Singular values of the Dirac operator at nonzero density

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Outline

1. Introduction
2. Eigenvalues and singular values of the Dirac operator
   - Index theorem for non-Hermitian Dirac operator
3. Phases of two-color QCD
4. Banks-Casher-type relation for diquark condensate
5. Low-energy effective theories with diquark sources at nonzero density
6. Smilga-Stern-type relations
7. Finite-volume analysis
   - ε-regime
   - Leutwyler-Smilga-type sum rules
   - Random matrix theories
8. Conclusions and outlook

- some QCD-like theories don't have a sign problem at nonzero density (with suitable choice of parameters, e.g., even N_f with degenerate m_f)

  examples:
  - QCD with gauge group $\mathrm{SU}(2)$ or $\mathrm{Sp}(2N_c)$ ($\beta = 1$)
  - three-color QCD at nonzero isospin density ($\beta = 2$)
  - QCD with gauge group $\mathrm{SO}(N_c)$ or QCD with adjoint fermions ($\beta = 4$)

  β = Dyson index (determined by anti-unitary symmetries, or equivalently by (pseudo-) reality of fermion representation)

- here I will concentrate on two-color QCD ($\beta = 1$)
  - Dirac eigenvalue spectrum studied in great detail in the past (related to $\langle\bar{\psi}\psi\rangle$ at low density and BCS gap Δ at high density)
  - today's topic: Dirac singular values (related to $\langle\psi\psi\rangle$ at all densities)

- see arXiv:1110.5858 for details and the isospin and adjoint cases

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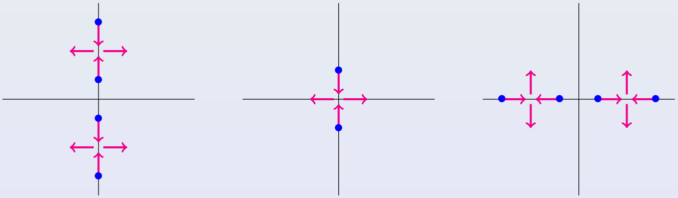
- Dirac eigenvalues:

$$D(\mu)\psi_n = \lambda_n \psi_n$$

$\mu = 0$: λ_n purely imaginary
because of $\{D, \gamma_5\} = 0$, nonzero eigenvalues come in pairs $\pm\lambda_n$

$\mu \neq 0$: λ_n generically complex
because of $[D, \gamma_5 C \tau_2 K] = 0$, nonzero eigenvalues come in quadruplets $\pm\lambda, \pm\lambda^*$ or purely real/purely imaginary pairs $\pm\lambda$

- eigenvalue flow as a function of μ (specific to $\beta = 1$):



(i) (ii) (iii)

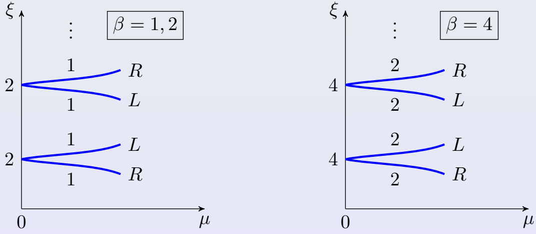
- defined by

$$D^\dagger D \varphi_n = \xi_n^2 \varphi_n$$

  - name comes from singular value decomposition of a non-Hermitian matrix
  - the ξ_n are real and nonnegative
  - $D^\dagger D$ and $DD^\dagger$ share all nonzero singular values
  - the states φ_n have definite chirality (eigenstates of γ_5)

- at $\mu = 0$ we trivially have $\xi_n = |\lambda_n|$, but for $\mu \neq 0$ the eigenvalues and singular values are unrelated (and live on different physical scales)

- singular value flow as a function of μ:



- D and $D^{\dagger}D$ have the same zero modes
- topological zero modes of $D(\mu)$ remain zero modes and change smoothly as a function of μ

- Dirac operator has the structure

$$D = \begin{pmatrix} 0 & D_L \\ D_R & 0 \end{pmatrix}$$

- index is defined as

$$\operatorname{ind} D = \dim\ker D_R - \dim\ker D_L$$

- non-Hermitian version of the index theorem (new result):

$$\frac{1}{32\pi^2}\int d^4x\, F\tilde{F} = \frac{1}{2}\left[\operatorname{ind} D(\mu) + \operatorname{ind} D(\mu)^\dagger\right]$$

  in the proof it is important to work with the eigenstates of $D^\dagger D$, since the eigenbasis of D (non-Hermitian!) can become incomplete

- if there are no accidental zero modes we have $\operatorname{ind} D(\mu) = \operatorname{ind} D(\mu)^\dagger$
  $\longrightarrow$ standard index theorem

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Two-color QCD at nonzero density

- Dirac operator (in Euclidean space) with τ_a = generators of SU(2):

$$D(\mu) = \underset{\substack{\uparrow \\ \text{anti-Hermitian}}}{\gamma_\nu D_\nu} + \underset{\substack{\uparrow \\ \text{Hermitian}}}{\mu\gamma_4} \quad \text{with} \quad D_\nu = \partial_\nu + iA_\nu^a \frac{\tau_a}{2}$$

- anti-unitary symmetry (C = charge conj., K = complex conj.):

Leutwyler-Smilga 1992

$$[C\tau_2 K, iD(\mu)] = 0 \quad \text{with} \quad (C\tau_2 K)^2 = \mathbb{1}$$

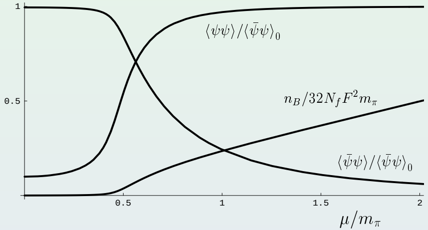
$\rightarrow \beta = 1$, $D(\mu)$ is real in a suitable basis, $\det D(\mu)$ is real
$\rightarrow$ no sign problem for even N_f (we don't consider odd N_f here)

- microscopic Lagrangian with mass term and diquark sources:

$$\mathscr{L}_f = \bar\psi[D(\mu) + MP_L + M^\dagger P_R]\psi + \left[\tfrac{1}{2}\psi^T C\tau_2(J_R P_R + J_L P_L)\psi + \text{h.c.}\right]$$

$P_{R/L} = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$, $J_{R/L}$ are complex anti-symmetric $N_f \times N_f$ matrices

- as a function of μ, m, and j, chiral symmetry is broken by $\langle\bar\psi\psi\rangle$ or $\langle\psi\psi\rangle$



Kogut-Stephanov-Toublan-Verbaarschot-Zhitnitsky 2000

- $\psi\psi$ is shorthand for $\psi^T C\gamma_5\tau_2 I\psi$

  this is the scalar, color- and flavor antisymmetric diquark condensate

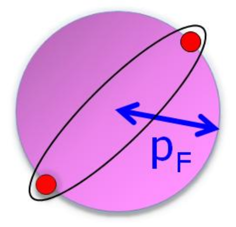
  - instanton-induced interaction and QCD inequalities favor scalar over pseudo-scalar condensate   Alford et al. 1998, Rapp et al. 1998, Kogut et al. 1999
  - single-gluon exchange favors this over the color-symmetric condensate

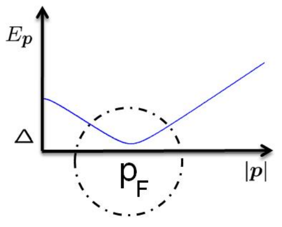
- perturbative calculations at large μ: Son 1999, T. Schäfer 2000

$$0 \approx \langle \bar{\psi}\psi \rangle \ll \langle \psi\psi \rangle$$

$\longrightarrow$ chiral symmetry and $\mathrm{U}(1)_B$ broken by the diquark condensate

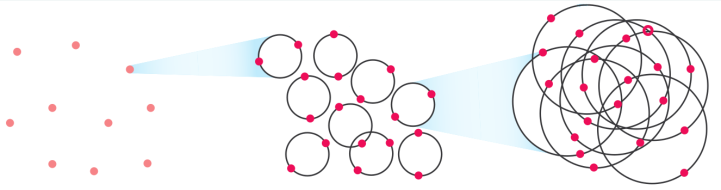
- for $\mu \gg \Lambda_{\mathrm{SU}(2)}$ we have BCS-type diquark pairing (since there is an attractive channel between quarks near the Fermi surface)
- diquarks are loosely bound in real space





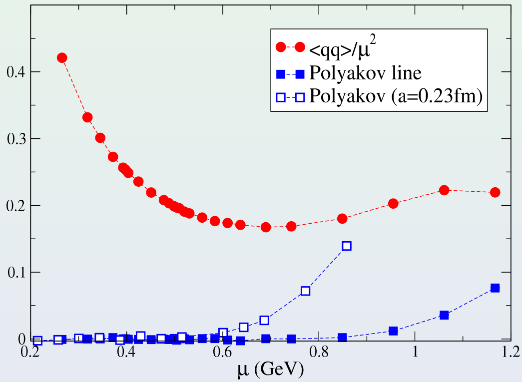
- BEC of tightly bound diquarks
- same quantum numbers as BCS superfluid at high density
  $\longrightarrow$ conjecture: BEC-BCS crossover

BEC $\longrightarrow$ BCS



(figure from Tin-Lun Ho, Science 305 (2004) 1114)

- diquark condensate in two-color QCD



Hands-Kim-Skullerud 2010

- we will construct alternative methods to obtain $\langle\psi\psi\rangle$ from the lattice

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- from now on: chiral limit, even N_f (odd N_f is a mystery), and $J_R = -J_L = jI$ with real j (to have positive definite fermionic measure)

$$Z(j) = \langle \det{}^{N_f/2}(D^\dagger D + j^2) \rangle_{\text{YM}} = \left\langle \prod_n (\xi_n^2 + j^2)^{N_f/2} \right\rangle_{\text{YM}}$$

- define density of singular values:

$$\rho_{\text{sv}}(\xi) = \lim_{V_4\to\infty} \frac{1}{V_4} \left\langle \sum_n \delta(\xi - \xi_n) \right\rangle_{j=0} \quad \text{for} \quad \xi > 0$$

- the scalar diquark condensate then follows by a standard calculation:

$$\begin{aligned}
\langle \psi^T C\gamma_5\tau_2 I\psi \rangle &= \lim_{j\to 0^+} \lim_{V_4\to\infty} \frac{1}{V_4} \frac{\partial}{\partial j} \ln Z(j) \\
&= \lim_{j\to 0^+} \lim_{V_4\to\infty} \frac{1}{V_4} \frac{N_f}{2} \left\langle \sum_n \frac{2j}{\xi_n^2 + j^2} \right\rangle_j \\
&= \frac{N_f}{2} \int_0^\infty d\xi\, \rho_{\text{sv}}(\xi) \lim_{j\to 0^+} \frac{2j}{\xi^2 + j^2} = \frac{N_f}{2} \pi \rho_{\text{sv}}(0)
\end{aligned}$$

$\longrightarrow$ diquark condensate can be obtained on the lattice from $\rho_{\text{sv}}(0)$

Comments

- result was already obtained by Fukushima 2008 (for $N_f = 2$)
- result holds at $\mu = 0$ and $\mu \neq 0$ (BC for QCD only at $\mu = 0$)
- integral over ξ needs UV regularization
  (UV-divergent part disappears in the limit $j \to 0^+$)
- contributions of zero modes were dropped (justified only if measure is positive definite)
- to derive the BC relation, the fermionic measure must be positive definite

  if it is not, we get $-\infty$ from the integral and $+\infty$ from zero modes (see Leutwyler-Smilga 1992)
  - sum is finite and gives the condensate
  - but $\rho_{\text{sv}}(0)$ is undefined $\to$ no BC relation

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three different density regimes: differ in their patterns of chiral symmetry breaking and therefore in the number of Nambu-Goldstone modes

- low density (theory **L**): start from symmetry breaking pattern at zero density

$$\mathrm{SU}(2N_f) \to \mathrm{Sp}(2N_f)$$

  then treat μ as a small perturbation (as in Kogut et al. 2000)
  some of the NG bosons acquire a mass as μ increases

- intermediate density (theory **I**): symmetry breaking pattern is

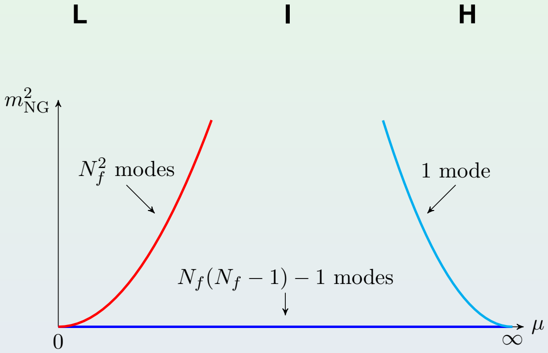
$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \to \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$$

  all NG modes massless for $j = 0$

- high density (theory **H**): instantons are screened
  $\to \mathrm{U}(1)_A$ not broken by anomaly but spontaneously by diquark condensate
  $\to$ one additional NG boson (η' becomes massless T. Schäfer 2003)
  symmetry breaking pattern is now

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \to \mathrm{Sp}(N_f)_L \times \mathrm{Sp}(N_f)_R$$

μ-dependence of the masses of the NG modes



- theory **I** is obtained from theory **L** or theory **H** by integrating out the massive mode(s)
- matching of the LECs similar to matching between SU(2) and SU(3) chPT
- domains of validity: see arXiv:1110.5858
  partial overlap of **L**/**I** (at low density) and of **I**/**H**(at high density)

- theory **L**:

$$\mathscr{L}_{\text{eff}}^{\mathbf{L}} = \frac{F^2}{2}\operatorname{tr}(\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger) - \Phi_{\mathbf{L}} \operatorname{Re}\operatorname{tr}(\bar{J}\Sigma)$$

with

$$\begin{aligned}
\nabla_\nu \Sigma &= \partial_\nu \Sigma - \mu\delta_{\nu 0}(B\Sigma + \Sigma B) \\
\nabla_\nu \Sigma^\dagger &= \partial_\nu \Sigma^\dagger + \mu\delta_{\nu 0}(\Sigma^\dagger B + B\Sigma^\dagger) \\
\Sigma &= U\Sigma_d U^T\,, \qquad \Sigma_d = \operatorname{diag}(I, -I)\,, \qquad I = \begin{pmatrix} 0 & -\mathbb{1}_{N_f/2} \\ \mathbb{1}_{N_f/2} & 0 \end{pmatrix} \\
B &= \operatorname{diag}(1_{N_f}, -\mathbb{1}_{N_f})\,, \quad \bar{J} = \operatorname{diag}(J_L, -J_R^\dagger)
\end{aligned}$$

  - Σ parametrizes the coset space $\mathrm{SU}(2N_f)/\mathrm{Sp}(2N_f)$
  - diquark condensate is assumed to form in the scalar channel
  - F and $\Phi_{\mathbf{L}} \mathrel{\widehat{=}} |\langle \psi^T C\gamma_5\tau_2 I\psi\rangle|/2N_f$ are low-energy constants (indep. of μ)

- two types of NG modes ($j \to 0$):

type 1: mass $= \sqrt{j\Phi_{\mathbf{L}}/F^2}$ $(N_f^2 - N_f - 1$ modes$)$

type 2: mass $= \sqrt{j\Phi_{\mathbf{L}}/F^2 + (2\mu)^2}$ $(N_f^2$ modes$)$

- theory **H**:

$$\mathcal{L}_{\text{eff}}^{\mathbf{H}} = \left\{ \frac{N_f \tilde{f}_0^2}{2} \left(|\partial_0 L|^2 + v_0^2 |\partial_i L|^2 \right) + \frac{\tilde{f}^2}{2} \operatorname{tr} \left(|\partial_0 \Sigma_L|^2 + v^2 |\partial_i \Sigma_L|^2 \right) + (L \longleftrightarrow R) \right\}$$

$$- \Phi_{\mathbf{H}} \operatorname{Re} \operatorname{tr}(J_L L \Sigma_L - J_R R \Sigma_R) - \frac{2\tilde{f}_0^2}{N_f} m_{\text{inst}}^2 \operatorname{Re}(L^\dagger R)^{N_f/2}$$

  - $\Sigma_{L/R}$ parametrize $\mathrm{SU}(N_f)_i/\mathrm{Sp}(N_f)_i$ $(i = L, R)$
  - L/R parametrize $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$
  - m_{inst} = single-instanton contribution to η'-mass (n-instanton vertices negligible) $\operatorname{Re}(L^\dagger R)^{N_f/2}$ is symmetric under anomaly-free subgroup $\mathbb{Z}_{2N_f} \subset \mathrm{U}(1)_A$
  - all LECs now depend on μ

- two types of NG modes ($j \to 0$):

type 1:	$m = \sqrt{j\Phi_{\mathbf{H}}/\tilde{f}^2}$	$(N_f^2 - N_f - 2$ modes$)$
	$m = \sqrt{j\Phi_{\mathbf{H}}/\tilde{f}_0^2}$	$(1$ mode $\widehat{=}\ \mathrm{U}(1)_B)$
type 2:	$m_{\eta'} = \sqrt{j\Phi_{\mathbf{H}}/\tilde{f}_0^2 + \underset{\substack{\uparrow \\ \to 0 \text{ for } \mu \to \infty}}{m_{\text{inst}}^2}}$	$(1$ mode $\widehat{=}\ \mathrm{U}(1)_A)$

- theory **I** is obtained from **L** or **H** by integrating out the type-2 NG modes:

$$\mathscr{L}_{\text{eff}}^{\mathbf{I}} = N_f f_0^2 \Big\{ |\partial_0 V|^2 + v_0^2 |\partial_i V|^2 \Big\} + \frac{f^2}{2} \operatorname{tr} \Big\{ |\partial_0 \Sigma_L|^2 + v^2 |\partial_i \Sigma_L|^2 + (L \longleftrightarrow R) \Big\}$$
$$- \Phi_{\mathbf{I}} \operatorname{Re} \big\{ \operatorname{tr}(J_L \Sigma_L - J_R \Sigma_R) V \big\}$$

  - L and R replaced by $V \ \widehat{=}\ \mathrm{U}(1)_B$
  - all LECs depend on μ

- now there are only type-1 NG modes (massless in the $j \to 0$ limit)

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- Smilga-Stern (1993) for massless three-color QCD at $\mu = 0$:

$$\rho(\lambda) = \frac{\Sigma}{\pi} + \frac{\Sigma^2}{32\pi^2 F^4} \frac{N_f^2 - 4}{N_f} |\lambda| + o(\lambda)$$

  - $\rho(\lambda)$ = spectral density (of Dirac eigenvalues), Σ = chiral condensate
  - first term: Banks-Casher relation
  - second term: slope of spectral density (Smilga-Stern relation)

- sketch of the calculation:
  - define a suitable susceptibility as a function of quark mass m
  - compute it in the microscopic theory (expressed in terms of ρ) and in the effective theory (to one loop)
  - both results contain logarithmic divergence for $m \to 0$
  - match the divergences

- role of quark mass now played by diquark source, parametrized as

$$J = jI + I\sum_a j_a t^a$$

with t^a the generators of $\mathrm{SU}(N_f)/\mathrm{Sp}(N_f)$

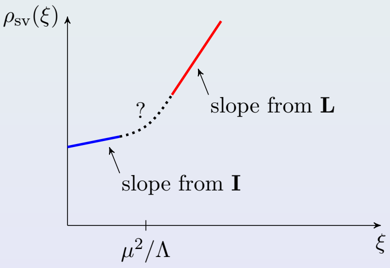
- going through similar steps as in the Smilga-Stern method we obtain

$$\rho'_{\mathrm{sv}}(0) = \begin{cases} \dfrac{(N_f-2)(N_f+1)}{N_f F^4}\dfrac{\Phi_{\mathbf{L}}^2}{16\pi^2} & \text{for } \mathbf{L} \text{ at } \mu = 0 \\ \left[\dfrac{(N_f-4)(N_f+2)}{2N_f f^4} + \dfrac{1}{N_f f_0^2 f^2}\right]\dfrac{\Phi_{\mathbf{I}}^2}{16\pi^2} & \text{for } \mathbf{I} \\ \left[\dfrac{(N_f-4)(N_f+2)}{2N_f f^4} + \dfrac{2}{N_f f_0^2 f^2}\right]\dfrac{\Phi_{\mathbf{H}}^2}{16\pi^2} & \text{for } \mathbf{H} \end{cases}$$

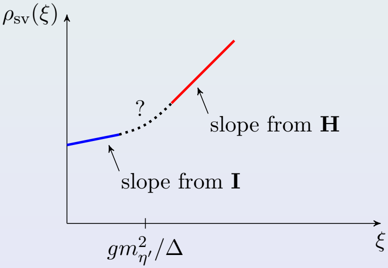
- note: method fails for $N_f = 2$, but results are expected to remain valid (slope can also be computed in partially quenched perturbation theory: more powerful method, but calculation much more complicated)

- puzzle: there are three different results, and it does not seem possible to interpolate them smoothly as a function of μ
- resolution: it depends on where you measure the slope
  can be understood by analogy to SU(2) $\longrightarrow$ SU(3) chPT (Zyablyuk 1999)
  or by employing partially quenched perturbation theory

low density



high density



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- in a finite box with $V_4 = L^4$, the ε-regime is defined by

$$\frac{1}{m_\ell} \ll L \ll \frac{1}{m_{\text{NG}}}$$

$\longrightarrow$ partition function dominated by zero-momentum modes of NG bosons

- m_ℓ = mass of lightest non-NG particle
  - theory **L**: $m_\ell(\mathbf{L}) \sim \Lambda$ = mass of lightest non-NG particle at zero density
  - theory **H**: $m_\ell(\mathbf{H}) \sim \Delta$ (since Δ plays the role of a constituent quark mass)
  - theory **I** at low density: $m_\ell(\mathbf{I}) \sim \mu$
  - theory **I** at high density: $m_\ell(\mathbf{I}) \sim \min\{m_{\eta'}(\mu), \Delta(\mu)\}$

- the ε-regimes of the three effective theories don't overlap

- idea:
  - expand the partition functions of the microscopic theory and of the static limit of the effective theory in powers of the quark mass
  - introduce a θ-angle and project onto sectors of fixed topology using

$$Z_\nu = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\nu\theta} Z(\theta)$$

  - matching the coefficients of the quark mass yields sum rules for the inverse Dirac eigenvalues in sectors of fixed topology
- here:
  - role of quark mass played by diquark sources
  - matching of the coefficients yields sum rules for the inverse singular values

Leutwyler-Smilga-type sum rules for theory **I**

- static limit of effective partition function for theory **I**:

$$Z^{\mathbf{I}}_{\text{eff}}(J_L,J_R)=\int_{\mathrm{SU}(N_f)/\mathrm{Sp}(N_f)} d\Sigma_L\, d\Sigma_R \int_{\mathrm{U}(1)} dV\, \exp\left\{V_4\Phi_{\mathbf{I}}\,\mathrm{Re}\,\mathrm{tr}(J_L\Sigma_L - J_R\Sigma_R)V\right\}$$

  introduce θ-dependence by $J_L \to J_L e^{-i\theta/N_f}$ and $J_R \to J_R e^{i\theta/N_f}$

- after some algebra (using results of Dalmazi-Verbaarschot 2001) we obtain, e.g.,

$$\left\langle \sum_n{}' \frac{1}{\xi_{Ln}^2} \right\rangle_\nu = \left\langle \sum_n{}' \frac{1}{\xi_{Rn}^2} \right\rangle_\nu = 2(V_4\Phi_{\mathbf{I}})^2 A_\alpha$$

$$\left\langle \left(\sum_n{}' \frac{1}{\xi_{Ln}^2}\right)^2 \right\rangle_\nu = \left\langle \left(\sum_n{}' \frac{1}{\xi_{Rn}^2}\right)^2 \right\rangle_\nu = 8(V_4\Phi_{\mathbf{I}})^4 B_\alpha$$

$$\left\langle \sum_n{}' \frac{1}{\xi_{Ln}^4} \right\rangle_\nu = \left\langle \sum_n{}' \frac{1}{\xi_{Rn}^4} \right\rangle_\nu = 4(V_4\Phi_{\mathbf{I}})^4 C_\alpha$$

$$\left\langle \left(\sum_m{}' \frac{1}{\xi_{Rm}^2}\right)\left(\sum_n{}' \frac{1}{\xi_{Ln}^2}\right) \right\rangle_\nu = 4(V_4\Phi_{\mathbf{I}})^4 A_\alpha^2$$

  with $A_\alpha = \frac{1}{2(\alpha+2)}$, $B_\alpha = \frac{\alpha+1}{8\alpha(\alpha+2)(\alpha+3)}$, $C_\alpha = \frac{1}{4\alpha(\alpha+2)(\alpha+3)}$, $\alpha = N_f + |\nu| - 3$

- after similar manipulations we obtain

$$Z_\nu^{\text{eff}}(J_L,J_R) \text{ of theory } \mathbf{H} = \left[Z_\nu^{\text{eff}}(J_L,J_R) \text{ of theory } \mathbf{I}\right]_{\Phi_{\mathbf{I}}\to\Phi_{\mathbf{H}}} \times \frac{I_\nu(\kappa)}{I_0(\kappa)}$$

with $\kappa = 2V_4 \tilde{f}_0^2 m_{\text{inst}}^2/N_f$

  - LS-type sum rules same as for theory **I**
  - relative factor goes to 1 for $\kappa \to \infty$
    for finite κ nontrivial topologies are suppressed (and eliminated for $m_{\text{inst}} = 0$)
  - in summations over ν, relative factor needs to be taken into account

- calculation for theory **L** involves integration over $\text{SU}(2N_f)/\text{Sp}(2N_f)$ and a more complicated integrand
  $\to$ only partial results for $N_f = 2$, where $\text{SU}(4)/\text{Sp}(4) \simeq \text{SO}(6)/\text{SO}(5) \simeq S^5$
  (Brauner 2006)

$$\left\langle {\sum_n}' \frac{1}{\xi_n^2} \right\rangle_{\nu=0} = 2(V_4\Phi_{\mathbf{L}})^2 \left(\frac{z}{e^z - z - 1} + 1 - \frac{2}{z}\right) \qquad \text{with} \quad z = 8\mu^2 F^2 V_4$$

- as usual, the ε-regimes can be described by RMT
- RMT for theory **I** ($A_{L/R}$ are real $N \times (N+\nu)$ matrices):

$$Z_\nu^{\mathrm{RMT}}(\hat{J}_L, \hat{J}_R, \hat{M}) = \int dA_L\, dA_R\, e^{-N\,\mathrm{tr}(A_L^T A_L + A_R^T A_R)}\, \mathrm{Pf}\begin{pmatrix} \hat{J}_L & A_L & -\hat{M}^T & 0 \\ -A_L^T & \hat{J}_L^\dagger & 0 & -\hat{M}^\dagger \\ \hat{M} & 0 & -\hat{J}_R^\dagger & -A_R \\ 0 & \hat{M}^* & A_R^T & -\hat{J}_R \end{pmatrix}$$

- relation between RMT sources and physical sources:
  - $\hat{J}_i = J_i V_4 \Phi_{\mathrm{I}} / \sqrt{2} N$ at all densities
  - at high density, $\hat{M} = M\sqrt{3V_4/N}\,\Delta/2\pi$ (Kanazawa-TW-Yamamoto 2009)
    at lower density, mass scale not known
- in the chiral limit:
  - RMT factorizes into right- and left-handed part
  - $\hat{\rho}_{\mathrm{sv}}(\xi)$ and higher-order correlations of singular values can be obtained by matching to results of Nagao-Forrester (1995) and Nagao-Nishigaki (2000)
    $\to$ determine Φ on the lattice by fit to $\hat{\rho}_{\mathrm{sv}}(\xi)$
  - LS method fails for some parameters, but sum rules can still be obtained as moments of the microscopic correlation functions

- RMT for theory **H** same as for **I** (for fixed ν)
- RMT for theory **L** (C and D are real $N \times (N+\nu)$ matrices):

$$Z_\nu^{\mathrm{RMT}}(\hat\mu, \hat J_L, \hat J_R, \hat M) = \int dC\, dD\, e^{-2N\,\mathrm{tr}(C^T C + D^T D)}$$
$$\times\, \mathrm{Pf}\begin{pmatrix} \hat J_L & C - \hat\mu D & -\hat M^T & 0 \\ -C^T + \hat\mu D^T & \hat J_L^\dagger & 0 & -\hat M^\dagger \\ \hat M & 0 & -\hat J_R^\dagger & -C - \hat\mu D \\ 0 & \hat M^* & C^T + \hat\mu D^T & -\hat J_R \end{pmatrix}$$

  - relation between RMT sources and physical sources:

$$\hat\mu^2 = 2\mu^2 F^2 V_4/N\,, \qquad \hat M = M V_4 \Phi_{\mathbf{L}}/2N\,, \qquad \hat J_i = J_i V_4 \Phi_{\mathbf{L}}/2N$$

  - for $\hat\mu = 1$ ("maximum non-Hermiticity"), RMT(**L**) reduces to RMT(**I**)
    consistency check: $\lim\limits_{z\to\infty}$(sum rule for **L**) $=$ (sum rule for **I**)

Outline

1. Introduction
2. Eigenvalues and singular values of the Dirac operator
    - Index theorem for non-Hermitian Dirac operator
3. Phases of two-color QCD
4. Banks-Casher-type relation for diquark condensate
5. Low-energy effective theories with diquark sources at nonzero density
6. Smilga-Stern-type relations
7. Finite-volume analysis
    - ε-regime
    - Leutwyler-Smilga-type sum rules
    - Random matrix theories
8. Conclusions and outlook

Conclusions and outlook

- rigorous index theorem for non-Hermitian Dirac operator
- new analytical results for the singular value spectrum in two-color QCD at $\mu \neq 0$:
  - effective theories at low, intermediate, and high density
  - Banks-Casher-type relation: $\langle\psi\psi\rangle \sim \rho_{\text{sv}}(0)$
  - Smilga-Stern-type relations: slope $\rho'_{\text{sv}}(0)$
  - Leutwyler-Smilga-type sum rules for inverse singular values
  - ε-regimes and random matrix theories
- results allow for alternative determination of $\langle\psi\psi\rangle$ on the lattice at any μ $\to$ conjectured BEC-BCS crossover could be confirmed numerically
- implications for three-color QCD?
  - diquark source no longer gauge-invariant (but $|\langle\psi\psi\rangle|$ is)
  - how about the gauge-invariant four-quark condensate?
  - how is spectrum of $D(\mu)^\dagger D(\mu)$ related to physical observables?