Wave function of the Proton

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Introduction

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- The standard creation and annihilation operators for the proton are given by

\[
\bar{\chi}(\vec{x}) = \epsilon^{abc} \bar{u}_a(\vec{x})(\bar{d}_b(\vec{x}) C\gamma^5 \bar{u}_c^T(\vec{x})),
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\chi(\vec{x}) = \epsilon^{abc} (u_a^T(\vec{x}) C\gamma^5 d_b(\vec{x})) u_c(\vec{x}).
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\begin{align*}
\bar{\chi}(\vec{x}) &= \epsilon^{abc} \bar{u}_a(\vec{x})(\bar{d}_b(\vec{x}) C \gamma_5 \tilde{u}^T_c(\vec{x})), \\
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- \( C = \gamma_2 \gamma_4 \) is the charge conjugation matrix.
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In order to create the wave function of the proton we fix the position of the \( u \) quarks and measure the \( d \) quark probability distribution.
\textbf{$u$ Quark Separation}

- Relative to a central spatial point $\vec{x}$, the vectors $\vec{d}_1$ and $\vec{d}_2$ describe the position of the two $u$ quarks respectively, and $\vec{y}$ describes the position of the $d$ quark. So we modify our annihilation operator to be

$$\chi(\vec{x}, \vec{d}_1, \vec{d}_2, \vec{y}) = \epsilon^{abc} (u^T_a (\vec{x} + \vec{d}_1) C \gamma_5 d_b (\vec{x} + \vec{y})) u_c (\vec{x} + \vec{d}_2).$$
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- The $u$ quarks are separated along the $x$–axis,

$$\vec{d}_1 = (d_1, 0, 0), \quad \vec{d}_2 = (d_2, 0, 0).$$
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$$\vec{d}_1 = (d_1, 0, 0), \quad \vec{d}_2 = (d_2, 0, 0).$$

- For even separations, $d_1 = -d_2$, and for odd separations, $d_1 + 1 = -d_2$. 
Gauge fixing

The wave function of the proton on the lattice is then defined to be proportional to the two-point correlation function at zero momentum in position space,

\[ G_{2\gamma\rho}(\vec{x}, \vec{d}_1, \vec{d}_2, \vec{y}, t) = \langle \Omega | T\{\chi(\vec{x}, \vec{d}_1, \vec{d}_2, \vec{y}, t)\bar{\chi}(0)\}|\Omega\rangle. \]
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- The two point function we have constructed is no longer gauge invariant.

- The simplest solution is to gauge fix. We study both the Landau and Coulomb gauges.
Landau Gauge

- Landau gauge, surface plot, D=0
Landau Gauge

- Landau gauge, surface plot, $D=7$
$u$ Quark Symmetrisation

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Solution: we symmetrise explicitly over $d_1$ and $d_2$,

$$\chi(\vec{x}, \vec{d}_1, \vec{d}_2, \vec{y}) =$$

$$\epsilon^{abc} \left[ (u^T_a (\vec{x} + \vec{d}_1) C\gamma_5 d_b (\vec{x} + \vec{y})) u_c (\vec{x} + \vec{d}_2) ight. + (u^T_a (\vec{x} + \vec{d}_2) C\gamma_5 d_b (\vec{x} + \vec{y})) u_c (\vec{x} + \vec{d}_1) \right].$$
Landau gauge wave function

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- “Peanut” shape.
- Diquark clustering clearly present.
Coulomb gauge wave function

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Coulomb gauge wave function

- See animation.
- “Ellipsoidal” shape.
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- Different gauge $\implies$ different shape.
Background $\vec{B}$ Field

- We also study the response of the proton to the presence of a constant background magnetic field.
Background $\vec{B}$ Field

- We also study the response of the proton to the presence of a constant background magnetic field.
- A constant background field in the $z$ direction (with continuum field strength $\omega$) is implemented as a $U(1)$ field, with

\[
U_1(x, y) = \exp(-i\omega y),
\]
\[
U_2(x, y) = 1,
\]

except for the boundary,

\[
U_2(x, n_y) = \exp(+i\omega n_y x).
\]
Background $\vec{B}$ Field

- The plaquettes in the $x$–$y$ plane are then constant,

$$P_{12}(x, y) = U_1(x, y)U_2(x + 1, y)U_1^\dagger(x, y + 1)U_2^\dagger(x, y)$$

$$= e^{-i\omega y}.1.e^{i\omega y+1}.1 = \exp(+i\omega).$$
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- The corner of the lattice requires special treatment,

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\[ = e^{-i\omega n_y}.e^{+i\omega n_y}.e^{+i\omega}.e^{-i\omega n_x n_y} \]

- A quantisation condition on the field strength is induced

\[ \exp(+i\omega n_x n_y) = 1 \implies \omega = \frac{2\pi k}{n_x n_y}. \]
Background Field

- Background field, $x$–$y$ plane.
There is clearly an $x$–$y$ asymmetry present.
\[ \vec{B} \text{ Asymmetry} \]

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  1. Our field description is manifestly asymmetric.
  2. The quark propagator contains the (lattice version of the) inverse of \( \partial_\mu + A_\mu \).
  3. We are looking at a gauge dependent quantity.
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Interpolating Field
\( u \) Symmetrisation
Landau Gauge
Coulomb Gauge

\( \vec{B} \) Field
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Summary

\( \vec{B} \) Asymmetry

- In the continuum it is possible to construct a manifestly (anti-)symmetric background gauge field,

\[
F_{12} = \partial_1 A_2 - \partial_2 A_1
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A_2 = \omega \frac{x}{2}, \quad A_1 = -\omega \frac{y}{2}
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- Thats bad...
Gauge Transformation

If we apply the gauge transformation

\[ G(x, y) = \exp(-i\omega xy), \]

using

\[ U'_\mu(x, y) = G(x, y)U_\mu(x, y)G^\dagger(x + \delta_{\mu 1}, y + \delta_{\mu 2}) \]

we obtain

\[ U'_1(x, y) = 1 \]

except on the boundary where

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- \( G(x, y) \) “rotates” the gauge potential 90°.
Gauge Transformation

- If we apply the gauge transformation
  \[ G(x, y) = \exp(-i\omega xn_y), \]
  we obtain
  \[ U_1(x, y) = \exp(-i\omega(n_y - y)) \]
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- \( G(x, y) \) “reverses” the gauge potential \( y \rightarrow (n_y - y) \).
Gauge Transformation

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this reverses the “rotated” potential, \( x \to (n_y - x) \).
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- We then average the wave function over all four gauge potentials to obtain symmetrised results (in the \( x-y \) plane).
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  this reverses the “rotated” potential, \( x \rightarrow (n_y - x) \).
- We then average the wave function over all four gauge potentials to obtain symmetrised results (in the \( x-y \) plane).
- All four gauge potentials produce the same field strength tensor.
Landau gauge, $B$ Field

▶ See animation.
Landau gauge, $B$ Field

- See animation.
- “Peanut” shape.
Landau gauge, $B$ Field

- See animation.
- “Peanut” shape.
- Diquark clustering still present.
Landau gauge, $B$ Field

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- “Peanut” shape.
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- Background field only induces a slight deformation.
Background Field

\( B = 0, \ x-z \ plane, \ d = 0. \)
Background Field

$B = 1$, $x$–$z$ plane, $d = 0$. 
Background Field

- $B = 0$, $x$–$z$ plane, $d = 5$. 

![Graph showing the $B = 0$ field in the $x$–$z$ plane with a value of $d = 5$.]
Background Field

- $B = 1$, $x$–$z$ plane, $d = 5$. 
Coulomb gauge, $B$ Field

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- Multiple symmetrisation techniques employed.
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- Diquark clustering present in Landau gauge.
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- Strong background field ($\mu B \approx 260$ MeV) induces a relatively small deformation,
- Multiple symmetrisation techniques employed.
- Happy Birthday Tony!