UV and IR behaviour for QFT and LCQFT with fields as Operator Valued Distributions: Epstein and Glaser revisited

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- Identification of divergences : propagators on the diagonnal *ie* $x = x'$.
- Link with Epstein and Glaser's analysis and role of the partition of unity in its extension.
- The method at work: UV and IR analysis.

Epstein and Glaser's analysis

- Euclidean massive scalar field as OPVD
  OPVD defines a functional with respect to a test function $\rho(x)$, C^∞ with compact support,

$$\Phi(\rho) \equiv < \varphi, \rho > = \int d^{(D)}y \varphi(y)\rho(y).$$

  More general interpretation: functional $\Phi(x, \rho)$ evaluated at $x = 0$.
  The translated functional is a well defined object such that

$$T_x\Phi(\rho) = < T_x\varphi, \rho > = < \varphi, T_{-x}\rho > = \int d^{(D)}y \varphi(y)\rho(x - y)$$

  Due to the properties of ρ $T_x\Phi(\rho)$ obeys the KG equation and is taken as the physical field $\phi(x)$

Fourier decomposition of $\rho(x-y)$

$$\rho(x-y) = \int \frac{d^{(D)}q}{(2\pi)^D} e^{iq(x-y)} f(q^2)$$

quantized form $\phi(x)$ follows

$$\phi(x) = \int \frac{d^{(D)}p}{(2\pi)^D} [a_p^+ e^{ipx} + a_p e^{-ipx}] f(p^2).$$

$f(p^2)$: partition of unity(paracompactness property of Euclidean manifold): ensures convergence of otherwise diverging integrals , plays no role on the reverse.

- Example:propagator

$$\Delta(x-y) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{[-ip.(x-y)]} f^2(p^2)}{(p^2+m^2)}$$

At $D = 2..4$ and for $x \neq y$ $\Delta(x-y)$ is finite and $f^2(p^2)$ may be taken to 1 everywhere.
Aim : understand the role of the partition of unity in the extension of $\Delta(x-y)$ to the diagonnal

- $E-G$'s analysis of singular distributions
  - $f(X)$: $\mathbb{C}^\infty(\mathbb{R}^d)$ test function $\in \mathbb{S}(\mathbb{R}^d)$
  - $T(X)$ distribution $\in \mathbb{S}'(\mathbb{R}^d)$
  - singular order k of $T(X)$ at the origin of ($\mathbb{R}^d$) such that

$$k = inf\{s : \lim_{\lambda\to 0} \lambda^s T(\lambda X) = 0\} - d$$

- $E-G$'s extension and magic of Lagrange's formula
  - Taylor series surgery : throw away the weigthed k-jet of $f(X)$ at the origin : $R_0^k f$ is the Taylor remainder

$$\mathbb{P}^w f(X) = (1-w(X))R_0^{k-1}f(X) + w(X)R_0^k f(X)$$

  - $w(X)$ $E-G$'s weight with properties $w(0)=1$, $w^{(\alpha)}(0)=0$,$0 <\mid \alpha \mid\leq k$

- $\widetilde{T}(X)$ extension of $T(X)$ such that

$$< \widetilde{T}, f > = < T, \mathbb{P}^w f > = \int d^d X T(X) \mathbb{P}^w f(X)$$

- Lagrange's formula for Taylor remainder

$$R_0^k f(X) = (k+1) \sum_{|\beta|=k+1} \partial^\beta [\frac{X^\beta}{\beta!} \int_0^1 dt (1-t)^k \partial^\beta_{(tX)} f(tX)]$$

- $\widetilde{T}(X)$ obtained by partial integration

$$\widetilde{T}(X) = (-)^k k \sum_{|\alpha|=k} \partial^\alpha [\frac{X^\alpha}{\alpha!} \int_0^1 dt \frac{(1-t)^{k-1}}{t^{k+d}} T(\frac{X}{t})(1-w(\frac{X}{t}))]$$

$$+(-)^{k+1}(k+1) \sum_{|\alpha|=k+1} \partial^\alpha [\frac{X^\alpha}{\alpha!} \int_0^1 dt \frac{(1-t)^{k-1}}{t^{k+d+1}} T(\frac{X}{t}) w(\frac{X}{t})]$$

- $f(X)$ partition of unity for $\|X\| \in [0, h]$, $f^{(\alpha)}(0) = f^{(\alpha)}(h) = 0$, $\forall \alpha \geq 0$
- $\Longrightarrow$ $\boldsymbol{f(X)} \equiv$ Taylor remainder
- at $\|X\| \approx 0$ Taylor remainder is:

$$f(X) \equiv f^{<}(X) \equiv (k+1) \sum_{|\beta|=k+1} [\frac{X^\beta}{\beta!} \int_0^1 dt (1-t)^k \partial^\beta_{(tX)} f(tX)] \forall k \geq 0$$

- at $\|X\| \approx h$ Taylor remainder is :

$$f(X) \equiv f^{>}(X) \equiv -(k+1) \sum_{|\beta|=k+1} [\frac{X^\beta}{\beta!} \int_1^\infty dt (1-t)^k \partial^\beta_{(tX)} f(tX)] \forall k \geq 0$$

Partition of unity : example,ppties

• $f^{>}(X) \Longrightarrow \widetilde{T}^{>}(X)$ extension of $T(X)$ singular of order k at $\|X\| = h$: UV extension

• $f^{<}(X) \Longrightarrow \widetilde{T}^{<}(X)$ extension of $T(X)$ singular of order k at $\|X\| = 0$: IR extension

- UV analysis
  • define $f^{>}(X)$

$$f^{>}(X) = \begin{cases} 1 & \text{for } \|X\| \leq 1 \\ \chi(\mathrm{X}, \mathrm{h}) & \text{for } 1 < \|X\| \leq 1 + h \\ 0 & \text{for } \|X\| > 1 + h \end{cases}$$

  • possible choice for $\chi(X, h)$

$$\chi(X, h) = \mathbb{N}_h \int_{\|X\|-1}^{h} e^{[-\frac{h^2}{v(v-h)}]} dv; \mathbb{N}_h^{-1} = \int_0^h e^{[-\frac{h^2}{v(v-h)}]} dv$$

- χ "builds up" 1 since

$$\text{for } 0 < \|X\| < h \quad \chi(X+1,h) + \chi(1+h-X,h) = 1$$

- h is a parameter: may depend on X. Consequences:

$-i)$ $\exists\ X_{max}$ such that
$X_{max} = 1 + h(X_{max}) \equiv \mu^2 X_{max} g(X_{max}) \Longrightarrow g(X_{max}) = \frac{1}{\mu^2}$

$-ii)$ $h > 0 \Longrightarrow \mu^2 X g(X) > 1\ \forall\ X \in [1, X_{max}] \Longrightarrow$
$g(1) > g(X_{max}) \Longrightarrow \mu^2 > 1$

$-iii)$ from $f^>(Xt)$ present in Lagrange's formula one has $t < \frac{1+h(X)}{X} = \mu^2 g(X) \Longrightarrow \widetilde{T}^>(X)$

UV behaviour

- UV extension of $T(X)$

$$< T, f^{>} > = \int d^d X T(X) \{ -(k+1) \sum_{|\beta|=k+1} [\frac{X^\beta}{\beta!} \int_1^{\mu^2 g(X)} dt \frac{(1-t)^k}{t^{(k+1)}} \partial_X^\beta f^{>}(tX)] \}$$

$$= \; < \widetilde{T}^{>}, 1 > \Longrightarrow \widetilde{T}^{>}(X) \quad \text{after partial integration}$$

$$\widetilde{T}^{>}(X) = (-)^k (k+1) \sum_{|\beta|=k+1} \partial_X^\beta [\frac{X^\beta}{\beta!} T(X) \int_1^{\mu^2 g(X)} dt \frac{(1-t)^k}{t^{(k+1)}}]$$

- back to propagator at $x = y$

$-i)$ $X = \frac{p^2}{\Lambda^2}; T(X) = \frac{1}{(X\Lambda^2+m^2)} \Longrightarrow \{D = 2, d = 1, k = 0\}$

$$
\begin{aligned}
\Big[\widetilde{\frac{1}{(p^2+m^2)}}\Big]_{\mu,D=2} &= \partial_X \Big[\frac{X}{(X\Lambda^2+m^2)} \int_1^{\mu^2 g(X)} \frac{dt}{t}\Big] \\
&= \frac{m^2 \log[\mu^2 g(X)]}{(X\Lambda^2+m^2)^2} + \frac{X g'(X)}{(X\Lambda^2+m^2) g(X)}
\end{aligned}
$$

the choice $g(x) = x^{(\alpha-1)}$ *ie* $h(x) = \mu^2 x^\alpha - 1$ with $0 < \alpha < 1$
is OK with the construction of $\chi(X, h)$
in the limit $\alpha \to 1$ $\quad \frac{g'(X)}{g(X)} = 0$ and $X_{max} = (\mu^2)^{\left(\frac{1}{(1-\alpha)}\right)} \to \infty$

$$
\begin{aligned}
\Delta(0) &= \int \frac{d^2p}{(2\pi)^2} \frac{f^2(p^2)}{(p^2+m^2)} = m^2 \log(\mu^2) \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2+m^2)^2} \\
&= \frac{1}{(4\pi)} \log(\mu^2)
\end{aligned}
$$

RG invariant w.r.t. scale μ

$-ii)\ X=\frac{p^2}{\Lambda^2}; T(X)=\frac{1}{(X\Lambda^2+m^2)} \Longrightarrow \{D=4, d=2, k=1\}$

$$\left[\widetilde{\frac{1}{(p^2+m^2)}}\right]_{\mu,D=4} = \lim_{\alpha\to 1} -\partial_X^{(2)}\left[\frac{X^2}{(X\Lambda^2+m^2)}\int_1^{\mu^2 g(X)} dt\frac{(1-t)}{t^2}\right]$$

$$= \frac{2m^4}{\mu^2}\frac{[1-\mu^2+\mu^2\log(\mu^2)]}{(X\Lambda^2+m^2)^3}$$

$-iii)$ alternate form of $\widetilde{T}^{>}(X)$: variable change $Xt\to Y$

$$\boldsymbol{\widetilde{T}^{>}(X) = (-)^k(k+1)\sum_{|\beta|=k+1}\partial_X^\beta\left[\frac{X^\beta}{\beta!}\int_1^{\mu^2} dt\frac{(1-t)^k}{t^{(k+d+1)}}T(X/t)\right]}$$

$$\left[\widetilde{\frac{1}{(p^2+m^2)}}\right]^{alter}_{\mu,D=2} = \partial_X\left[X\int_1^{\mu^2}\frac{dt}{t}\frac{1}{(X\Lambda^2+m^2t)}\right]$$

$$= \frac{1}{(p^2+m^2)} - \frac{1}{(p^2+m^2\mu^2)}$$

overall results unchanged after p-integration

IR behaviour

- IR extension of $T(X)$

  $-i)$ for $\|X\| \approx 0$ $\quad f^{<}(X) = w(X) f^{>}(X)$ with
$w(X) = \chi(h - \|X\| + 1, h)$

  $-ii)$ $T(X)$ is homogeneous near $\|X\| = 0 \Longrightarrow$
$T(\frac{X}{t}) = t^{(k+d)} T(X)$

$$<T, f^{<}>=(-)^{(k+1)}(k+1)\sum_{|\beta|=k+1}\int d^d X \partial_X^\beta \left[\frac{X^\beta}{\beta!} T(X) \int_0^1 dt \frac{(1-t)^k}{t} w(\frac{X}{t})\right] f^{>}(X)$$

$-iii)$ $w(\frac{X}{t})$ effectively cuts the t-integration *ie*
$\|X\|(\mu^2 - 1) \equiv \tilde{\mu}\|X\| < t < 1$

$$<\widetilde{T}^{<}, 1>=(-)^{k+1}(k+1)\sum_{|\beta|=k+1}\int d^d X \partial_X^\beta \left[\frac{X^\beta}{\beta!} T(X) \int_{\tilde{\mu}\|X\|}^1 dt \frac{(1-t)^k}{t}\right]$$

IR behaviour:continued

- The t-integration is trivial $\Longrightarrow \widetilde{T}^{<}(X)$

$$\widetilde{T}^{<}(X) = (-)^{k}(k+1)\sum_{|\beta|=k+1} \partial_X^{\beta}\left[\frac{X^{\beta}}{\beta!}T(X)\log(\tilde{\mu}\|X\|)\right] + \frac{(-)^{k}}{k!}H_k \sum_{|\beta|=k} C^{\beta}\delta^{(\beta)}(X)$$

Here $H_k = \sum_{p=1}^{k} \frac{(-1)^{(p+1)}}{p} \begin{pmatrix} k \\ p \end{pmatrix} = \gamma + \psi(k+1)$

and $C^{\beta} = \int_{(\|X\|=1)} T(X) X^{\beta} dS$

IR behaviour : example

- massive scalar field propagator from perturbative mass expansion
- $D_F^0(x) =< \phi(x), \phi(0) >$ known from CFT $= \lim_{m\to 0} K_0(mr)$

$$D_F(x) = D_F^0(x) - m^2 \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip.x}}{p^4} (f^<(p^2))^4 + m^4 \int \frac{d^2p}{(2\pi)^2} \frac{e^{ip.x}}{p^6} (f^<(p^2))^6 + ...$$

from $\widetilde{T}^<(X)$ with $X = \frac{p^2}{\Lambda^2}$ one finds

$$\left[\widetilde{\frac{1}{(p^2)^{(k+1)}}}\right] = \frac{(-)^k}{k!} \frac{\partial^{k+1}}{\partial(p^2)^{k+1}} \left[\log(\frac{p^2}{\Lambda^2})\right] + 2\frac{(-)^k}{k!} H_k \delta^{(k)}(p^2)$$

IR example continued

Taking Fourier transform gives

$$\int \frac{d^2p}{(2\pi)^2}\frac{e^{ip.x}}{\left[\widetilde{(p^2)^{(k+1)}}\right]} = \frac{(-)^k}{2\pi(k!)^2}(\frac{\mid x\mid^2}{4})^k\left[\psi(k+1)-\log(\frac{\Lambda\mid x\mid}{2})\right]$$

for $k=0$ this is $-\frac{1}{2\pi}\left[\gamma+\log(\frac{\Lambda|x|}{2})\right] \equiv D_F^0(x) \Longrightarrow \Lambda \equiv m$
the overall expression for $D_F(x)$ is then

$$\begin{aligned} D_F(x) &= \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\left[\psi(k+1)-\log(\frac{m|x|}{2})\right]}{(k!)^2}\left[\frac{m^2\mid x\mid^2}{4}\right]^k \\ &= \frac{1}{2\pi}K_0(m\mid x\mid) \end{aligned}$$

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- Towards a finite LCQFT for the S-matrix repesented in terms of the light-front time $\sigma = \omega.x$ (counterterms avoided)

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