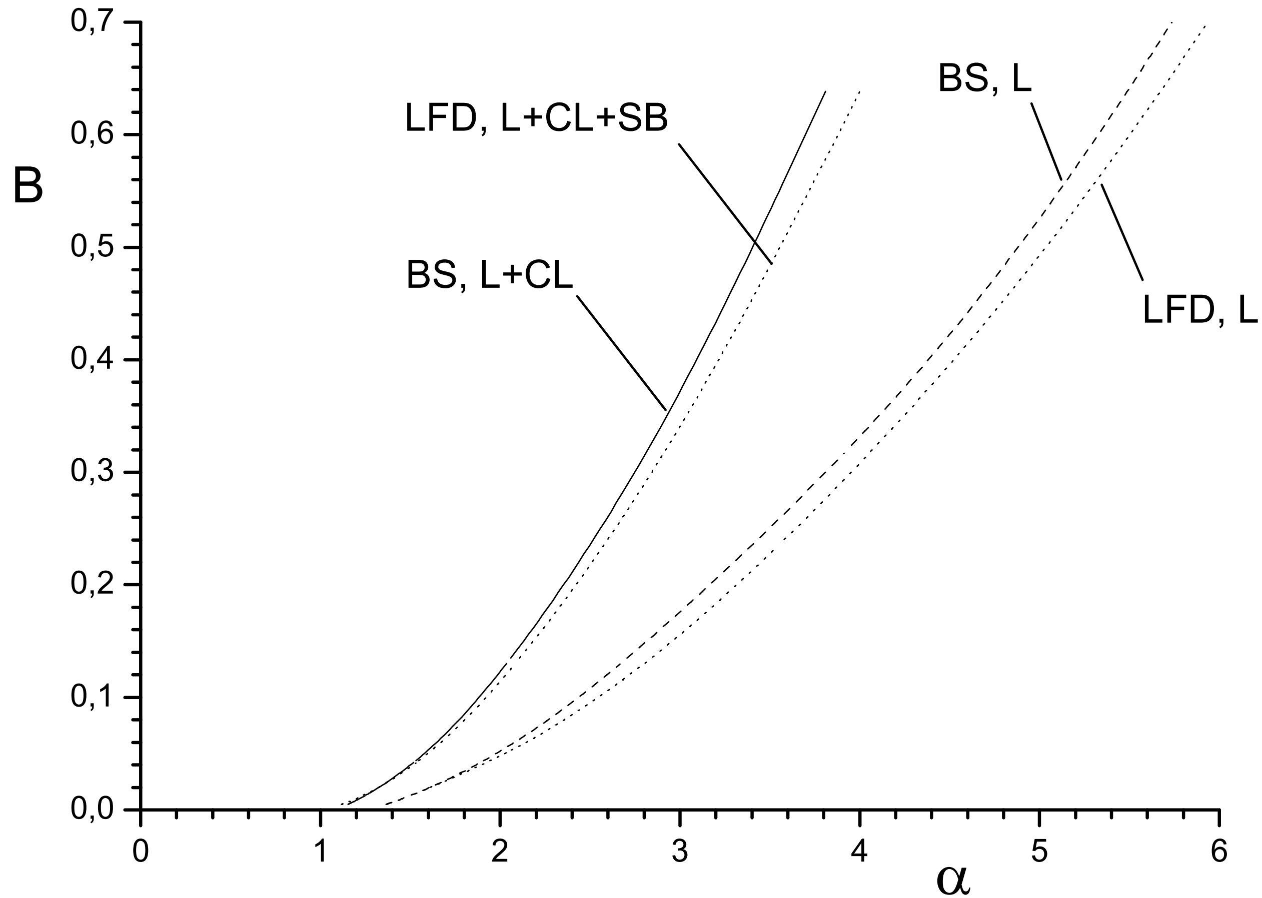
Stretched box $V^{(str.box)}$ is small (*N.C.J. Schoonderwoerd, B.L.G. Bakker and V.A. Karmanov, Phys. Rev.* **C 58** *(1998) 3093.*)

• Numerical results (L +CL), $\mu = 0.15$



Binding energy B vs. coupling constant α for BS and LFD equations with the ladder (L) kernels only and with the ladder +cross-ladder (L+CL) one for exchange mass $\mu = 0.15$.

• Numerical results (L +CL +SB), $\mu = 0.5$



Binding energy B vs. coupling constant α for BS and LFD equations with the ladder (L) kernels only, with the ladder +cross-ladder (L+CL) and with the ladder +cross-ladder +stretched boxes (L+CL+SB) for exchange mass $\mu = 0.5$.

- Stretched box $V^{(str.box)}$ is small

  N.C.J. Schoonderwoerd, B.L.G. Bakker and V.A. Karmanov, Phys. Rev. **C 58** *(1998) 3093.*

  Dae Sung Hwang and V.A. Karmanov, Nucl.Phys. **B696** *(2004) 413-444; hep-th/0405035.*

- Cross ladder $V^{(cr.ladder)}$ is large

• Wick rotation for cross ladder

(method of brute force)

R.h.s of the BS equation:

$$I \sim \int \frac{d^4k'}{(2\pi)^4} K(k,k',p)\Phi(k',p).$$

Substitute here the Nakanishi representation:

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' \frac{-ig(\gamma',z')}{\left[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\ z' - i\epsilon\right]^3}.$$

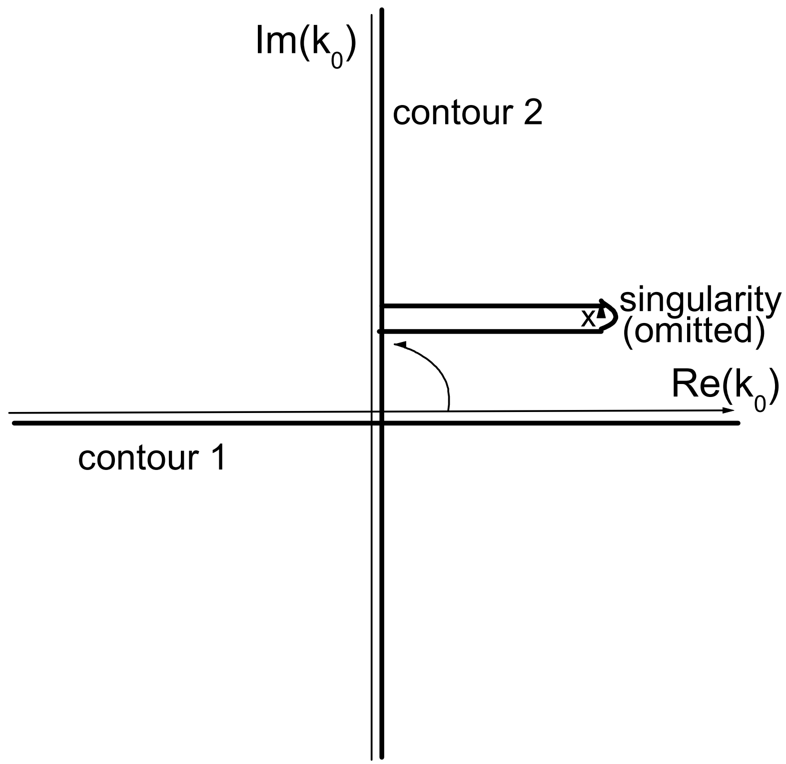
I calculate the integral over d^4k' (with a test function $g(\gamma,z)$) by two methods:

(1) Directly, without any Wick rotation;

(2) By rotating the contour: $k_0 \to ik_4$;

(3) Compare results.

If contour crosses singularities (if any), then Wick rotation is impossible. The results (1) and (2) will be different.



Contour 1 is integration in Minkowski space.
Contour 2 is integration in Euclidean space.

Result ($k_0 = 0.1i$, $k = 0.05$, $M = 1.5$)

$$\left.\begin{array}{ll} (1) & I^{exact} = 0.0169730 \\ (2) & I^{Wick} = 0.0169728 \end{array}\right\} \quad \text{coincide!}$$

But for $M = 2.1 + 0.1i$ (i.e., $|M| > 2m$):

$$\left.\begin{array}{ll} (1) & J^{exact} = 0.060007 + i0.0406 \\ (2) & J^{Wick} = 0.054764 + i0.0424 \end{array}\right\} \quad \text{different!}$$

- Wick rotation may be possible if $M < 2m$.

• BS, L+CL, by Wick rotation

(M. Mangin-Brinet and V.A.K.)

$$\left[\left(\vec{k}^2+k_4^2+m^2-\frac{M^2}{4}\right)^2+M^2k_4^2\right]\Phi_E(\vec{k},k_4)=\int\frac{d^3k'dk'_4}{(2\pi)^4}K_E(\vec{k},k_4;\vec{k}',k'_4)\Phi_E(\vec{k}',k_4),$$

Coupling constant α vs. binding energy B for $\mu=0.5$ calculated with BS equation for the ladder +cross-ladder (L+CL) kernel in Minkowski and in Euclidean spaces.

B	BS(L+CL), Minkowski	BS(L+CL), Euclid
0.01	1.21	1.21
0.05	1.62	1.61
0.10	1.93	1.93
0.20	2.42	2.42
0.50	3.47	3.46
1.00	4.56	4.56
		error: ± 0.01

Wick rotation is valid for cross-ladder! Coincidence shows that both methods work. Binding energies are the same, but BS amplitudes Φ and Φ_E are different.

• Conclusions

- A method to find Bethe-Salpeter amplitude in Minkowski space is developed.
- It gives, as a by-product, LF wave function.
- It is tested in the ladder model.
- The method is applied to ladder +cross ladder kernel.
  - Cross ladder contribution is large (and attractive).
  - Stretched box contribution is small (and attractive).
  - Difference between BS and LFD is small.
  - BS binding energies found in Minkowski and Euclidean spaces are the same (but BS amplitudes are different).
- The method can be applied to any kernel.
- Generalization for the fermions seems possible.