LIGHT-FRONT SINGULARITIES IN THE YUKAWA MODEL

Ben Bakker

Department of Physics and Astronomy
Vrije Universiteit, Amsterdam, The Netherlands

I Introduction

- Light-front singularities
- Yukawa model

II First reminder: singularities

III Second reminder: Yukawa model

IV Work in progress

- Box diagram: form factors
- Box diagram: singularities

I Introduction

Strengths and weaknesses of light-front (LF) approaches

Does LF dynamics respect Poincaré symmetry?

Convergence of Fock-space expansion (neglect small terms)

Miranda van Iersel et al., Few-Body Systems **33**, 27 - 47 (2003)

Occurrence of LF singularities (neglect infinite terms)

Cheung Ji et al., Phys. Rev. D **62**, 074014 (2000) and this meeting

This work: Yukawa model, spin-1/2 – spin-0, as testbed

Previous calculation in naive *LFD using* regularization *and* renormalization

Stan Głazek et al., Phys. Rev. D **47** 1599 (1993)

Manifestly covariant Yukawa is finite; LF diagrams divergent; divergences cancel

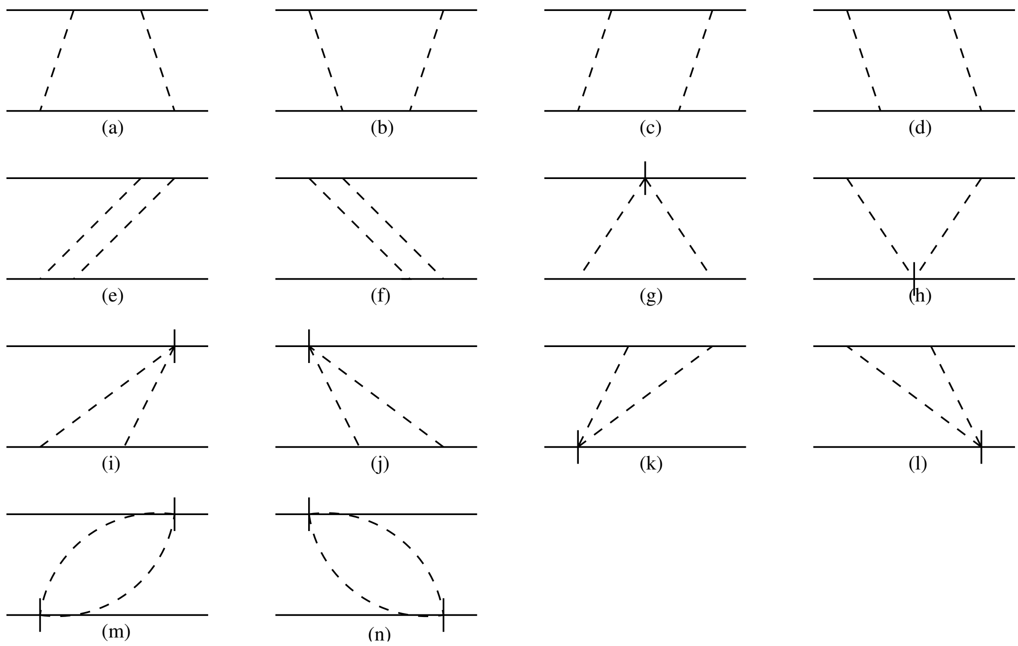
Miranda van Iersel (LC2004) Few-Body Systems **36**, 133 - 136 (2005)

A complete calculation of bound states in the Yukawa model in instant-form *dynamics exists*

Axel Weber and Norbert Ligterink, hep-ph/0506123

II First reminder: singularities

Results of Miranda van Iersel et al.



All LF time ordered boxes are divergent. The divergences are due to the p^- dependence of the fermion propagators

$$\frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon} = \frac{i(\not p_{\rm on} + m)}{p^2 - m^2 + i\varepsilon} + \frac{i(\not p - \not p_{\rm on})}{p^2 - m^2 + i\varepsilon} = \frac{i\sum |u >< u|}{p^2 - m^2 + i\varepsilon} + \frac{i\gamma^+}{2p^+}$$

$$p_{\rm on} = (p^-_{\rm on}, p^+, \vec{p}_\perp), p^-_{\rm on} = \frac{\vec{p}_\perp^{\,2} + m^2}{2p^+}$$

The sum of the divergences vanishes.

If the LF time ordered boxes with instantaneous parts, $(g) \ldots (n)$, are dropped, the singularities do not cancel.

If the stretched box,(e), (f), and $(i) \ldots (n)$, is dropped, the singularities do not cancel either.

III Second reminder: Yukawa model

Model: Spin-1/2 particles exchanging spinless bosons.

SPIN-1/2

In LFD we use the Kogut and Soper spinors

$$u_{\rm LF}(\vec{p};+) = \frac{1}{\sqrt{4\sqrt{2}\,p^+M}} \begin{pmatrix} \sqrt{2}\,p^+ + M \\ p^r \\ \sqrt{2}\,p^+ - M \\ p^r \end{pmatrix}, u_{\rm LF}(\vec{p};-) = \frac{1}{\sqrt{4\sqrt{2}\,p^+M}} \begin{pmatrix} -p^l \\ \sqrt{2}\,p^+ + M \\ p^l \\ -\sqrt{2}\,p^+ + M \end{pmatrix}.$$

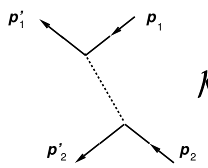
At the $f\bar{f}b$-vertex the inner product of two spinors occurs

$$\langle \vec{p}' s' | \vec{p}; s \rangle := I = \begin{pmatrix} I_a & I_b \\ -I_b^* & I_a^* \end{pmatrix},$$

The spin-flip matrix element I_b is responsible for the divergences in LFD.

$$I_a = \frac{p'^+ + p^+}{2\sqrt{p'^+ p^+}}, \quad I_b = \frac{-p'^+ p^l + p^+ p'^l}{2M\sqrt{p'^+ p^+}}$$

Using this representation one finds that the one boson exchange matrix element $\mathcal{K}$ has the following structure



$$\mathcal{K} = \frac{I(1)\otimes I(2)}{D} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \mathcal{K}_{13} & \mathcal{K}_{14} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{23} & \mathcal{K}_{24} \\ -\mathcal{K}_{24}^* & \mathcal{K}_{23}^* & \mathcal{K}_{22}^* & -\mathcal{K}_{21}^* \\ \mathcal{K}_{14}^* & -\mathcal{K}_{13}^* & -\mathcal{K}_{12}^* & \mathcal{K}_{11}^* \end{pmatrix}$$

D is the free LF energy denominator and we use the numbering

$$\{1,\ldots,4\} \leftrightarrow \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$$

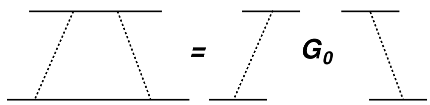
DIVERGENCES

Głazek et al. (1993) and Miranda van Iersel (Thesis, 2004)

$$\mathcal{K}_{14}, \mathcal{K}_{41}, \mathcal{K}_{23}, \mathcal{K}_{32} \to \text{ constant } \quad \text{for} \quad p'_\perp/p_\perp \quad \text{or} \quad p'_\perp/p_\perp \to \infty$$

The connection to the box is (schematically)

$$\mathcal{M}_{\alpha\beta} \propto \sum_\gamma \mathcal{K}_{\alpha\gamma} \frac{1}{D} \mathcal{K}_{\gamma\beta} \leftrightarrow$$



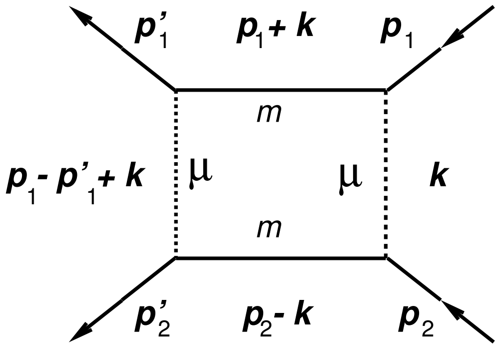
Głazek et al. subtract the divergent parts in specific spin channels. (asymptotic counter term).
This does not completely remove the singularities in the LF Yukawa model, because
  Instantaneous contributions are not included,
  Stretched box is not included (higher Fock space components).

Instantaneous contributions can be included by the $u - w$ formalism
C.-Y. Pang and C.-R. Ji, J. Comp. Phys. **115**, 267 (1994)
Miranda van Iersel LC 2004

$$\sum |u><u| = \not{p}_{\rm on} + m, \quad \sum |w><w| = (p^- - p^-_{\rm on})\gamma^+$$

The spinors $|w\rangle$ have zero norm.

Work in progress on the Yukawa box



<u>MANIFESTLY COVARIANT APPROACH: FORM FACTORS</u>

Scattering of two spin-1/2 paricles with masses M, that exchange scalar particles of mass μ. Amplitude:

$$\mathcal{T}_{fi} = \bar{u}(p_1', s_1')\bar{u}(p_2', s_2')\mathcal{M}u(p_1, s_1)u(p_2, s_2).$$

$\mathcal{M}$ is a matrix in spin space and depends in the invariants that can be built from the momenta p_1, p_2, p_1', and p_2'.
Using the standard Feynman rules:

$$\mathcal{M} = N \int \frac{d^4k}{(2\pi)^4} \frac{[\gamma(1)\cdot(p_1+k)+m]\,[\gamma(2)\cdot(p_2-k)+m]}{(k^2-\mu^2)[(p_1-p_1'+k)^2-\mu^2][(p_1+k)^2-m^2][(p_2-k)^2-m^2]}$$

Using Feynman parameters α_i, a Wick rotation and shift

$$k \to k' = k + (\alpha_3+\alpha_4)p_1 - \alpha_2 p_2 - \alpha_3 p_1',$$

the denominator becomes $[k'^2 - M^2_{\text{cov}}]^4$, with the invariant mass function given by

$$M^2_{\text{cov}} = (\alpha_2+\alpha_4)m^2 + (\alpha_1+\alpha_3)\mu^2$$

$$-\left[\alpha_1\alpha_2 p_2^2 + \alpha_1\alpha_3(p_1-p_1')^2 + \alpha_1\alpha_4 p_1^2 + \alpha_2\alpha_3 p_2'^2 + \alpha_2\alpha_4(p_1+p_2)^2 + \alpha_3\alpha_4 p_1'^2\right].$$

Owing to the symmetries of the diagram the mass function is symmetric under the transpositions $\alpha_1 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \alpha_4$.

Spin structure

$$\mathcal{M} = -\gamma(1)\cdot\gamma(2)\frac{1}{4}D_2 + m\gamma(1)\cdot D(1) + m\gamma(2)\cdot D(2) + \gamma(1)_\mu\gamma(2)_\nu D(12)^{\mu\nu} + m^2 D_0,$$

vectors

$$\begin{aligned} D(1) &= p_1(D_0 - D_{\alpha_1} - D_{\alpha_2}) + p_2 D_{\alpha_2} + p_1' D_{\alpha_1}, \\ D(2) &= p_1(D_{\alpha_1} + D_{\alpha_2}) + p_2(D_0 - D_{\alpha_2}) - p_1' D_{\alpha_1}, \end{aligned}$$

tensor

$$\begin{aligned} D(12)^{\mu\nu} &= p_1^\mu p_1^\nu \left[D_{\alpha_1} + D_{\alpha_2} - D_{(\alpha_1+\alpha_2)^2}\right] \\ &+ p_1^\mu p_2^\nu\, D_{\alpha_1} + p_1^{\{\mu} p_2^{\nu\}}\, [D_{\alpha_1\alpha_2} + D_{\alpha_2\alpha_4}] \\ &+ p_2^\mu p_2^\nu\, [D_{\alpha_2} - D_{\alpha_2^2}] \\ &+ p_1^\mu p_1'^\nu\, [-D_{\alpha_1}] + p_1^{\{\mu} p_1'^{\nu\}}\, [D_{\alpha_1^2} + D_{\alpha_1\alpha_2}] \\ &+ p_1'^\mu p_2^\nu\, D_{\alpha_1} - p_1'^{\{\mu} p_2^{\nu\}}\, D_{\alpha_1\alpha_2} - p_1'^\mu p_1'^\nu\, D_{\alpha_1^2}. \end{aligned} \tag{1}$$

Notation: $p^{\{\mu}q^{\nu\}}$ is the tensor element symmetrized over the indices. $\gamma(i)$ is associated with the internal line connecting p_i and p_i'.

$$\begin{aligned} D_0 &= \int_T d\alpha \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - M_{\rm cov}^2)^4} = \frac{i}{(4\pi)^2}\frac{1}{6}\int_T d\alpha \frac{1}{M_{\rm cov}^4}, \\ D_{\alpha_i} &= \int_T d\alpha \int \frac{d^4k}{(2\pi)^4}\frac{\alpha_i}{(k^2 - M_{\rm cov}^2)^4} = \frac{i}{(4\pi)^2}\frac{1}{6}\int_T d\alpha \frac{\alpha_i}{M_{\rm cov}^4}, \\ D_{\alpha_i\alpha_j} &= \int_T d\alpha \int \frac{d^4k}{(2\pi)^4}\frac{\alpha_i\alpha_j}{(k^2 - M_{\rm cov}^2)^4} = \frac{i}{(4\pi)^2}\frac{1}{6}\int_T d\alpha \frac{\alpha_i\alpha_j}{M_{\rm cov}^4}, \\ D_2 &= \int_T d\alpha \int \frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2 - M_{\rm cov}^2)^4} = \frac{-i}{(4\pi)^2}\frac{1}{3}\int_T d\alpha \frac{1}{M_{\rm cov}^2}. \end{aligned}$$

Using the Dirac equation to simplify the matrix elements we find the following structure

$$\mathcal{M} = O_1 F_1 + O_2 F_2 + O_3 F_3 + O_4 F_4,$$

Operators

$$\begin{aligned}
O_1 &= I(1) \otimes I(2), \\
O_2 &= [I(1) \otimes (p_1 \cdot \Gamma(2)) + (p_2 \cdot \Gamma(1)) \otimes I(2)], \\
O_3 &= (p_2 \cdot \Gamma(1)) \otimes (p_1 \cdot \Gamma(2)), \\
O_4 &= \Gamma(1) \cdot \Gamma(2)
\end{aligned}$$

Form factors

$$\begin{aligned}
F_1 &= 2(m+M)^2 D_{\alpha_1} + 2m(m+M) D_{\alpha_2} + M^2 D_{\alpha_2 \alpha_4}, \\
F_2 &= (m+M) D_{\alpha_2} - 2M D_{\alpha_2 \alpha_4}, \\
F_3 &= D_{\alpha_2^2}, \\
F_4 &= -D_2/4.
\end{aligned}$$

Conclusion: The box diagram is fully described by *four* independent invariant form factors.

The amplitude $\mathcal{M}$ is a covariant object; after taking matrix elements with respect to the spinors we obtain noncovariant objects, the actual amplitudes.

SPIN DEGREES OF FREEDOM

In the instant form we use the canonical Dirac spinors

$$u(\vec{p};s) = \sqrt{\frac{E+M}{2M}} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M}\chi_s \end{pmatrix}$$

In LFD we use the Kogut and Soper spinors

$$u_{\mathrm{LF}}(\vec{p};+) = \frac{1}{\sqrt{4\sqrt{2}\,p^+M}} \begin{pmatrix} \sqrt{2}\,p^+ + M \\ p^r \\ \sqrt{2}\,p^+ - M \\ p^r \end{pmatrix}, u_{\mathrm{LF}}(\vec{p};-) = \frac{1}{\sqrt{4\sqrt{2}\,p^+M}} \begin{pmatrix} -p^l \\ \sqrt{2}\,p^+ + M \\ p^l \\ -\sqrt{2}\,p^+ + M \end{pmatrix}.$$

In either representation:

$$\langle \vec{p}' s' | \vec{p}; s \rangle := I = \begin{pmatrix} I_a & I_b \\ -I_b^* & I_a^* \end{pmatrix},$$

$$\langle \vec{p}'; s' | \gamma^\mu | \vec{p}; s \rangle := \Gamma^\mu = \begin{pmatrix} \Gamma_a^\mu & \Gamma_b^\mu \\ -\Gamma_b^{\mu\,*} & \Gamma_a^{\mu\,*} \end{pmatrix}.$$

Using this representation one finds that the matrix $\mathcal{M}$ has the following structure

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{M}_{14} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \mathcal{M}_{24} \\ -\mathcal{M}_{24}^* & \mathcal{M}_{23}^* & \mathcal{M}_{22}^* & -\mathcal{M}_{21}^* \\ \mathcal{M}_{14}^* & -\mathcal{M}_{13}^* & -\mathcal{M}_{12}^* & \mathcal{M}_{11}^* \end{pmatrix}.$$

There are 8 independent complex matrix elements depending on 4 form factors, so there exist 4 complex angular conditions.

EXTRACTION OF FORM FACTORS AND ANGULAR CONDITIONS

Angular conditions are linear relations between matrix elements of an operator expressing the conditions of rotational symmetry. Sometimes also parity invariance and time reversal invariance are included in the definition of angular condition.

One can easily see that the spin matrix elements of the identity operator and the gamma matrices produce Hermitian matrices, if together with the spins also the momenta are involved in the Hermitian conjugation.

In a manifestly covariant calculation the *form factors* can be calculated directly, but in a LF calculation only the *matrix elements* are calculated. The form factors must be determined by solving a sufficient number of matrix elements for the invariant form factors.

Form factors can be extracted from matrix elements once the kinematics is fixed. Not all kinematics are fit for extraction of *all* the form factors.

Examples of kinematics (CMS)

Forward

$$p_1 = p_1' = (E, p\hat{e}_z), \quad p_2 = p_2' = (E, -p\hat{e}_z).$$

Backward

$$p_1 = p_2' = (E, p\hat{e}_z), \quad p_2 = p_1' = (E, -p\hat{e}_z).$$

Perpendicular xy

$$p_1 = (E, p\hat{e}_x), \quad p_2 = (E, -p\hat{e}_x), \quad p_1' = (E, p\hat{e}_y), \quad p_2' = (E, -p\hat{e}_y).$$

MATRIX ELEMENTS (IFD)

Forward

In forward kinematics the matrix $\mathcal{M}$ is proportional to the 4×4 indentity matrix

$$\mathcal{M}_{\rm f} = A_{\rm f} \, \mathbb{1}_4$$

with the coefficient given by

$$A_{\rm f} = F_1 + \left[1 + \frac{2p^2}{M^2}\right] 2MF_2 + \left[1 + \frac{2p^2}{M^2}\right]^2 M^2 F_3 + \left[1 + \frac{2p^2}{M^2}\right] F_4.$$

Backward

In backward kinematics the matrix $\mathcal{M}$ is more complicated

$$\mathcal{M}_{\rm b} = \begin{pmatrix} A_{\rm b} & 0 & 0 & 0 \\ 0 & A_{\rm b} & E_{\rm b} & 0 \\ 0 & E_{\rm b} & A_{\rm b} & 0 \\ 0 & 0 & 0 & A_{\rm b} \end{pmatrix}$$

with the entries given by

$$A_{\rm b} = \left(\frac{E}{M}\right)^2 \left(F_1 + 2MF_2 + M^2 F_3\right) + F_4, \quad E_{\rm b} = -2\frac{p^2}{M^2} F_4.$$

Perpendicular xy

In perpendicular kinematics xy the matrix $\mathcal{M}$ is still relatively simple

$$\mathcal{M}_{xy} = \begin{pmatrix} A_{xy} & 0 & 0 & C_{xy} \\ 0 & D_{xy} & E_{xy} & 0 \\ 0 & E_{xy} & D_{xy} & 0 \\ C^*_{xy} & 0 & 0 & A^*_{xy} \end{pmatrix}$$

with the entries given by

$$\begin{aligned} A_{xy} &= \frac{E}{M^2}\left[MF_1 + 2(M^2+p^2)F_2 + (M^2+2p^2)MF_3 + MF_4\right], \\ &\quad + i\frac{p^2}{2M^2}\left[F_1 - 2MF_2 - (3M^2+4p^2)MF_3 - 3F_4\right], \\ C_{xy} &= i\frac{p^2}{2M^2}F_4, \\ D_{xy} &= \frac{1}{2M^2}\left[(2M^2+p^2)F_1 + 2M(2M^2+3p^2)F_2 \right. \\ &\qquad \left. + (2M^4+5M^2p^2+4p^4)F_3 + (2M^2+3p^2)F_4\right], \\ E_{xy} &= iC_{xy} = -\frac{p^2}{2M^2}F_4. \end{aligned}$$

In other perpendicular kinematics, e.g., xz and yz, the matrix $\mathcal{M}$ attains its full complexity,

$$\mathcal{M}_{ij} = \begin{pmatrix} A_{ij} & B_{ij} & B'_{ij} & C_{ij} \\ \bar{B}_{ij} & D_{ij} & E_{ij} & -\tilde{B}^*_{ij} \\ \tilde{B}_{ij} & E_{ij} & D_{ij} & -\bar{B}^*_{ij} \\ C^*_{ij} & -B'*_{ij} & -B^*_{ij} & A^*_{ij} \end{pmatrix}$$

but we shall not need these kinematics.

EXTRACTIONS (IFD)

Forward $\rightarrow A_{\mathrm{f}}$.

Backward $\rightarrow F_1 + 2MF_2 + M^2F_3$ and F_4.

Perpendicular $xy \rightarrow F_1, \ldots, F_4$

Clearly, only in nonforward of nonbackward kinematics all form factors can be determined separately.

LIGHT-FRONT REPRESENTATION

Forward

In forward kinematics we find again that $\mathcal{M}$ is diagonal

$$\mathcal{M}_\mathrm{f} = A_\mathrm{f}\, \mathbb{1}_4$$

with the coefficient given by

$$A_\mathrm{f} = F_1 + \left[1 + \frac{2p^2}{M^2}\right] 2MF_2 + \left[1 + \frac{2p^2}{M^2}\right]^2 M^2 F_3 + \left[1 + \frac{2p^2}{M^2}\right] F_4.$$

This is the same as in the IFD calculation, because in the forward kinematics the spinors are the same in LFD and IFD. (The *Melosh transformation* reduces to the identity.)

Backward

In backward kinematics the matrix $\mathcal{M}$ is now given by

$$\mathcal{M}_\mathrm{b} = \begin{pmatrix} A_\mathrm{b} & 0 & 0 & 0 \\ 0 & A_\mathrm{b} & E_\mathrm{b} & 0 \\ 0 & E_\mathrm{b} & A_\mathrm{b} & 0 \\ 0 & 0 & 0 & A_\mathrm{b} \end{pmatrix}$$

with the entries given by

$$A_\mathrm{b} = \left(\frac{E}{M}\right)^2 \left(F_1 + 2MF_2 + M^2 F_3\right) + F_4, \quad E_\mathrm{b} = -2\frac{p^2}{M^2} F_4.$$

Also in the backward kinematics the Melosh transformations are just the identity.

Perpendicular xy

In perpendicular kinematics xy the matrix $\mathcal{M}$ is

$$\mathcal{M}_{xy} = \begin{pmatrix} A_{xy} & B_{xy} & -B_{xy} & C_{xy} \\ iB_{xy} & D_{xy} & E_{xy} & -iB^*_{xy} \\ -iB_{xy} & E_{xy} & D_{xy} & iB^*_{xy} \\ C^*_{xy} & B^*_{xy} & -B^*_{xy} & A^*_{xy} \end{pmatrix}$$

with the entries given by

$$\begin{aligned}
A_{xy} &= F_1 + \frac{2E^2}{M}F_2 + (M^2+2p^2)F_3 + F_4 \\
&\quad -i\frac{2p^2}{M}\left[F_2 + \frac{E^2}{M}F_3 + F_4\right], \\
B_{xy} &= \frac{p}{2M}\left[F_1 + \frac{2E^2}{M}F_2 + (M^2+2p^2)F_3 + F_4\right] \\
&\quad +i\frac{p}{2M}\left[F_1 + 2MF_2 + M^2F_3 + F_4\right], \\
C_{xy} &= -i\frac{p^2}{2M^2}\left[F_1 + 2MF_2 + M^2F_3\right], \\
D_{xy} &= F_1 + \frac{2E^2}{M}F_2 + \frac{(E^4+p^4)}{M^2}F_3 + \frac{E^2}{M^2}F_4, \\
E_{xy} &= iC_{xy} = \frac{p^2}{2M^2}\left[F_1 + 2MF_2 + M^2F_3\right].
\end{aligned}$$

Extraction

Again, all four form factors can be extracted from perpendicular xy kinematics.

Outlook

To do:

Calculate the amplitude in LFD and check with the covariant calculation

Check Głazek et al. asymptotic regularization in 4th order

Find general nth order regularization in LFD that preserves covariance