Lattice Investigations of Nucleon Structure at Light Quark Masses

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Tony’s 60th B’day Party, Adelaide, February 15-19, 2010
Topics to Cover

✦ Form Factors
  ✦ Provide information on size, shape and internal (charge) densities
  ✦ eg. Neutron has charge zero, but charge density? +/-?
  ✦ Good place to search for chiral non-analytic behaviour

✦ Nucleon Axial Charge, $g_A$
  ✦ Neutron beta decay, chiral symmetry breaking
  ✦ Study finite size effects

✦ Transversity (Tensor charge, $g_T$)
  ✦ Not yet measured experimentally (lattice prediction)
  ✦ $\langle x \rangle_q, \langle x \rangle_{\Delta q}$: Look for curvature at light quark masses
Challenges from Tony for the Lattice

- Magnetic moments
  - PRD60:034014 (1999)

- Radii
  - PRD79:094001 (2009)

- $g_A$
  - hep-lat/0502002

- $\langle x \rangle_q, \langle x \rangle_{\Delta q}$
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The Lattice

- Discretise space-time with lattice spacing $a$ volume $L^3 \times T$
- Quark fields reside on sites
- Gauge fields on the links

$L=Na$
The Lattice

- Discretise space-time with lattice spacing $a$ volume $L^3 \times T$
- Quark fields reside on sites
- Gauge fields on the links

- Lattice simulations for QCD give first principle results
- but need to have control of [ideally in this order]:
  - Statistical errors, $N_{\text{conf}} \sim O(1000)$
  - Volume: $L \sim 1.5 \text{ fm} \rightarrow 3 \text{ fm}$
  - Continuum limit: $a \sim 0.1 \text{ fm} \rightarrow 0.04 \text{ fm}$
  - Chiral extrapolation: $m_\pi \sim 500 \text{ MeV} \rightarrow 200 \text{ MeV}$ $m_\pi \rightarrow 140 \text{ MeV}$
- difficult, need Tflop++ machines to approach the theoretical goal

\[ L=Na \]
Lattice Techniques - QCDSF

- $O(a)$-improved Wilson (Clover) fermions
- Wilson gauge action
- $N_f = 2$ dynamical configurations
- 4 $\beta$ values $\rightarrow$ $0.07 \text{ fm} < a < 0.12 \text{ fm}$
  
  $150 \text{ MeV} < m_\pi < 1.2 \text{ GeV}$

  $1.1 \text{ fm} < L < 3.2 \text{ fm}$
Approaching The Chiral Limit

Nucleon Mass

$r_0 = 0.467 \text{ fm}$
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Nucleon Mass

$r_0 = 0.467 \text{ fm}$
Nucleon Form Factor
Electromagnetic Form Factors

\[ \langle p', s'| J^\mu(q) | p, s \rangle = \bar{u}(p', s') \left[ \gamma^\mu F_1(q^2) + i \sigma^{\mu\nu} \frac{q_\nu}{2m} F_2(q^2) \right] u(p, s) \]

Quark (charge) distribution in transverse plane

\[ q(b^2_\perp) = \int d^2 q_\perp e^{-i \vec{b}_\perp \cdot \vec{q}_\perp} F_1(q^2) \]

Distance of (active) quark to the centre of momentum in a fast moving nucleon
Electromagnetic Form Factors

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Distance of (active) quark to the centre of momentum in a fast moving nucleon

Provide information on the size and internal charge densities
Scaling of Form Factors

From dimensional counting [Brodsky & Farrar, 1973]

\[ F_1 \propto \frac{1}{Q^4} \quad \text{(dipole?)} \]

\[ F_2 \propto \frac{1}{Q^6} \quad \text{(tripole?)} \]

For \( Q^2 > \zeta_{pQCD} \)

\[ Q^2 \frac{F_2}{F_1} \propto \text{const} \]

\[ \frac{G_E}{G_M} \propto \text{const} \]
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\[ JLab \]

\[ Q \frac{F_2}{F_1} \propto \]
Scaling of Form Factors

From dimensional counting

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(dipole?)

\[ F_2 \propto \frac{1}{Q^6} \]  
(triplon?)

For \( Q^2 > \zeta_{PQCD} \)

\[ Q^2 \frac{F_2}{F_1} \propto \text{const} \]

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\[ \frac{F(0)}{(1 + Q^2/M^2)^p} \]

[Brodsky & Farrar, 1973]
Form Factor Radii & Magnetic Moments

Search for non-analytic behaviour predicted by Chiral Perturbation Theory

- **Form factor radii:**
  \[ r_i^2 = -6 \left. \frac{dF_i(q^2)}{dq^2} \right|_{q^2=0} \]

- **Magnetic moment \( \mu \) /anomalous magnetic moment \( \kappa \)
  \[ \mu = 1 + \kappa = G_m(0) \]
Form Factors: $F_1^{(p)}$
Comparison with experiment
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Comparison with experiment

charge radius

$$r_i^2 = -6 \frac{dF_i(q^2)}{dq^2} \bigg|_{q^2=0}$$
Form Factor Radii

\[ \langle r^2 \rangle \text{ [fm}^2] \]

\[ m^2 \pi \text{ [GeV}^2] \]

Symbols and Values:
- ▼: 5.25
- ▲: 5.29
- ◇: 5.4
Form Factor Radii
Form Factor Radii

![Graph showing form factor radii vs. $m_{\pi}^2$ (GeV^2)]
Chiral perturbation theory: Dramatic non-analytic predicted 
Need \( m_\pi < 300 \text{ MeV} \)

Evidence for divergence in \( r_2 \) but not \( r_1 \)

Radius measures slope at \( Q^2=0 \), but smallest \( Q^2>0.25\text{GeV}^2 \)

Twisted boundary conditions

Finite volume effects?
Anomalous Magnetic Moment

\[ \kappa_{\nu} \left[ \mu_N \right] \]

\[ m_{\pi}^2 \left[ GeV^2 \right] \]
Anomalous Magnetic Moment

- **Chiral perturbation theory:** Dramatic non-analytic predicted in the infinite volume

- **Finite volume effects should suppress magnetic moment**

hep-lat/0406001
[Young, Leinweber, Thomas]
Accessing Small $Q^2$: Partially Twisted Boundary Conditions

* On a periodic lattice with spatial volume $L^3$, momenta are discretised in units of $2\pi/L$

* Modify boundary conditions on the valence quarks

\[ \psi(x_k + L) = e^{i\theta_k} \psi(x_k), \quad (k = 1, 2, 3) \]

* allows to tune the momenta continuously

\[ \vec{p}_{FT} + \vec{\theta}/L \]

* Introduces additional finite volume effect \( \sim e^{-m_\pi L} \)

\[ q^2 = (p_f - p_i)^2 = \left\{ [E_f(\vec{p}_f) - E_i(\vec{p}_i)]^2 - \left[ (\vec{p}_{FT,f} + \vec{\theta}_f/L) - (\vec{p}_{FT,i} + \vec{\theta}_i/L) \right]^2 \right\} \]
Pion Dispersion Relation

\[
E(\vec{p}) = \sqrt{m^2 + \vec{p}^2}
\]

- \(E(\vec{p}_\Theta)\)
- \(E(\vec{p}_{FM})\)
Accessing Small $Q^2$: Partially Twisted Boundary Conditions

$Q_{CDSF}: N_f=2$ Clover

We need to extrapolate $F_2(q^2)$ to $q^2=0$

Model dependence

$m_\pi \approx 750$ MeV
Accessing Small $Q^2$: Partially Twisted Boundary Conditions

$QCDSF: N_f=2$ Clover

We need to extrapolate $F_2(q^2)$ to $q^2=0$ ☢️

Model dependence

$m_\pi \approx 750$ MeV
Neutron Form Factors

\[ F^n = -\frac{1}{3} F^u - \frac{2}{3} F^d \]

\[ G_E(q^2) = F_1(q^2) - \frac{q^2}{(2m)^2} F_2(q^2) \]

\[ m_\pi \approx 900 \text{ MeV} \]

\[ F_1 \text{ neutron negative at small } Q^2 \]

How does “hump” change with quark mass?
Axial charge, $g_A$

- Governs neutron $\beta$ decay
- Given by the forward nucleon matrix elements
  
  $$\langle p, s | \bar{u} \gamma_\mu \gamma_5 u - \bar{d} \gamma_\mu \gamma_5 d | p, s \rangle = 2g_A s_\mu$$

- $p$ - nucleon momentum
- $s$ - spin vector, $s^2 = -m_N^2$
- $g_A = \Delta u - \Delta d$

- Renormalised improved axial vector current
  
  $$A_\mu(x) = Z_A (1 + b_A a m_q) (\bar{q}(x) \gamma_\mu \gamma_5 q(x) + a c_A \partial_\mu \bar{q}(x) \gamma_5 q(x))$$

- $m = (1/\kappa - 1/\kappa_c)/(2a)$ is the bare quark mass
- derivative operator vanishes for forward matrix elements
- $b_A$ is only known perturbatively
$g_A$ vs. $m_{\pi}^2$ [GeV$^2$]
$a \approx 0.081 \text{ fm}$
\[ g_A \approx 0.081 \text{ fm} \]
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$g_A \approx 0.081 \text{ fm}$

$m_\pi L = 2.7$
$m_\pi L = 3.7$
$m_\pi L = 4.5$
$m_\pi L = 5.5$
$m_\pi L = 6.4$
$m_\pi L = 7.5$
For $m_\pi < 300$ MeV require $L > 3$ fm

$m_\pi < 200$ MeV $L > 3.5$ fm ?

Overall all trend still low.
  * Renormalisation?
  * Chiral physics?

QCDSF (preliminary)
$g_A/f_\pi \ [\text{GeV}^{-1}]$
$O_{i4}^\sigma = \bar{q}\gamma_5\sigma_{i4}q$
[Detmold, Melnitchouk, Thomas]
• Forward MEs with no momentum transfer provide moments of quark distributions (or structure functions)

\[
\langle N(p) \rangle \left[ \mathcal{O}_q^{\{\mu_1 \cdots \mu_n\}} - \text{Tr} \right] | N(p) \rangle^S = 2\nu^{(q)S} n [p^{\mu_1} \cdots p^{\mu_n} - \text{Tr}]
\]

\[
\int_0^1 dx \, x^{n-1} [q(x) + (-1)^n \bar{q}(x)] = \nu^{(q)S}
\]

• Consider u-d

• Nonperturbative renormalisation using Rome-Southampton method (RI’-MOM), then convert to \( \overline{\text{MS}} \) at 2 GeV

\[
\mathcal{O}_q^{\mu_1 \cdots \mu_n} = \bar{q} \gamma^{\mu_1} \overrightarrow{D}^{\mu_2} \cdots \overrightarrow{D}^{\mu_n} q \quad \overrightarrow{D} = \frac{1}{2}(\overrightarrow{D} - \overrightarrow{D})
\]
QCDSF (preliminary)
\( \langle x \rangle \)

\( \langle x_\nu^N \rangle \) versus \( m^2_{\pi} \) [GeV\(^2\)]

- \( L = \infty \)
- \( L = 6 \text{ fm} \)
- \( L = 4 \text{ fm} \)
- \( L = 3 \text{ fm} \)

\( a \approx 0.081 \text{ fm} \)

\( a_{\text{MS}}(u-d) \approx 0.081 \text{ fm} \)
• Results confirm predictions that “flat” behaviour will persist on small volumes

• Evidence for curvature with large volume results
$\langle x \rangle \Delta u - \Delta \Delta d$
\[ \langle x \rangle \Delta u - \Delta d \]

\[ \langle x \rangle \overset{MS}{\rightarrow} (\Delta u - \Delta d)(2 \text{GeV}) \]

\[ m_{\pi}^2 \text{ [GeV}^2\text{]} \]

Legend:
- ▼ 5.25
- ▲ 5.29
- ⌈ 5.4
\[ \langle x \rangle_{\Delta u - \Delta d} \]

The figure shows the dependence of the quantity \( \langle x \rangle_{\Delta u - \Delta d} \) on the scale parameter \( m_\pi L \). The data points are represented by different symbols for different values of \( m_\pi L \) (5.2, 5.25, 5.29, 5.4), with error bars indicating the statistical uncertainties.
\[ \langle x \rangle \Delta u - \Delta d \]

\[
\frac{\langle x \rangle^{\text{MS}}}{(\Delta u - \Delta d)} (2 \text{GeV})
\]

\[
m_{\pi}^2 [\text{GeV}^2]
\]

- \( m_{\pi} L = 2.7 \)
- \( m_{\pi} L = 3.7 \)
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- \( m_{\pi} L = 5.5 \)
- \( m_{\pi} L = 6.4 \)
- \( m_{\pi} L = 7.5 \)
FIG. 9: The lowest three moments of the helicity distribution $\Delta u - \Delta d$, extrapolated using a linear extrapolation (short-dashed lines) and the improved chiral extrapolation described in the text. In each panel, the long-dashed lines correspond to fits with no $\Delta$ and the LNA coefficient determined from $\chi^2_{PT}$, while the solid lines are fits obtained using $g_{\pi N L}/g_{\pi N N} = 2$ (upper solid curves) and $\sqrt{72}/25$ (lower solid curves). The lattice data are taken from the sources listed in Table I.
Generalised Parton Distributions
Generalised Parton Distributions

**Basic properties**

* M. Diehl (2001): 8 real functions needed for a complete description of the nucleon quark structure at twist 2

\[
\begin{align*}
H(x, \xi, t), E(x, \xi, t) & , \quad \tilde{H}(x, \xi, t), \tilde{E}(x, \xi, t) \\
H_T(x, \xi, t), E_T(x, \xi, t) & , \quad \tilde{H}_T(x, \xi, t), \tilde{E}_T(x, \xi, t)
\end{align*}
\]

- **Forward limit (t=0):** reproduces the parton distributions

\[
\begin{align*}
H(x, 0, 0) & = q(x) \\
\tilde{H}(x, 0, 0) & = \Delta q(x) \\
H_T(x, 0, 0) & = \delta q(x)
\end{align*}
\]
Generalised Parton Distributions

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\[ H_T(x, \xi, t), E_T(x, \xi, t) \quad , \quad \tilde{H}_T(x, \xi, t), \tilde{E}_T(x, \xi, t) \]

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  \[ \tilde{H}(x, 0, 0) = \Delta q(x) \]
  
  \[ H_T(x, 0, 0) = \delta q(x) \]

- **\( \int dx \)**: Form factors
  
  - **Dirac**: \( \int dx \ H(x, \xi, t) = F_1(t) \)
  
  - **Pauli**: \( \int dx \ E(x, \xi, t) = F_2(t) \)
  
  - **Axial**: \( \int dx \ \tilde{H}(x, \xi, t) = g_A(t) \)
  
  - **Pseudo-scalar**: \( \int dx \ \tilde{E}(x, \xi, t) = g_P(t) \)
  
  - **Tensor**: \( \int dx \ H_T(x, \xi, t) = g_T(t) \)
Generalised Parton Distributions

\[ H(x, \xi, t), E(x, \xi, t) \quad , \quad \tilde{H}(x, \xi, t), \tilde{E}(x, \xi, t) \]
\[ H_T(x, \xi, t), E_T(x, \xi, t) \quad , \quad \tilde{H}_T(x, \xi, t), \tilde{E}_T(x, \xi, t) \]

Construct Mellin moments \( \int \, dx \, x^{n-1} \)

Non-forward MEs of tower of local twist-2 operators

\[ \mathcal{O}_q^{\{\mu_1 \cdots \mu_n\}} = \bar{q} \gamma^{\{\mu_1 \overset{\rightarrow}{\mathcal{D}} \mu_2 \cdots \overset{\rightarrow}{\mathcal{D}} \mu_n\}} q \]

\[ \langle P' | \mathcal{O}_q^{\{\mu_1 \cdots \mu_n\}} | P \rangle \propto A_{ni}(t), B_{ni}(t), C_n(t) \]

\[ \int_{-1}^{1} dx \, x^{n-1} \, H_q(x, \xi, t) = \sum_{i=0}^{\frac{n-1}{2}} A_{qn,2i}(t)(-2\xi)^{2i} + C_{qn}(t)(-2\xi)^n \bigg|_{n \text{ even}} \]
\[ \int_{-1}^{1} dx \, x^{n-1} \, E_q(x, \xi, t) = \sum_{i=0}^{\frac{n-1}{2}} B_{qn,2i}(t)(-2\xi)^{2i} - C_{qn}(t)(-2\xi)^n \bigg|_{n \text{ even}} \]
Quark densities in the transverse plane

Quark (charge) distribution in transverse plane

\[ q(b^2) = \int d^2 \Delta \ e^{-i\vec{b} \cdot \Delta} F_1(\Delta^2) \]

Distance of (active) quark to the centre of momentum in a fast moving nucleon
Impact Parameter GPDs (M. Burkardt, 2000)

Quark densities in the transverse plane

**Quark (charge) distribution in transverse plane**

\[
q(b^2_{\perp}) = \int d^2 \Delta_{\perp} e^{-i \vec{b}_{\perp} \cdot \Delta_{\perp}} F_1(\Delta^2)
\]

[Probabilistic interpretation of \( H(x, \xi, t), \tilde{H}(x, \xi, t), H_T(x, \xi, t) \) at \( \xi = 0 \)]

\[
q(x, \vec{b}_{\perp}) = \frac{1}{(2\pi)^2} \int d^2 \Delta_{\perp} e^{-i \vec{b}_{\perp} \cdot \Delta_{\perp}} H(x, 0, \Delta^2_{\perp})
\]

Distance of (active) quark to the centre of momentum in a fast moving nucleon

Decompose into contributions from individual quarks with momentum fraction, \( x \)
Transverse Spin Structure of the Nucleon

**Transverse densities:**

\[
\rho^n(b_\perp, s_\perp, S_\perp) = \int_{-1}^{1} dx \, x^{n-1} \rho(x, b_\perp, s_\perp, S_\perp) = \frac{1}{2} \left\{ A_{n0}(b_\perp^2) + s_\perp^i S_\perp^i \left( A_{Tn0}(b_\perp^2) - \frac{1}{4m^2} \Delta b_\perp \tilde{A}_{Tn0}(b_\perp^2) \right) \right. \\
+ \left. \frac{b_\perp^j e^{ji}}{m} \left( S_\perp^i B_{n0}'(b_\perp^2) + s_\perp \overline{B}_{Tn0}'(b_\perp^2) \right) + s_\perp (2b_\perp^i b_\perp^j - b_\perp^2 \delta^{ij}) S_\perp^j \frac{1}{m^2} \tilde{A}_{Tn0}''(b_\perp^2) \right\}
\]

[Diehl & Haegler, 2005] [Burkardt, 2005]

\[
F(b_\perp^2) = \int d^2 \Delta \_ \_ e^{-i\vec{b}_\perp \cdot \Delta} F(\Delta_\perp^2) = \int d^2 \Delta \_ \_ e^{-i\vec{b}_\perp \cdot \Delta} \frac{F(0)}{(1 - \Delta_\perp^2/M^2)^p}
\]
Tensor Form Factors

$\bar{B}_{Tn0}(t)$ are remarkably large
$A^{d}_{T\pi 0}(t)$ is sizeable while $A^{u}_{T\pi 0}(t) \approx 0$
Anomalous tensor magnetic moment

\[ \kappa = \int dx E(x, \xi, 0) = B_{10}(0) = F_2(0) \]

\[ \kappa_{\text{exp}}^u \approx 1.67 \]
\[ \kappa_{\text{exp}}^d \approx -2.03 \]

\[ \kappa_T = \int dx E_T(x, \xi, 0) = B_{T10}(0) \]

\[ \kappa_{\text{latt}}^{T_u} \approx 3.13 \]
\[ \kappa_{\text{latt}}^{T_d} \approx 1.94 \] 

Both positive
Deformed Spin Densities

Ph. Hägler (QCDSF) [PRL 98, 222001 (2007)]

\[ n=1 \]
Deformed Spin Densities

Nucleon

Ph. Hägler (QCDSF) [PRL 98, 222001 (2007)]

\( n=1 \)
Conclusion & Outlook

- Calculations of hadronic quantities becoming available close to the physical masses (beware finite size effects)
- Lattice provides a useful tool for investigating FFs/GPDs
- $q^2$ scaling of $F_1, F_1, F_2/F_1, G_E, G_M \ (F_1^n \ negative)$
  - Twisted b.c.s give access to small $q^2$
- Moments of Generalised Parton Distributions
  - Quark contribution to nucleon spin and angular momentum
  - Non-trivial transverse spin densities in pion and nucleon
- $g_A$ still a challenge.
  - Finite volume effects go in the right direction, but are they enough?
  - Renormalisation? Discretisation?
- $\langle x \rangle$ finite volume analysis at light quark masses indicate results might be going in the right direction indicating “bending down”
Currently improving the $40^3 \times 64$ results at the physical point

Simulations with the same parameters but $64^3 \times 96$ volume are starting

So far have only used $N_f=2$, now starting simulations with $N_f=2+1$ flavours of $O(a)$-improved Clover (Wilson) fermions
Happy Birthday Tony!