Probability distribution functions in the finite density lattice QCD

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WHOT-QCD collaboration

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Histogram method

- Problem of Complex Determinant at μ≠0
  - Boltzmann weight: complex at μ≠0
    - Monte-Carlo method is not applicable.

$$\langle O\rangle_{(m,T,\mu)}=\frac{1}{Z}\int DU\, O\,\underbrace{\left(\det M(m,\mu)\right)^{N_{\rm f}}}_{\text{complex}}e^{-S_g}$$

- Distribution function in Density of state method (Histogram method)

  X: order parameters, total quark number, average plaquette etc.

$$\det M=\left|\det M\right|e^{i\theta}$$

$$W(X;m,T,\mu)\equiv\int DU\;\delta\left(X\text{-}\hat{X}\right)\left(\det M(m,\mu)\right)^{N_{\rm f}}\;e^{-S_g}\;=W_0\times\underbrace{\left\langle e^{i\theta}\right\rangle_{X:\text{fixed}}}_{\text{Complex phase factor}}$$

$$W_0(X;T,m,\mu)=\int DU\;\delta\left(X\text{-}\hat{X}\right)\left|\det M(m,\mu)\right|^{N_{\rm f}}e^{-S_g}$$

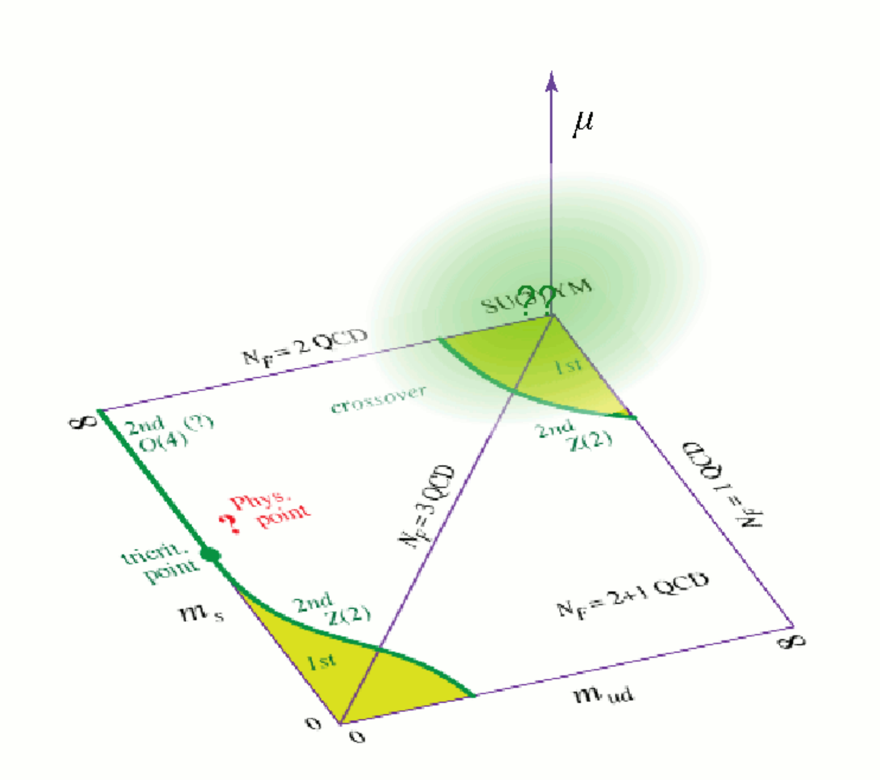
histogram in phase-quenched simulations

  - Expectation values

$$\langle O[X]\rangle_{(m,T,\mu)}=\frac{1}{Z}\int dX\,O[X]\,W(X,m,T,\mu)$$

$$Z(m,T,\mu)=\int dX\,W(X,m,T,\mu)$$

Distribution function in the heavy quark region



- We study the properties of $W(X)$ in the heavy quark region.
- Performing quenched simulations + Reweighting.
- We find the critical surface.
- Standard Wilson quark action + plaquette gauge action, $S_g = -6N_{\text{site}}\beta P$
- lattice size: $24^3 \times 4$
- 5 simulation points; β=5.68-5.70.

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

Hopping parameter expansion

$$N_f \ln\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right) = N_f\left(288 N_{\text{site}} K^4 P + 12 \cdot 2^{N_t} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R + \underbrace{i\sinh(\mu/T)\Omega_I}_{\text{phase}}\right) + \cdots\right)$$

P: plaquette, Ω=ΩR+iΩI : Polyakov loop $\det M(0,0) = 1$

Distribution function for P and Ω_R

$$\boxed{W(P,\Omega_R,\beta,\kappa)=\int DU\ \delta\left(P-\hat{P}\right)\delta\left(\Omega_R-\hat{\Omega}_R\right)\left(\det M(\kappa)\right)^{N_f}\ e^{-6N_{\rm site}\hat{P}}}$$

$$\frac{W(\beta,K,\mu)}{W(\beta_0,0,0)}=\frac{\left\langle \delta\left(P-\hat{P}\right)\delta\left(\Omega_R-\hat{\Omega}_R\right)e^{6N_{\rm site}(\beta-\beta_0)P}\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right)^{N_f}\right\rangle_{(\beta_0,K=\mu=0)}}{\left\langle \delta\left(P-\hat{P}\right)\delta\left(\Omega_R-\hat{\Omega}_R\right)\right\rangle_{(\beta_0,K=\mu=0)}}\equiv\left\langle e^{6N_{\rm site}(\beta-\beta_0)P'}\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right)^{N_f}\right\rangle_{P,\Omega_R}$$

- Effective potential $\quad V_{\rm eff}(P,\Omega_R;\beta,\kappa)=-\ln W(P,\Omega_R;\beta,\kappa)$
- Hopping parameter expansion

$$V_{\rm eff}(\beta,\kappa)-V_{\rm eff}(\beta_0,0)=-\left(6(\beta-\beta_0)+288N_f K^4\right)N_{\rm site}P-12\times 2^{N_t}N_f N_s^3 K^{N_t}\cosh(\mu/T)\Omega_R-\ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R}$$

$$\equiv V_0(\beta,\kappa)\qquad -\ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R}\qquad \left(\theta=12\cdot 2^{N_t}N_s^3 N_f K^{N_t}\sinh(\mu/T)\hat{\Omega}_I\right)$$

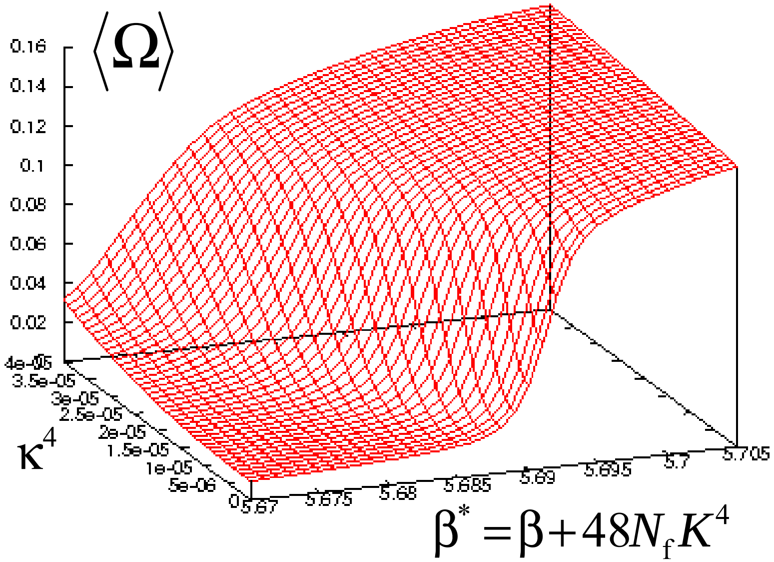
Phase-quenched part

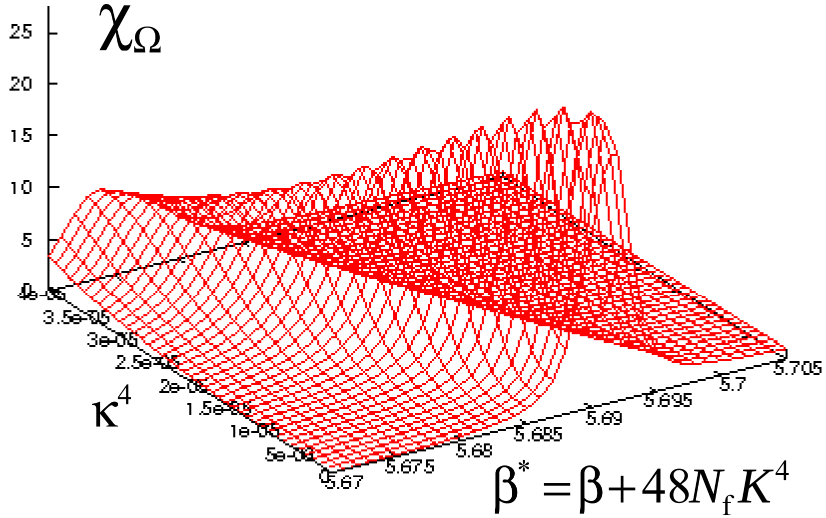
- 2 parameters in V_0: $\boxed{\beta+48N_f K^4\equiv\beta^*,}$ $\boxed{K^{N_t}\cosh(\mu/T)}$
  - V_0 is the same as $V_{\rm eff}(\mu=0)$ when $K^{N_t}\Rightarrow K^{N_t}\cosh(\mu/T)$
- 1 parameter in θ: $\boxed{K^{N_t}\sinh(\mu/T)=K^{N_t}\cosh(\mu/T)\tanh(\mu/T)<K^{N_t}\cosh(\mu/T)}$

Distribution function for P and Ω_R

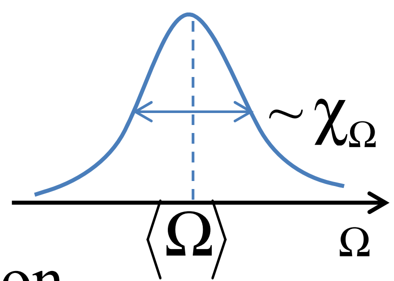
Expectation value of Polyakov loop and its susceptibility by the reweightuing method at μ=0.

$24^3 \times 4$ lattice





- If $W(P,\Omega)$ is a Gaussian distribution,
  - The peak position of $W(P,\Omega)$ ⇨ $(\langle P\rangle,\langle\Omega\rangle)$
  - The width of $W(P,\Omega)$ ⇨ susceptibilities χ_P, χ_Ω
- If $W(P,\Omega)$ have two peaks, ⇨ first order transition

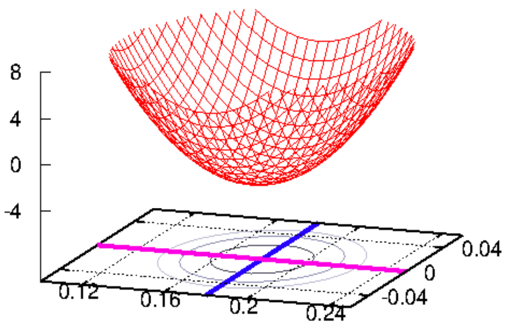


Order of phase transitions and Distribution function

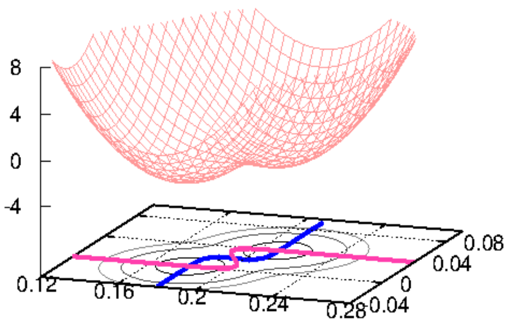
$$\boxed{W(P,\Omega_R,\beta,\kappa)=\int DU\ \delta(P-\hat{P})\delta(\Omega_R-\hat{\Omega}_R)(\det M(\kappa))^{N_f}\ e^{-6N_{site}P}}$$

$$V_{eff}(P,\Omega_R;\beta,\kappa)=-\ln W(P,\Omega_R;\beta,\kappa)$$

- Peak position of W: $\dfrac{dV_{eff}}{dP}=\dfrac{dV_{eff}}{d\Omega_R}=0$

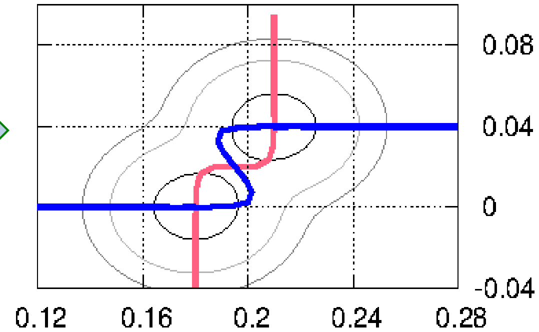


crossover
1 intersection



first order transition
3 intersections

Lines of zero derivatives
for first order



Derivatives of V_{eff} in terms of P and Ω_R

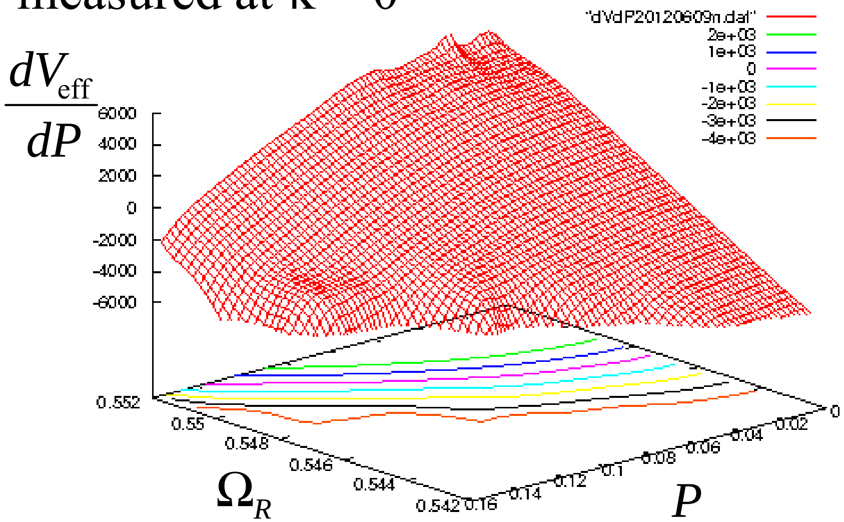
Phase-quenched part: when $\ln\langle e^{i\theta}\rangle$ is neglected,

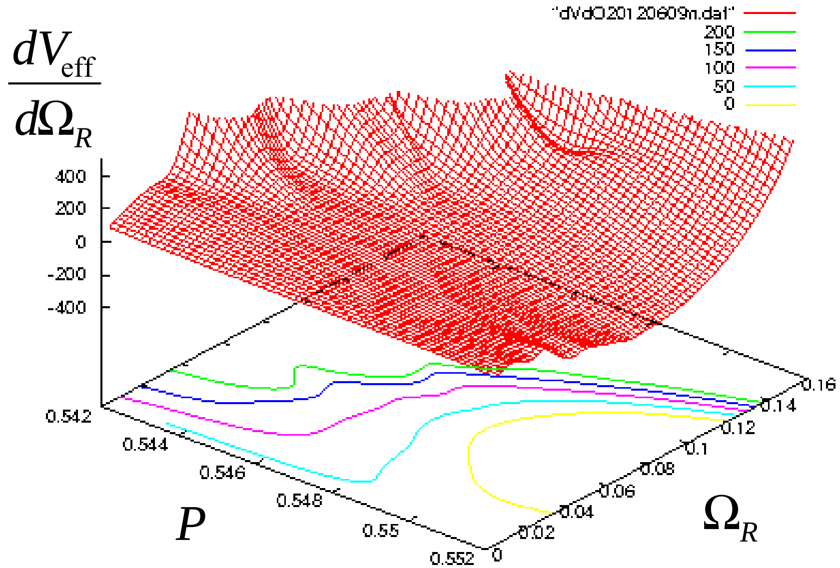
$$V_{\mathrm{eff}}(\beta,\kappa)-V_{\mathrm{eff}}(\beta_0,0)=-\left(6(\beta-\beta_0)+288N_{\mathrm{f}}K^4\right)N_{\mathrm{site}}P-12\times 2^{N_t}N_{\mathrm{f}}N_s^3K^{N_t}\cosh(\mu/T)\Omega_R$$

$$\frac{dV_{\mathrm{eff}}(\beta,K)}{dP}-\frac{dV_{\mathrm{eff}}(\beta_0,0)}{dP}=\underbrace{-\left(6(\beta-\beta_0)+288N_{\mathrm{f}}K^4\right)N_{\mathrm{site}}}_{\text{constant shift}}$$

$$\frac{dV_{\mathrm{eff}}(\beta,K)}{d\Omega_R}-\frac{dV_{\mathrm{eff}}(\beta_0,0)}{d\Omega_R}=\underbrace{-12\times 2^{N_t}N_{\mathrm{f}}N_s^3K^{N_t}\cosh\left(\frac{\mu}{T}\right)}_{\text{constant shift}}$$

measured at $\kappa = 0$



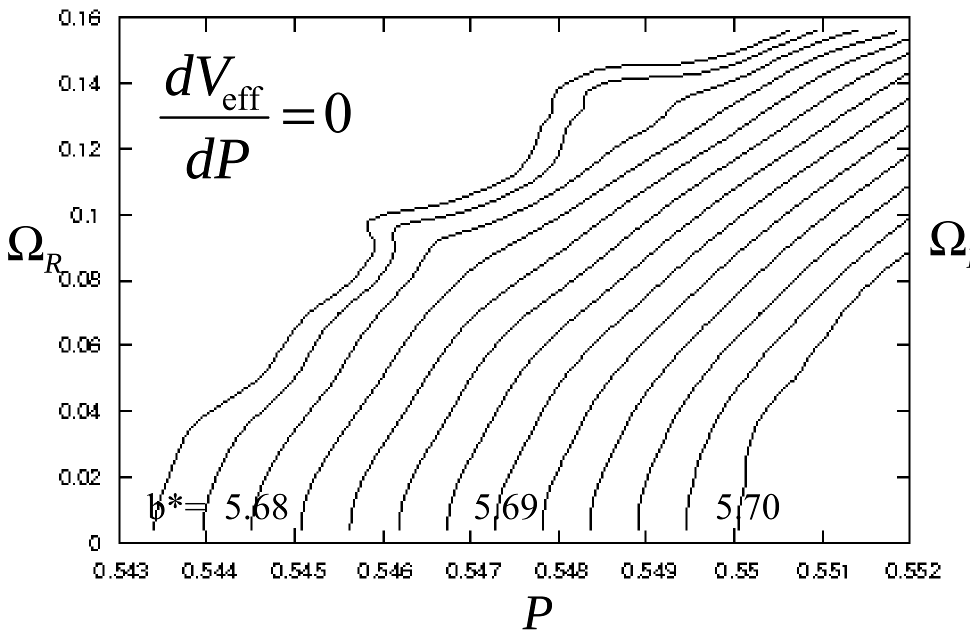


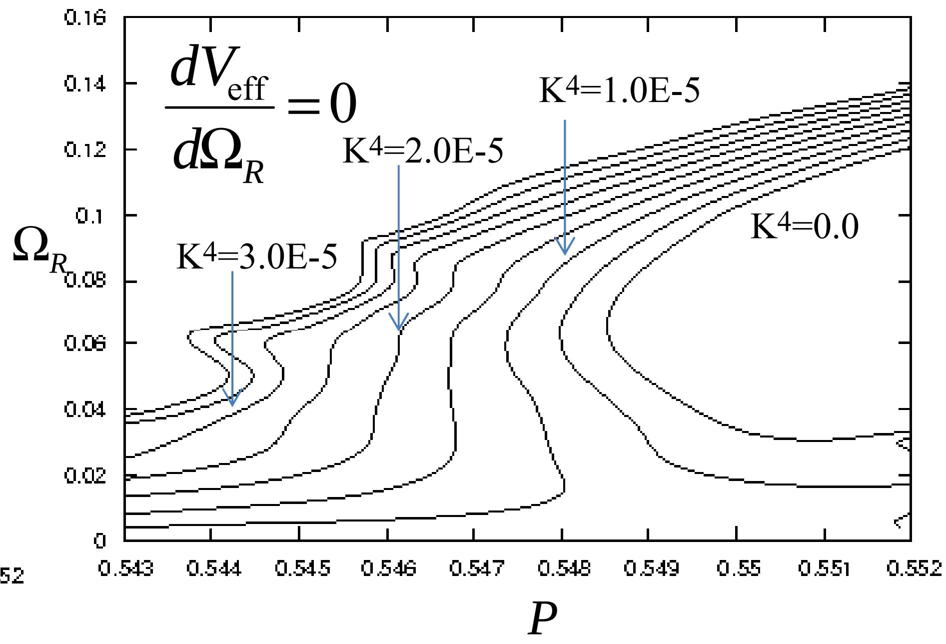
- Contour lines of $\frac{dV_{\mathrm{eff}}}{dP}$ and $\frac{dV_{\mathrm{eff}}}{d\Omega_R}$ at $(\beta,\kappa) = (\beta_0,0)$ correspond to the lines of the zero derivatives at (β,κ).

lines of $\frac{dV_{\text{eff}}}{dP}=0$ and $\frac{dV_{\text{eff}}}{d\Omega_R}=0$ in the (P,Ω) plane

$$\frac{dV_{\text{eff}}(\beta_0,0)}{dP}=6N_{\text{site}}\left(\beta-\beta_0+48N_f K^4\right)=6N_{\text{site}}\left(\beta^*-\beta_0\right)$$

$$\frac{dV_{\text{eff}}(\beta_0,0)}{d\Omega_R}=12\times 2^{N_t}N_f N_s^3 K^{N_t}\cosh\left(\frac{\mu}{T}\right)$$

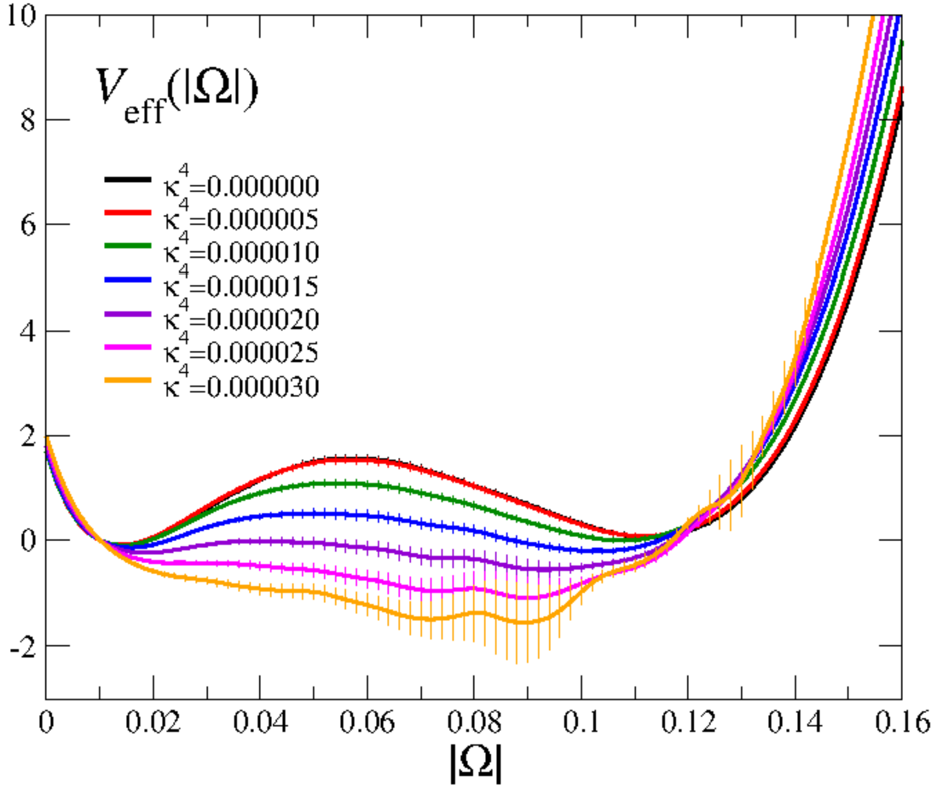




- Small K: lines of $\frac{dV_{\text{eff}}}{d\Omega_R}=0$: S-shape ➡ first order
- Large K: lines of $\frac{dV_{\text{eff}}}{d\Omega_R}=0$: straight line ➡ crossover

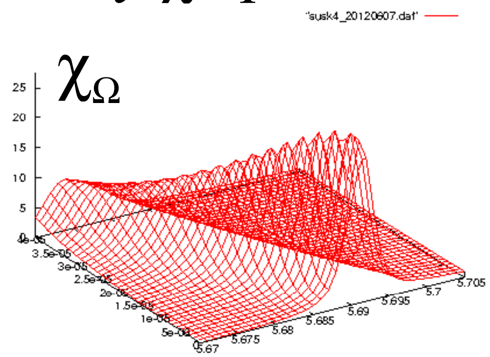
Order of the phase transition
Polyakov loop distribution

Effective potential of $|\Omega|$
on the pseudo-critical line at $\mu=0$



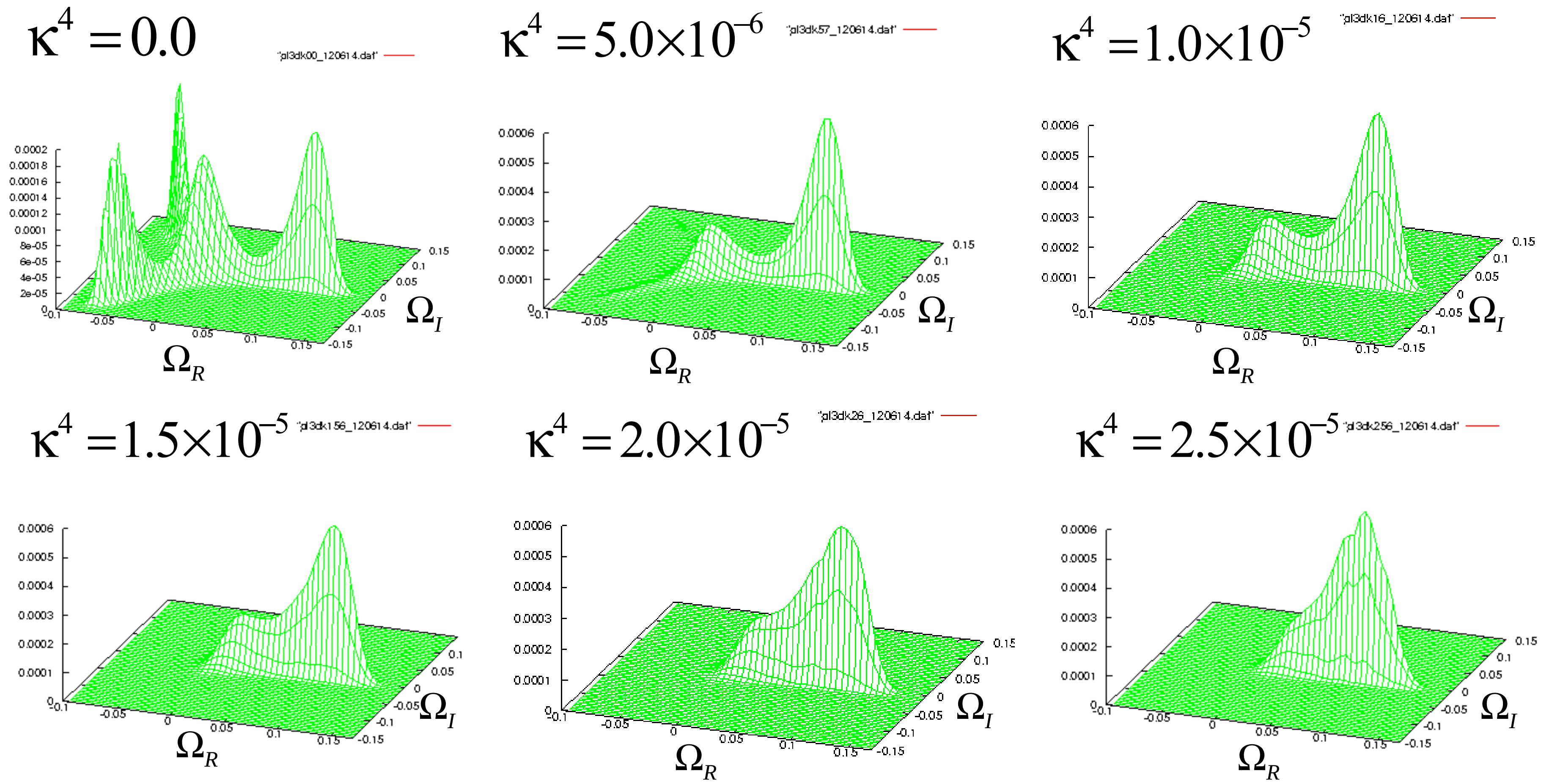
Critical point: $\kappa^4 \approx 2.0\times10^{-5}$

- The pseudo-critical line is determined by χ_Ω peak.



- Double-well at small K
  – First order transition
- Single-well at large K
  – Crossover

Polyakov loop distribution in the complex plane
(μ=0)



critical point

- on β_{pc} measured by the Polyakov loop susceptibility.

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\theta \equiv \mathrm{Im} \ln \det M \quad \approx 12 \cdot 2^{N_t} N_s^3 N_f K^{N_t} \sinh(\mu/T) \Omega_I$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R \text{ fixed}} \ll \text{(statistical error)}$$

- Cumulant expansion $\quad$ $<..>_{P,\Omega R}$: expectation values fixed P and ΩR.

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[\underbrace{\cancel{i\langle\theta\rangle_C}}_{\to 0} - \frac{1}{2}\left\langle\theta^2\right\rangle_C \underbrace{\cancel{- \frac{i}{3!}\left\langle\theta^3\right\rangle_C}}_{\to 0} + \frac{1}{4!}\left\langle\theta^4\right\rangle_C \cancel{+ \cdots} \right]$$

cumulants

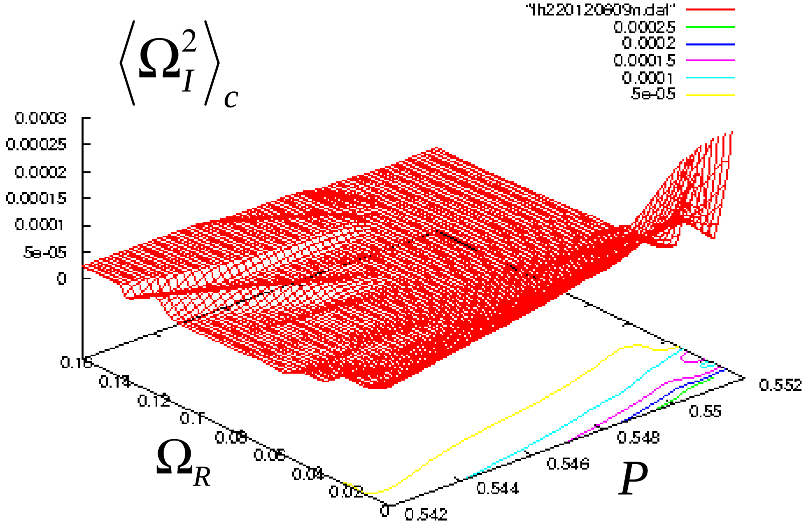
$$\langle\theta\rangle_C = \langle\theta\rangle_{P,\Omega_R}, \quad \left\langle\theta^2\right\rangle_C = \left\langle\theta^2\right\rangle_{P,\Omega_R} - \langle\theta\rangle_{P,\Omega_R}^2, \quad \left\langle\theta^3\right\rangle_C = \left\langle\theta^3\right\rangle_{P,\Omega_R} - 3\left\langle\theta^2\right\rangle_{P,\Omega_R}\langle\theta\rangle_{P,\Omega_R} + 2\langle\theta\rangle_{P,\Omega_R}^3, \quad \left\langle\theta^4\right\rangle_C = \cdots$$

- Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)

  Source of the complex phase

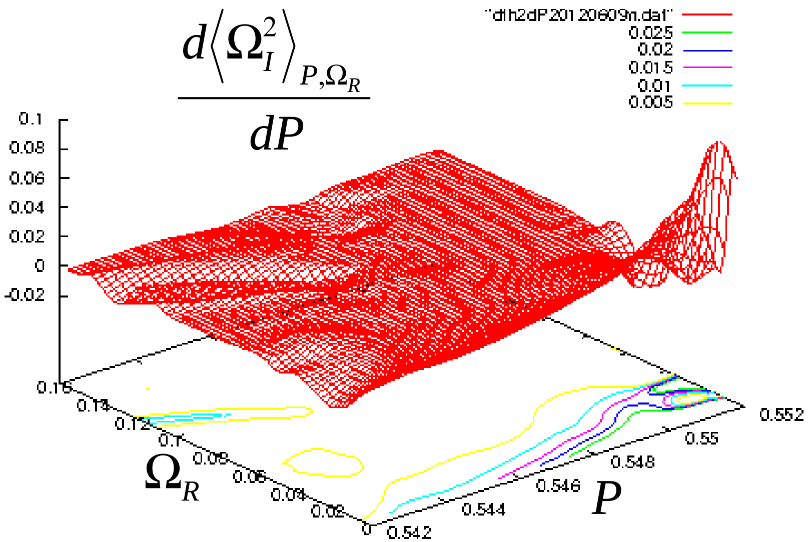
If the cumulant expansion converges, No sign problem.

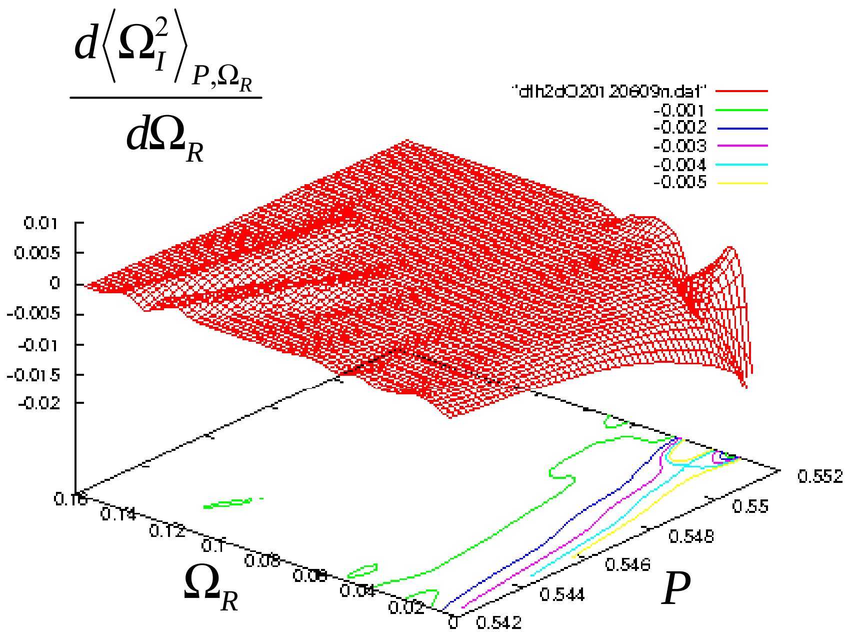
Effect from the complex phase



$$\theta \approx 12 \cdot 2^{N_t} N_s^3 N_f K^{N_t} \sinh(\mu/T)\Omega_I$$

- The complex phase fluctuation is small except near the Ω_R=0 axis.
- Ω_R-dependence is important.



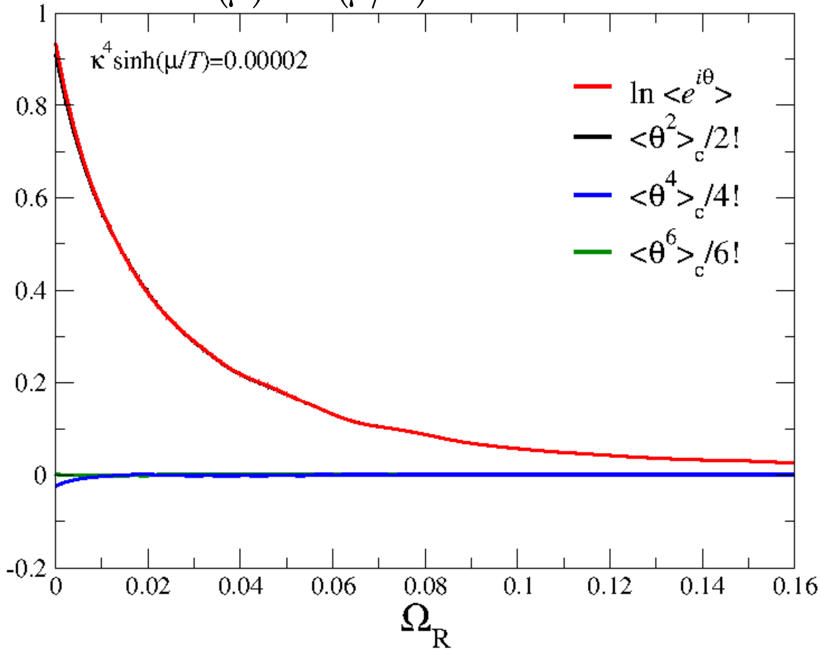


Cumulant expansion

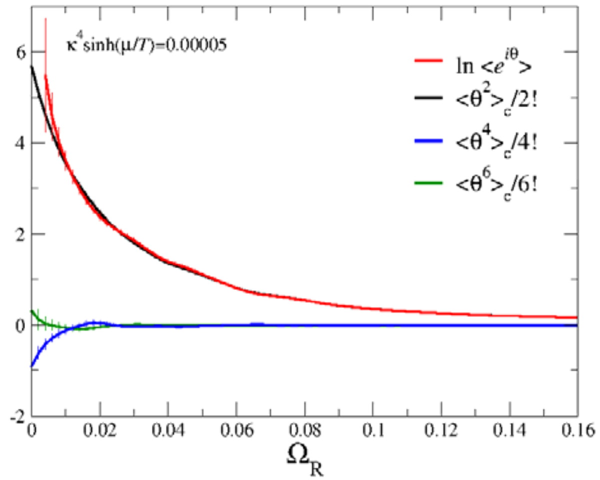
$\beta^* = 5.69$, independent of κ.

$$\ln\left\langle e^{i\theta}\right\rangle_{P,F} = -\frac{1}{2}\left\langle\theta^2\right\rangle_C + \frac{1}{4!}\left\langle\theta^4\right\rangle_C - \frac{1}{6!}\left\langle\theta^6\right\rangle_C + \cdots$$

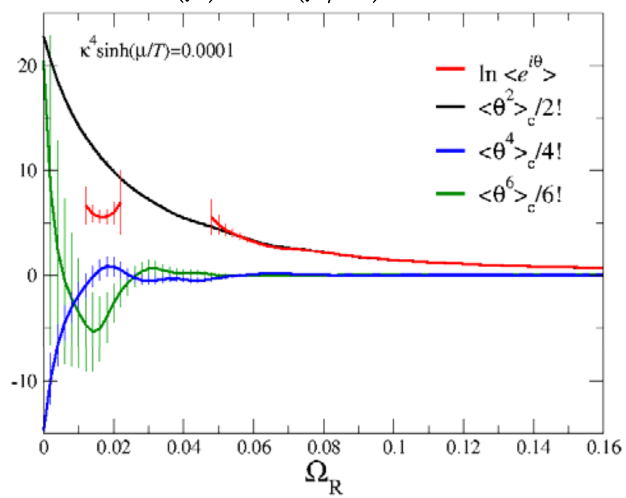
$K^4(\mu)\sinh(\mu/T) = 0.00002$



$K^4(\mu)\sinh(\mu/T) = 0.00005$



$K^4(\mu)\sinh(\mu/T) = 0.0001$



- The effect from higher order terms is small near the critical point.

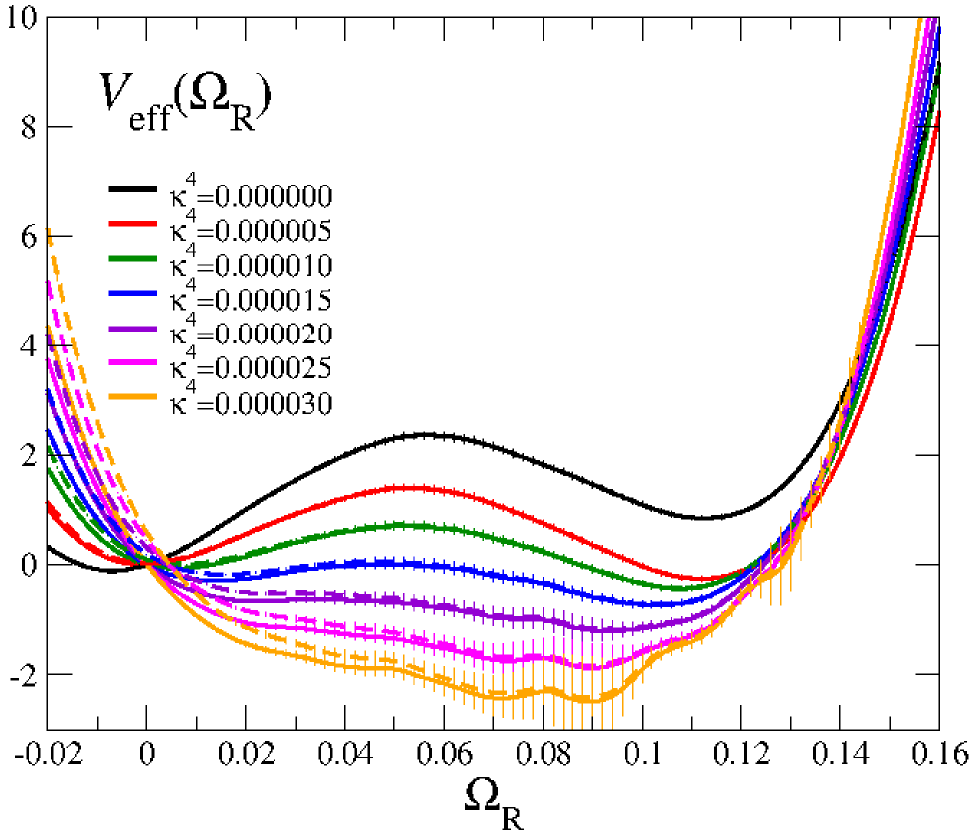
$$K_{\rm cp}^{N_t}(0) = K_{\rm cp}^{N_t}(\mu)\cosh(\mu/T) > K_{\rm cp}^{N_t}(\mu)\sinh(\mu/T)$$

~0.00002

- Phase fluctuations
  - large in the confinement phase
  - small in the deconfinement phase

Effect from the complex phase factor

- Polyakov loop effective potential for each $K^{N_t}\cosh(\mu/T)$
  at the pseudo-critical (β, K).
  - Solid lines: complex phase omitted, i.e., $\tanh(\mu/T)=0$
  - Dashed lines: complex phase is estimated from $\langle\theta^2\rangle_c/2$
    with $\tanh(\mu/T)=1$



$$V_{\rm eff}(\Omega_R)=V_0(\Omega_R)-\ln\langle e^{i\theta}\rangle_{\Omega_R:\text{fixed}}$$

$$\approx V_0(\Omega_R)+\frac{1}{2}\langle\theta^2\rangle_{\Omega_R:\text{fixed}}$$

The effect from the complex phase factor is very small except near Ω_R=0.

Critical line in 2+1-flavor finite density QCD

- The effect from the complex phase is very small for the determination of K_{cp}.

N_f=2 at μ=0: K_{cp}=0.0658(3)(8)

(WHOT-QCD, Phys.Rev.D84, 054502(2011))

$$2\ln\left(\frac{\det M(K)}{\det M(0)}\right) = 2\left(288 N_{\rm site} K^4 P + \underline{12\times 2^{N_t} N_s^3 K^{N_t} \Omega_R} + \cdots\right)$$

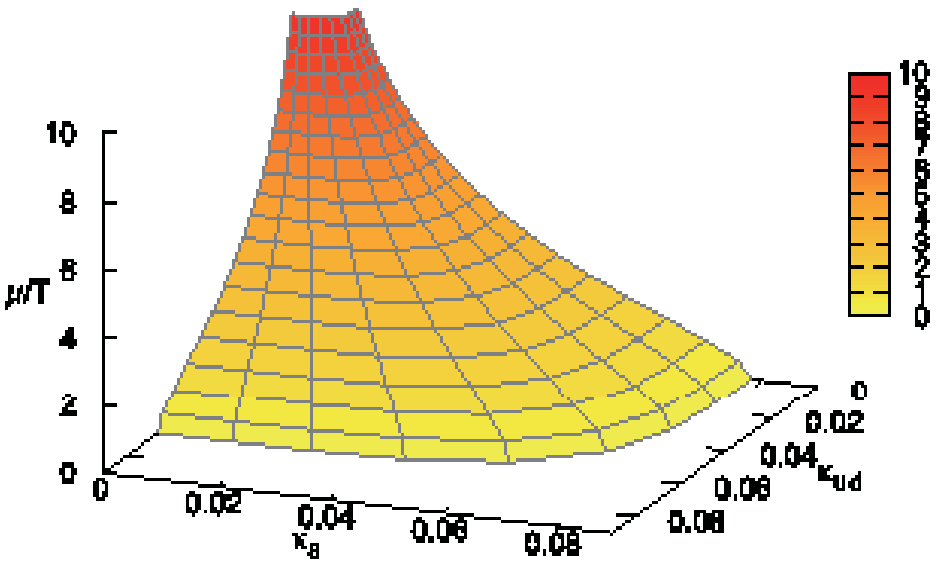
$$K_{\rm cp}^{N_t}(0) = K_{\rm cp}^{N_t}(\mu)\cosh(\mu/T)$$

N_f=2+1

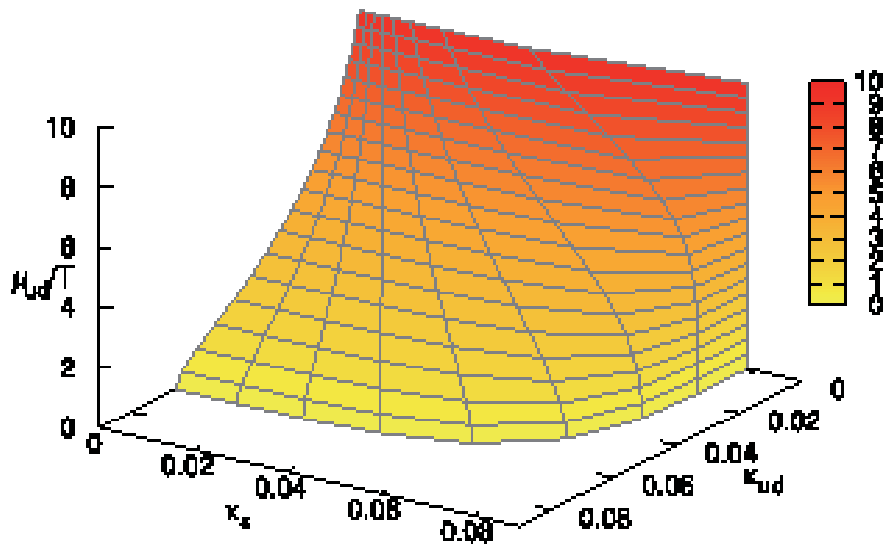
$$\ln\left[\frac{(\det M(K_{ud}))^2 \det M(K_s)}{(\det M(0))^3}\right] = 288 N_{\rm site}(2K_{ud}^4 + K_s^4)P + \underline{12\times 2^{N_t} N_s^3 \left(2K_{ud}^{N_t}\cosh\left(\frac{\mu_{ud}}{T}\right) + K_s^{N_t}\cosh\left(\frac{\mu_s}{T}\right)\right)\Omega_R} + \cdots$$

The critical line is described by $\boxed{2K_{ud}^{N_t}\cosh\left(\frac{\mu_{ud}}{T}\right) + K_s^{N_t}\cosh\left(\frac{\mu_s}{T}\right) = 2K_{{\rm cp}(Nf=2)}^{N_t}}$

Critical line for μu=μd=μs=μ



Critical line for μu=μd=μ, μs=0



Summary

- We studied the quark mass and chemical potential dependence of the nature of QCD phase transition in the heavy quark region.

- The shape of the probability distribution function changes as a function of the quark mass and chemical potential.

- To avoid the sign problem, the method based on the cumulant expansion of θ is useful.

- To find the critical point at finite density, studies in light quark region are important applying this method.

  ⟹ Nakagawa's talk (Thursday, 2:50-)