
Auxiliary field Monte-Carlo study of the QCD phase diagram at strong coupling

Akira Ohnishi (YITP)

Terukazu Ichihara (Kyoto U.)

Takashi Z. Nakano (YITP/Kyoto U.)

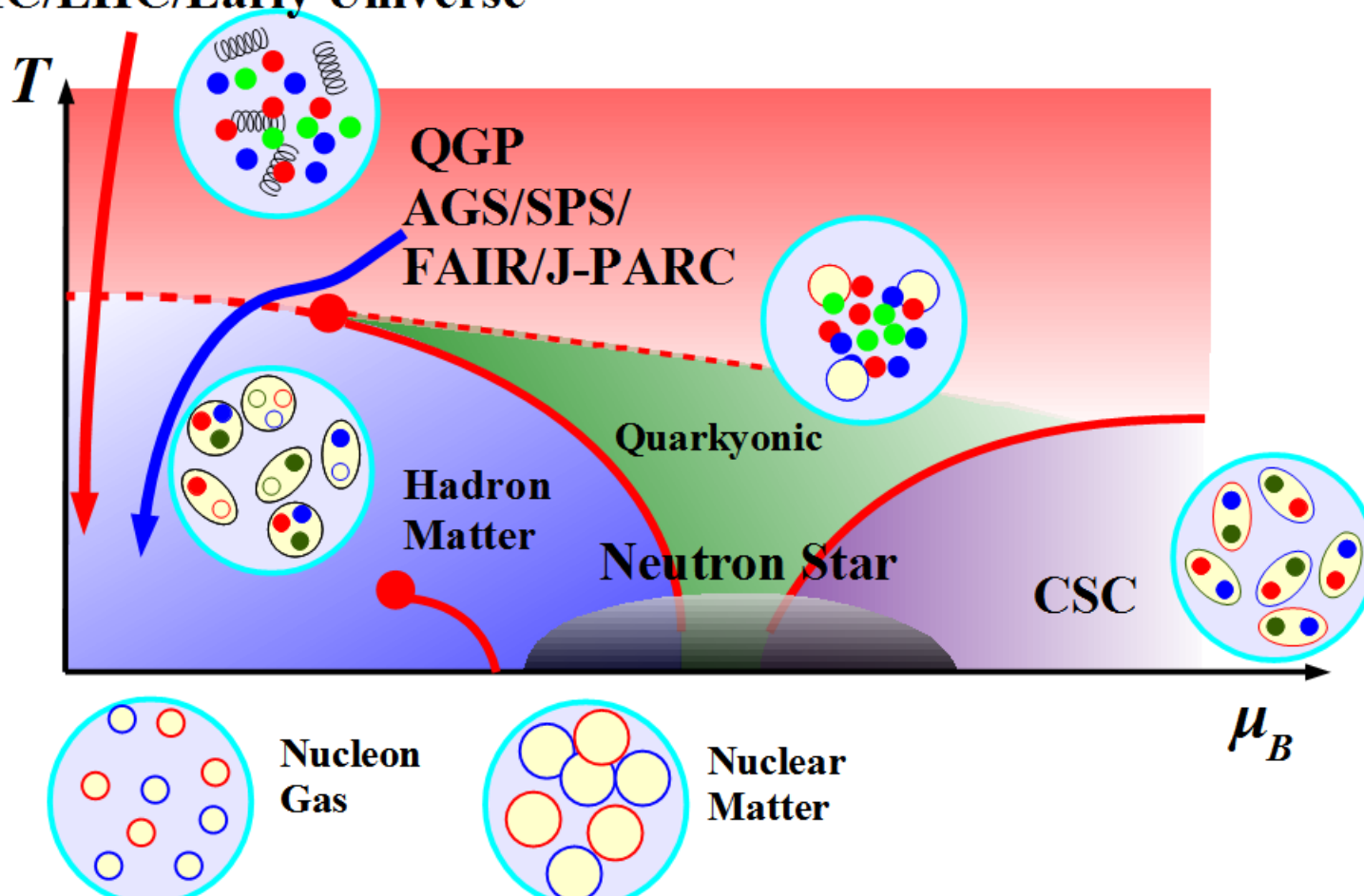
- **Introduction**
- **Auxiliary field effective action in the Strong Coupling Limit**
- **AFMC phase diagram**
- **Summary**

AO, T. Ichihara, T. Z. Nakano, in prep.

QCD phase diagram

- Various phases, rich structure (conjectured)
Related to early universe and compact star phenomena,
and CP may be reachable in RHIC.

RHIC/LHC/Early Universe



Can we understand QCD phase diagram in lattice QCD ?

Lattice QCD at Finite Density

■ Dream

Ab initio calc. of phase diagram and nuclear matter EOS

■ Unreachable ?

Sign prob. is severe at low T & high μ

● No go theorem

Han, Stephanov ('08), Hanada, Yamamoto ('11), Hidaka, Yamamoto ('11)

Phase quenched sim. at finite quark $\mu \sim$ Finite isospin μ

(No flavor mixing, as justified in large N_c)

→ Average sign factor vanishes at low T & high μ due to π cond.

■ Hope ?

● Sampling method other than phase quenched simulation ?

● Strong coupling lattice QCD

→ Mean field approximation

Monomer-Dimer-Polymer simulation

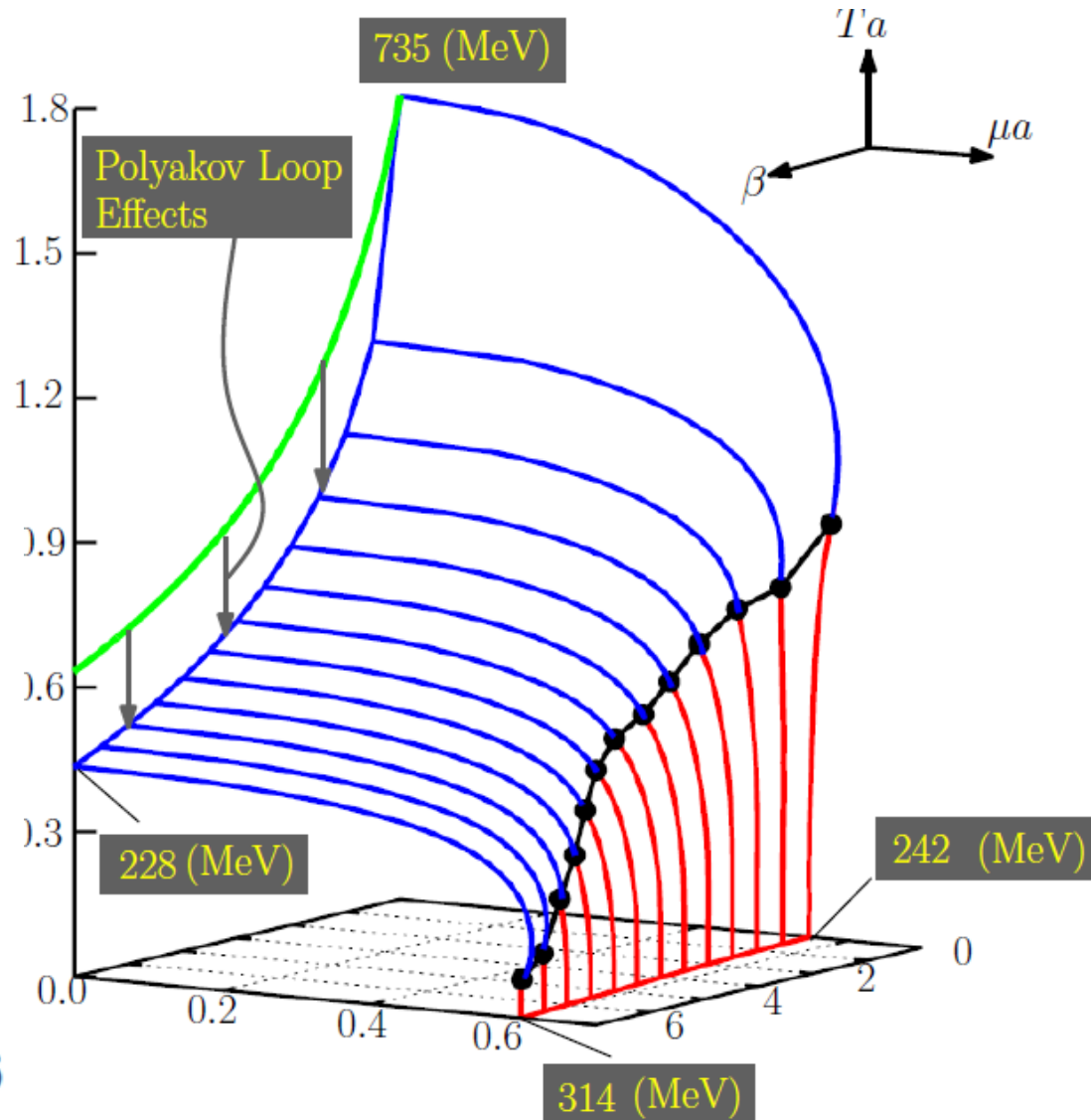
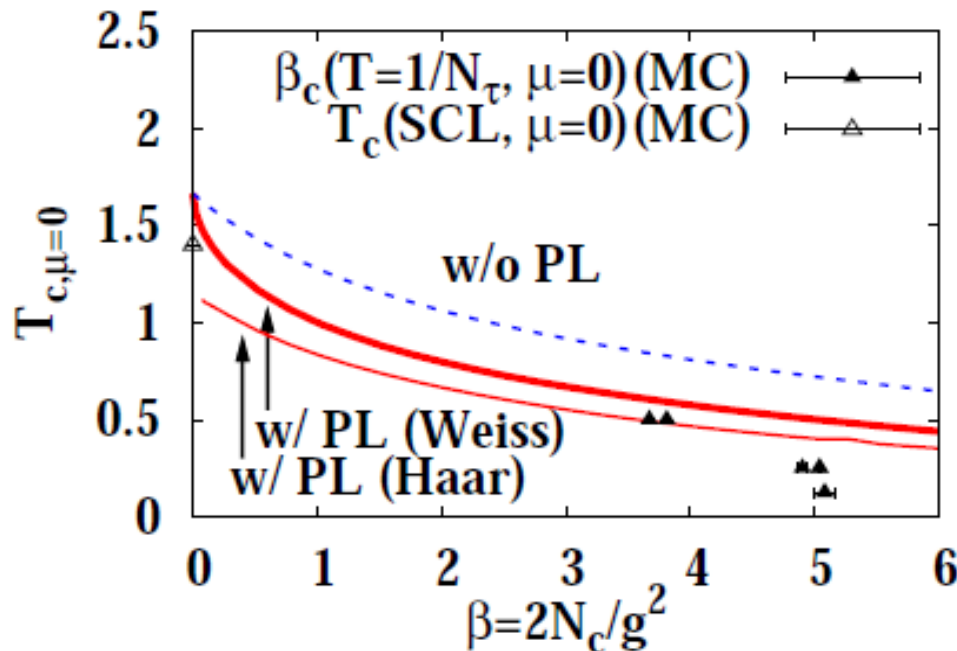
Strong Coupling Lattice QCD

- **Successful from the dawn of lattice gauge theory**
 - **Pure YM: Area Law, MC calc. of string tension, $1/g^2$ expansion**
Wilson ('74), Creutz ('80), Munster ('80)
- **Strong Coupling Lattice QCD with quarks**
 - **Spontaneous breaking of chiral sym. in vacuum, Chiral transition**
Kawamoto, Smit ('81), Damgaard, Kawamoto, Shigemoto ('84)
→ **Utilized in constructing effective models**
Gocksch, Ogilvie (85), Fukushima ('03), Ratti, Thaler, Weise ('06), ...
 - **Phase diagram in the strong coupling limit (mean field)**
Bilic, Karsch, Redlich ('92), Fukushima ('04), Nishida ('04)
 - **Finite coupling and Polyakov loop effects (mean field)**
*Faldt, Petersson ('86), Miura, Nakano, AO ('09),
Miura, Nakano, AO, Kawamoto ('09), Nakano, Miura, AO ('10),
Nakano, Miura, AO, Kawamoto ('11)*
 - **Fluctuation effects via MDP simulation (mean field)**
Karsch, Mutter('89); de Forcrand, Fromm ('10); de Forcrand, Unger ('11)

Finite Coupling and Polyakov Loop Effects

*Miura, Nakano, AO ('09), Miura, Nakano, AO, Kawamoto ('09),
Nakano, Miura, AO ('10), Nakano, Miura, AO, Kawamoto ('11)*

- Finite coupling & Pol. loop reduces T_c while μ_c is stable.
 - MC results of T_c at $\mu=0$ are explained at $\beta_g=2 N_c/g^2 < 4$.
 - Compatible with empiricals.



Nakano et al. ('11)

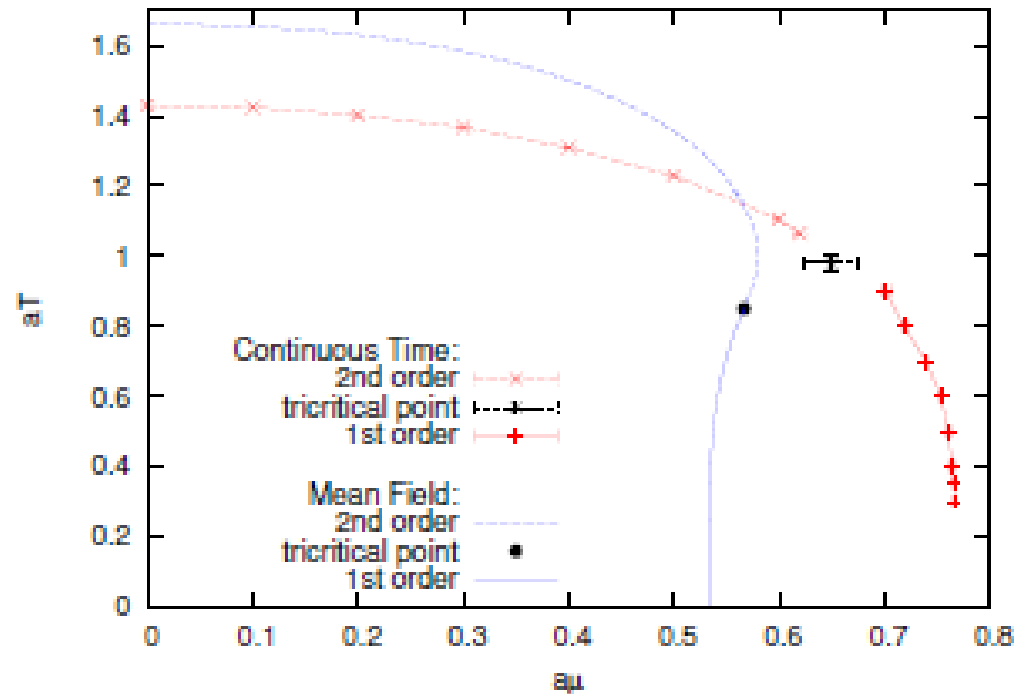
Miura et al., in prep.

Monomer-Dimer-Polymer phase diagram

■ MDP simulation

Karsch, Mutter('89); de Forcrand, Fromm ('10); de Forcrand, Unger ('11)

- Partition function
= sum of config. weights
of various loops.
- Extension to finite coupling
($1/g^2 \neq 0$) is not straightforward.

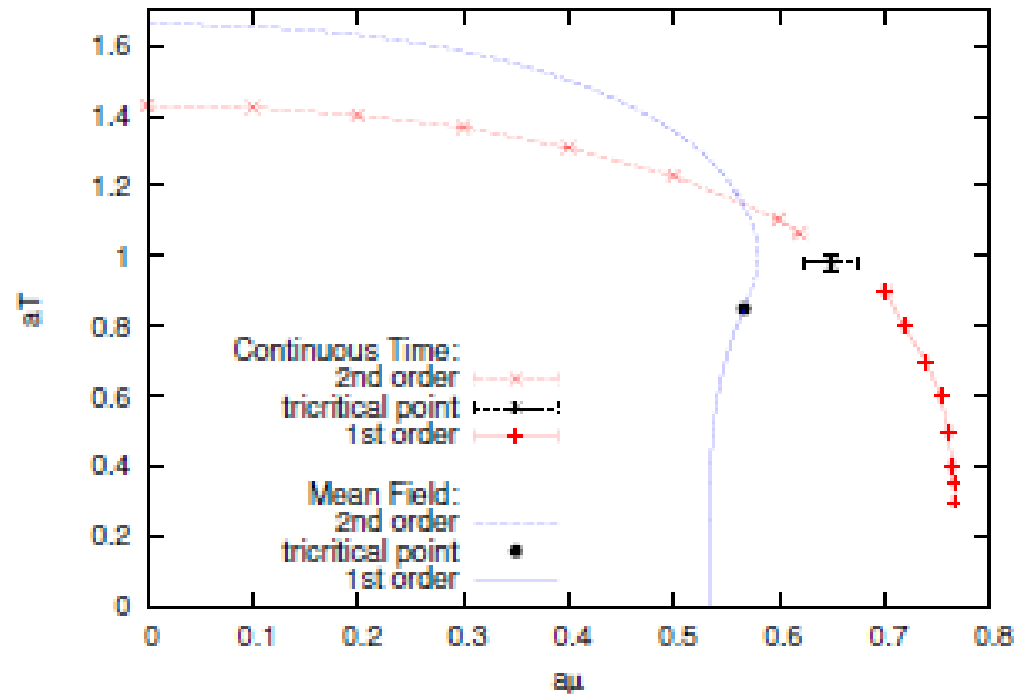


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*Both finite coupling and fluctuation effects are important.
Is there any way to include both of these ?
→ Auxiliary Field Monte-Carlo method*

Auxiliary Field Monte-Carlo in the Strong Coupling Limit

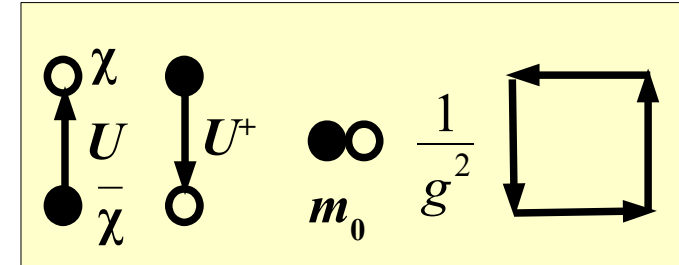
Strong Coupling Effective Action

■ Strong Coupling Limit Lattice QCD action

- no plaquette action, aniso. lattice $a_\tau = a_s / \gamma$, unrooted stag. fermion

$$S_{\text{LQCD}} = \frac{1}{2} \sum_x \left[e^{\mu/\gamma^2} \bar{\chi}_x U_{0,x} \chi_{x+\hat{0}} - e^{-\mu/\gamma^2} \bar{\chi}_{x+\hat{0}} U_{0,x}^+ \chi_x \right]$$

$$+ \frac{1}{2\gamma} \sum_{x,j} \eta_j(x) \left[\bar{\chi}_x U_{j,x} \chi_{x+\hat{j}} - \bar{\chi}_{x+\hat{j}} U_{j,x}^+ \chi_x \right] + \frac{m_0}{\gamma} \sum_x \bar{\chi}_x \chi_x$$



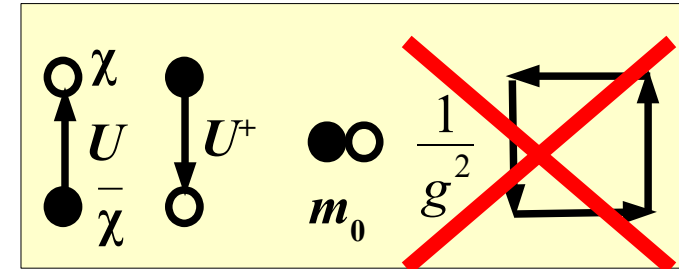
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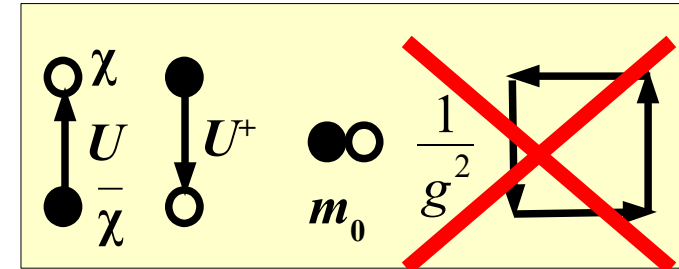
anisotropy

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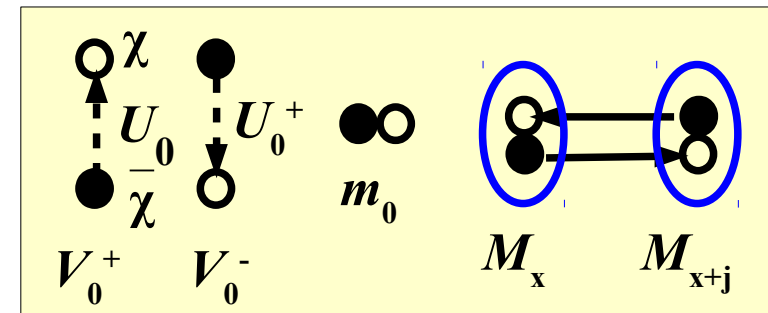
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anisotropy

Strong Coupling Limit Effective Action

- Leading orders in $1/g^2$ and $1/d$
+ Spatial link integral

→ Eff. action of mesonic composites



(d=spatial dim.)

$$S_{\text{eff}} = \frac{1}{2} \sum_x \left[V^+(x) - V^-(x) \right] - \frac{1}{4 N_c \gamma^2} \sum_{x,j} M_x M_{x+\hat{j}} + \frac{m_0}{\gamma} \sum_x M_x$$

$$V^+(x) = e^{\mu/\gamma^2} \bar{\chi}_x U_{x,0} \chi_{x+\hat{0}}, \quad V^-(x) = e^{-\mu/\gamma^2} \bar{\chi}_{x+\hat{0}} U_{x,0}^+ \chi_x, \quad M_x = \bar{\chi}_x \chi_x$$

Introduction of Auxiliary Fields

■ Bosonization of MM term in Mean Field Treatment

$$-\alpha \sum_{j,x} M_x M_{x+\hat{j}} \rightarrow \alpha d \sigma^2 + 2d\alpha \sigma \sum_x M_x$$

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Const. quark mass

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Const. quark mass

■ More rigorous treatment

$$-\alpha \sum_{j,x} M_x M_{x+\hat{j}} = -\alpha \sum_{x,y} M_x V_{x,y} M_y = -\alpha L^3 \sum_{k,\tau} f(k) M_{-k,\tau} M_{k,\tau}$$

● Meson hopping matrix has positive and negative eigen values

$$V_{x,y} = \frac{1}{2} \sum_j (\delta_{x+\hat{j},y} + \delta_{x-\hat{j},y}), \quad f_M(k) = \sum_j \cos k_j,$$

$$f_M(\bar{k}) = -f_M(k) \quad [\bar{k} = k + (\pi, \pi, \pi)]$$

● Extended Hubbard-Stratonotic (HSMNO) transf.

→ Introducing “ i ” gives rise to the sign problem.

Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)

$$\exp(\alpha A B) = \int d\varphi d\phi \exp[-\alpha(\varphi^2 - (A+B)\varphi + \phi^2 - i(A-B)\phi)]$$

Auxiliary Field Effective Action

■ Bosonized effective action

$$\begin{aligned} S_{\text{eff}}(\sigma, \pi, \chi, \bar{\chi}, U_0) = & \frac{1}{2} \sum_x [V^+(x) - V^-(x)] + \sum_x \bar{\chi}_x \chi_x \Sigma_x \\ & + \frac{L^3}{4 N_c \gamma^2} \sum_{\mathbf{k}, \tau, f_M(\mathbf{k}) > 0} f_M(\mathbf{k}) [\sigma_k^* \sigma_k + \pi_k^* \pi_k] \\ \Sigma_x = & \frac{1}{4 N_c \gamma^2} \sum_j [(\sigma + i \varepsilon \pi)_{x+\hat{j}} + (\sigma + i \varepsilon \pi)_{x-\hat{j}}] + \frac{m_0}{\gamma} \end{aligned}$$

Auxiliary Field Effective Action

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Const. quark mass

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 & + \frac{L^3}{4 N_c \gamma^2} \sum_{\mathbf{k}, \tau, f_M(\mathbf{k}) > 0} f_M(\mathbf{k}) [\sigma_{\mathbf{k}}^* \sigma_{\mathbf{k}} + \pi_{\mathbf{k}}^* \pi_{\mathbf{k}}] \\
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$$+ \frac{L^3}{4 N_c \gamma^2} \sum_{\mathbf{k}, \tau, f_M(\mathbf{k}) > 0} f_M(\mathbf{k}) [\sigma_k^* \sigma_k + \pi_k^* \pi_k]$$

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Nearest Neighbor Interaction

Negative mode $\rightarrow$ high k modes

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Nearest Neighbor Interaction

Negative mode → high k modes

■ Grassmann and Temporal Link Integral (analytic)

Faldt, Petersson ('86), Nishida ('04)

$$S_{\text{eff}}(\sigma, \pi) = \frac{L^3}{4 N_c \gamma^2} \sum_{\mathbf{k}, f_M(\mathbf{k}) > 0} f_M(\mathbf{k}) [\sigma_{\mathbf{k}}^* \sigma_{\mathbf{k}} + \pi_{\mathbf{k}}^* \pi_{\mathbf{k}}]$$

$$- \sum_{\mathbf{x}} \log [X_N(\mathbf{x})^3 - 2 X_N(\mathbf{x}) + 2 \cosh(3 N_\tau \mu / \gamma^2)]$$

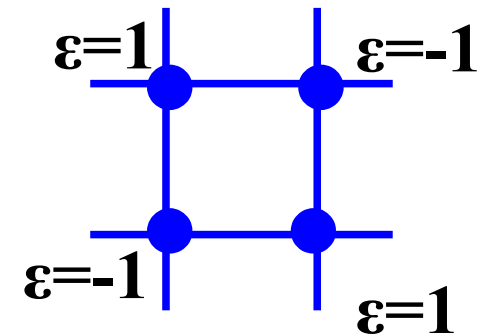
$$X_N(\mathbf{x}) = X_N[\Sigma(\mathbf{x}, \tau)] \quad (\text{known function})$$

$$= 2 \cosh(N_\tau \text{arcsinh } \Sigma / \gamma^2) \quad (\text{for const. } \Sigma)$$

Merits of Auxiliary Field Monte-Carlo

- Fermion matrix is spatially separated
→ Fermion det. at each point
- Imaginary part (π) involves

$$\varepsilon = (-1)^{x_0+x_1+x_2+x_3} = \exp(i \pi(x_0+x_1+x_2+x_3))$$
 → Phase cancellation of nearest neighbor spatial site det for π field having low k
- Phase appears only from the $\log(\det)$ term,



$$\log \left[\underbrace{X_N(\mathbf{x})^3 - 2 X_N(\mathbf{x})}_{\text{Complex}} + \underbrace{2 \cosh(3 N_\tau \mu / \gamma^2)}_{\text{Real}} \right]$$

Complex

Real

→ Less phase at larger μ !

While we have sign problem, it should be suppressed especially at larger μ . Let's try

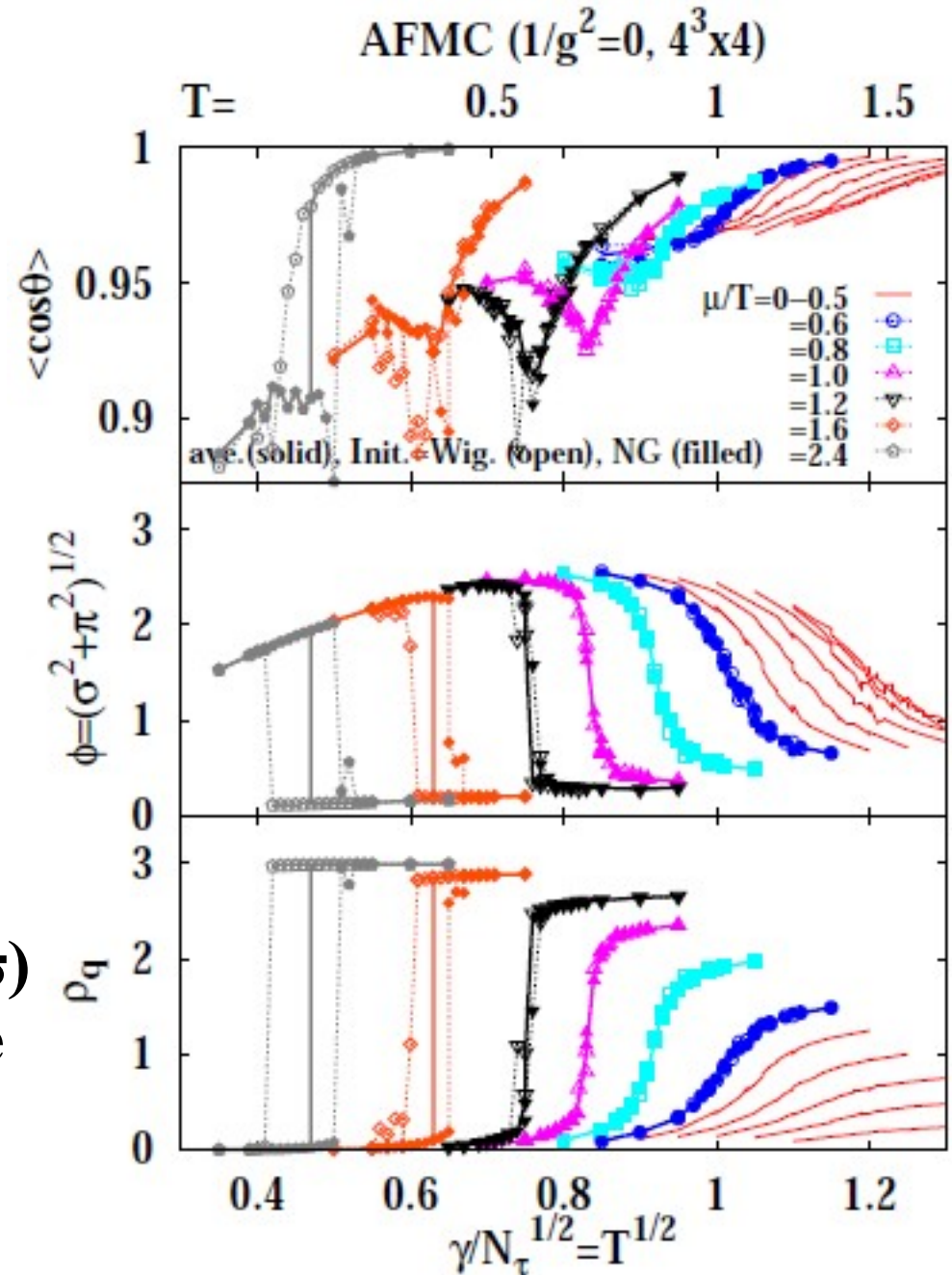
AFMC phase diagram

AFMC Simulations

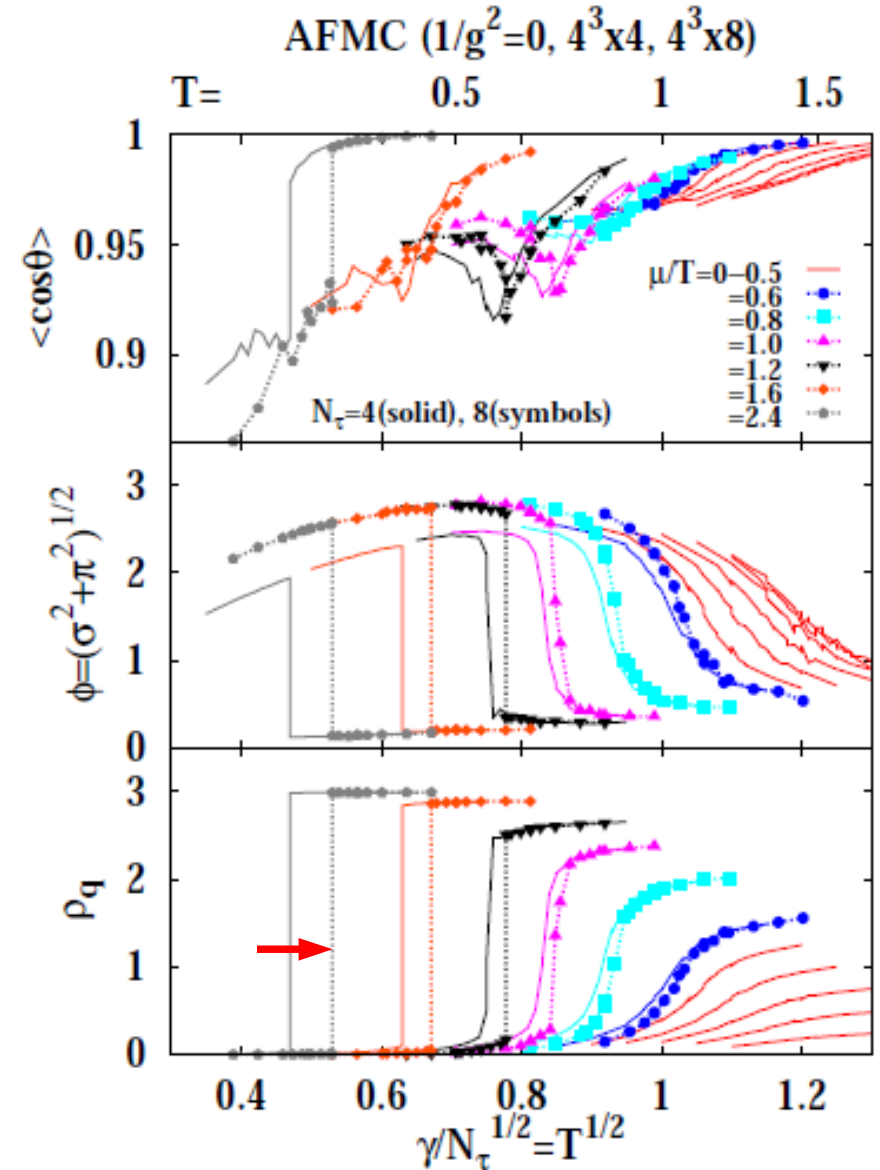
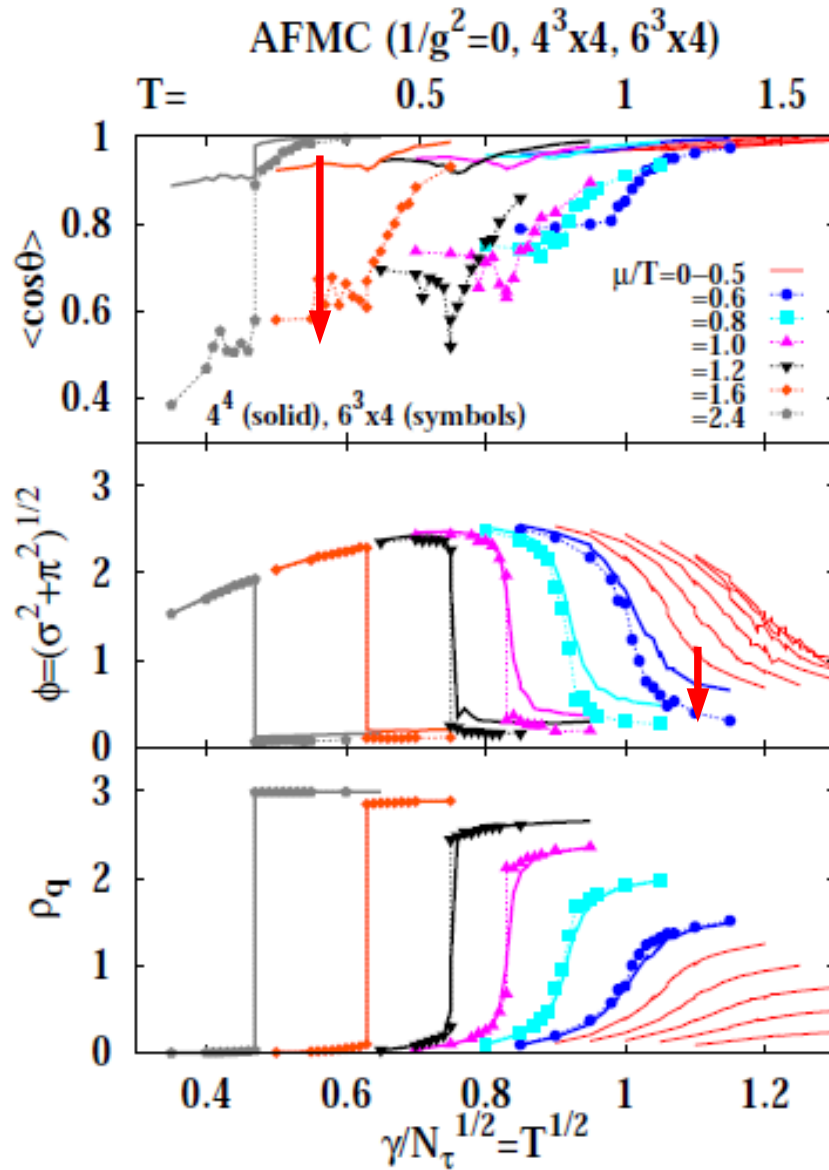
- Unrooted staggered fermion in the chiral limit ($m_0=0$)
- Lattice size: $4^3 \times 4$, $6^3 \times 4$, $4^3 \times 8$, $4^3 \times 12$
- Fixed fugacity: $\mu/T = 0, 0.1, \dots, 0.5, 0.6, 0.8, 1.2, 1.6, 2.4$
- Temperature assignment
 $T = \gamma^2 / N_\tau$ (rather than $T = \gamma / N_\tau$)
Bilic, Karsch, Redlich ('92), Bilic, Demeterfi, Petersson ('92)
- MC samples : 200 k $\sim$ 1 M sweeps
 - Dynamical var. = $\sigma(k, \tau), \pi(k, \tau)$
Det. is evaluated from $\sigma(x, \tau), \pi(k, \tau)$
→ Generate new $\sigma(k, \tau), \pi(k, \tau)$ for a given τ ,
and Metropolis sampling is carried out.
- Machine = Core i7 PC
- To do: Parallel computing, FFT, Jack knife error estimate, larger lattice,

Average Sign Factor, Chiral Condensate, Quark Density

- $4^3 \times 4$ results
- Average sign factor
 $\langle \cos \theta \rangle \geq 0.9$
in $4^3 \times 4$ lattice.
 - $\langle \cos \theta \rangle$ becomes small
in transition region.
- Chiral condensate
quickly decreases around γ_c .
- Quark number density
quickly increases around γ_c .
- Results from “NG start” (large σ)
and “Wigner start” (small σ) are
different with small sampling #.



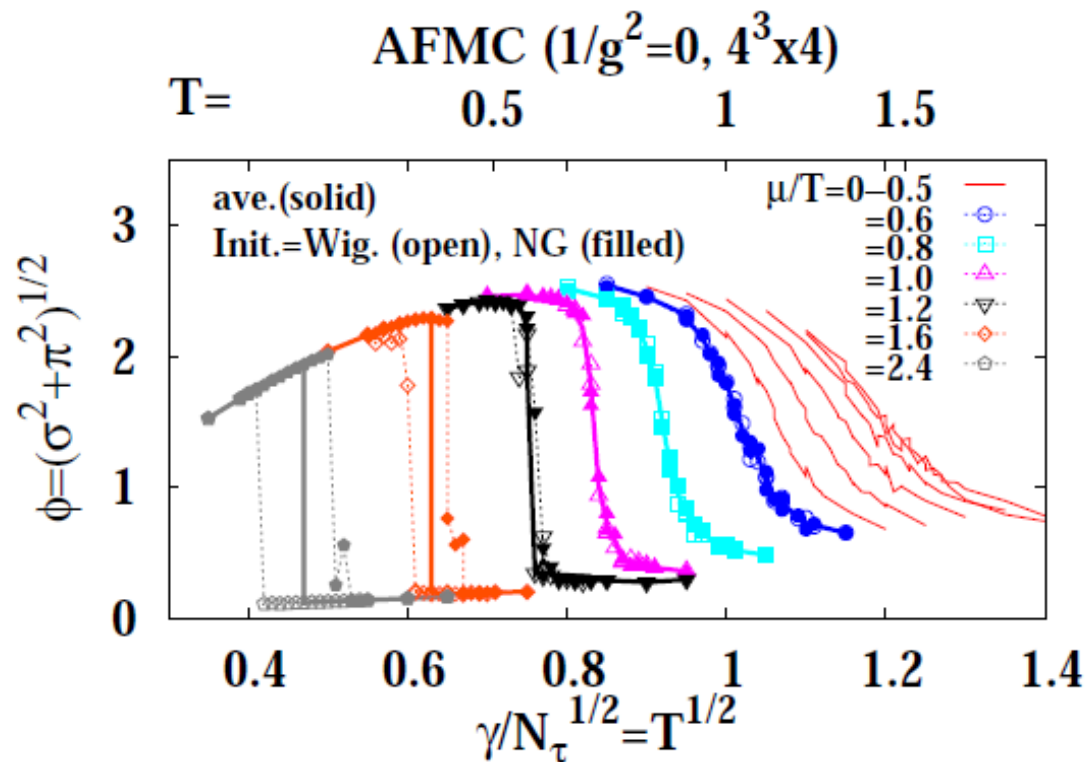
“Larger” Lattice Results



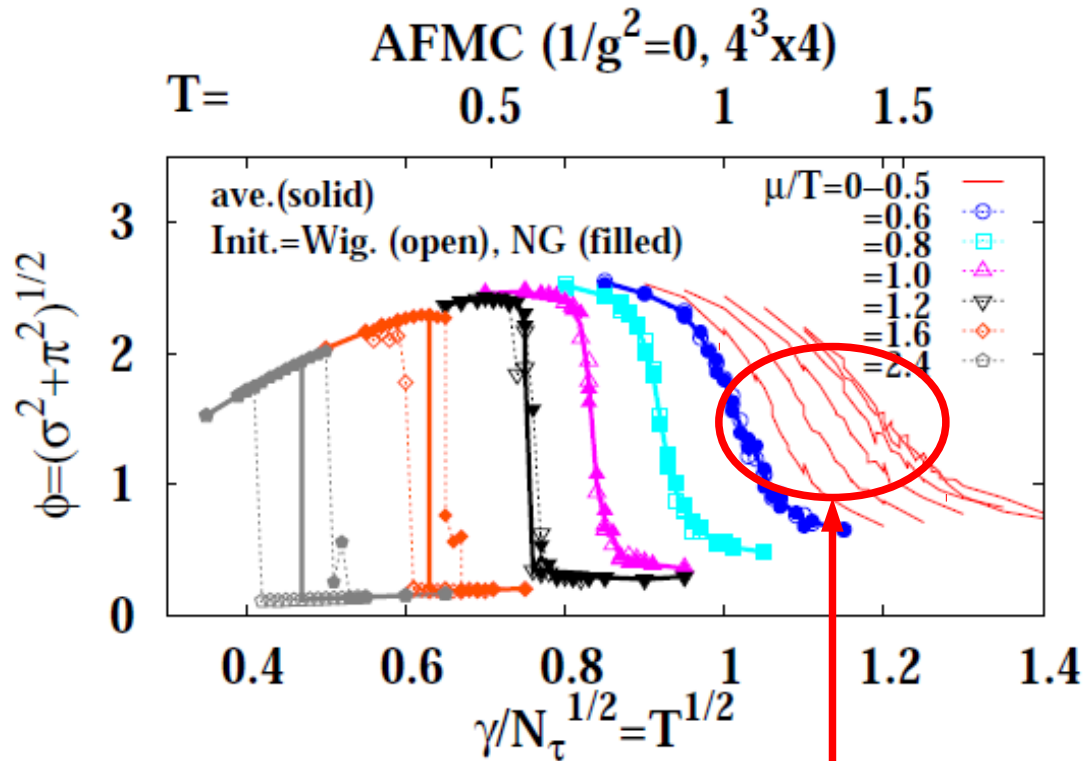
$6^3 \times 4$: Smaller $\langle \cos \theta \rangle$, Sharper trans.,
small fluc. in Wig. phase

$4^3 \times 8$: Sharper trans.,
Larger σ , Larger T_c ,

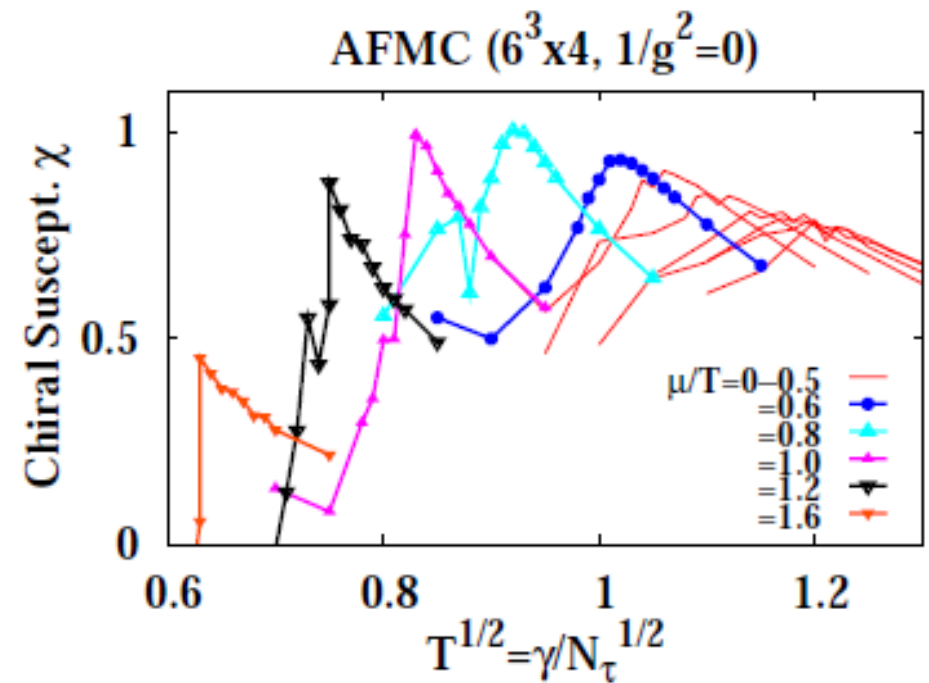
How to determine the phase boundary ?



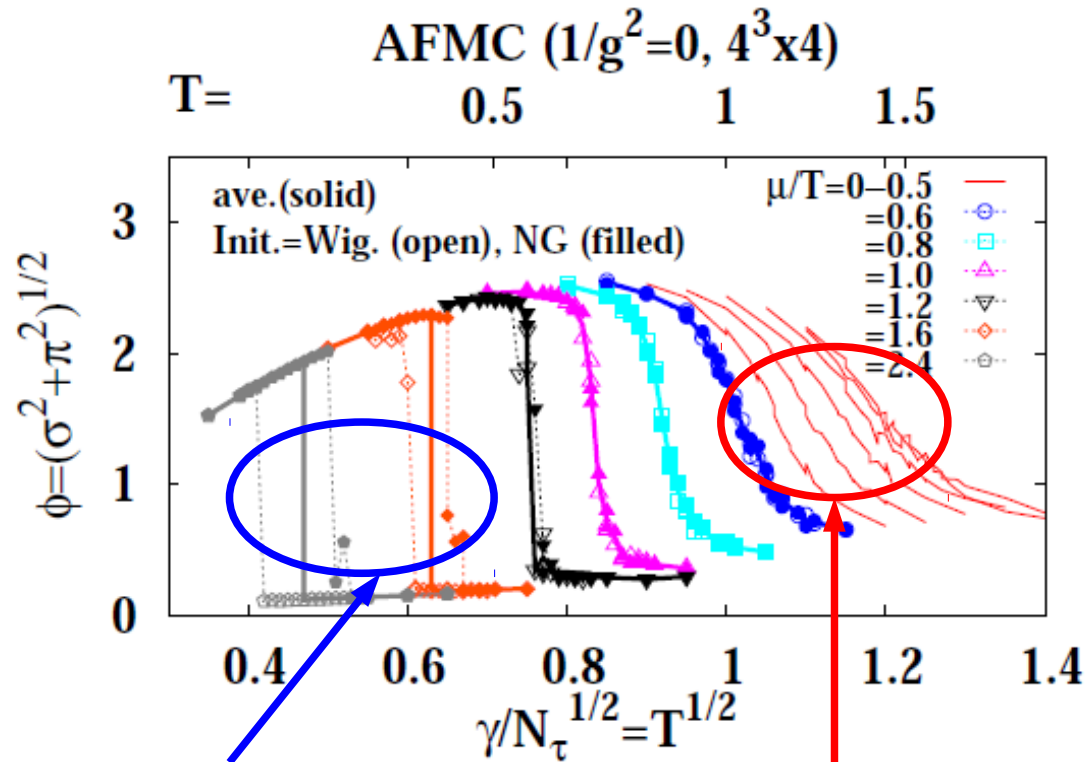
How to determine the phase boundary ?



(would-be)
2nd order
→ Suscept.
peak

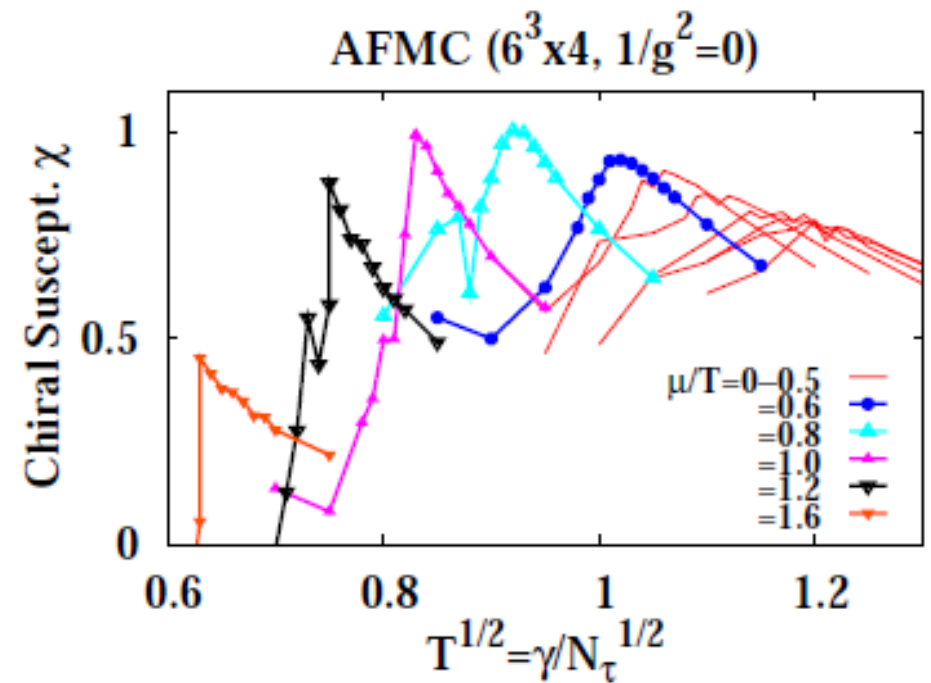
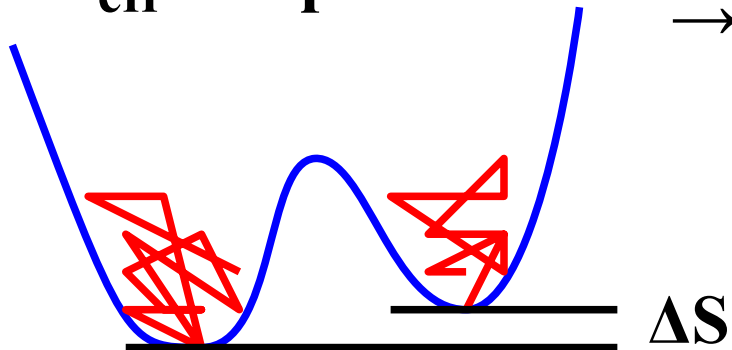


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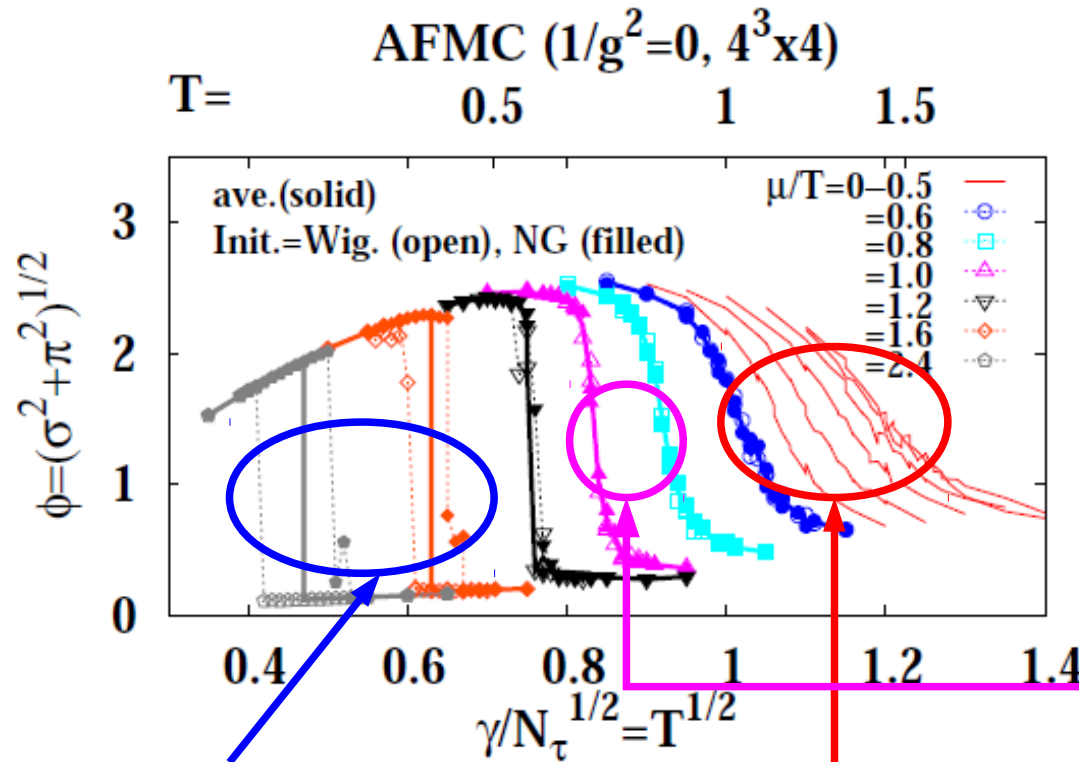


Strong 1st order
→ S_{eff} comparison

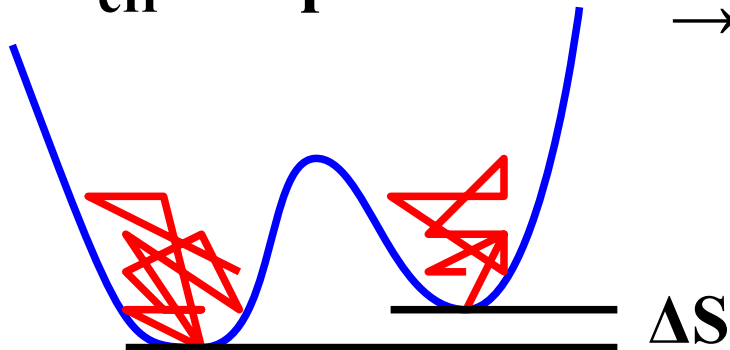
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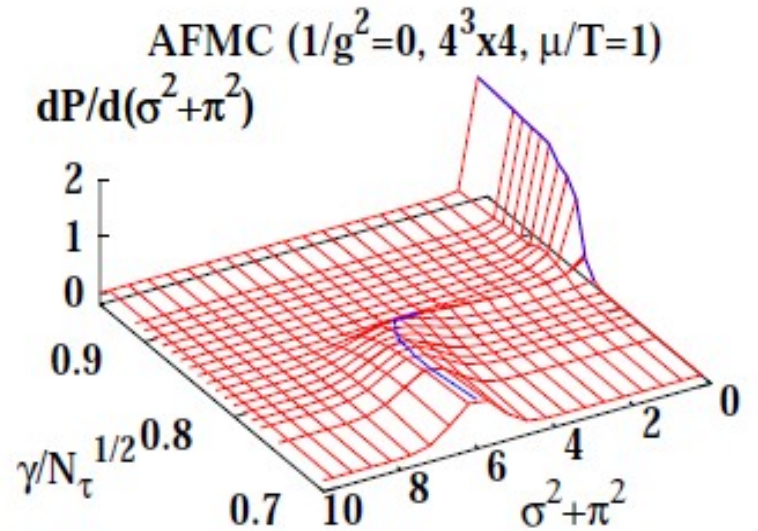
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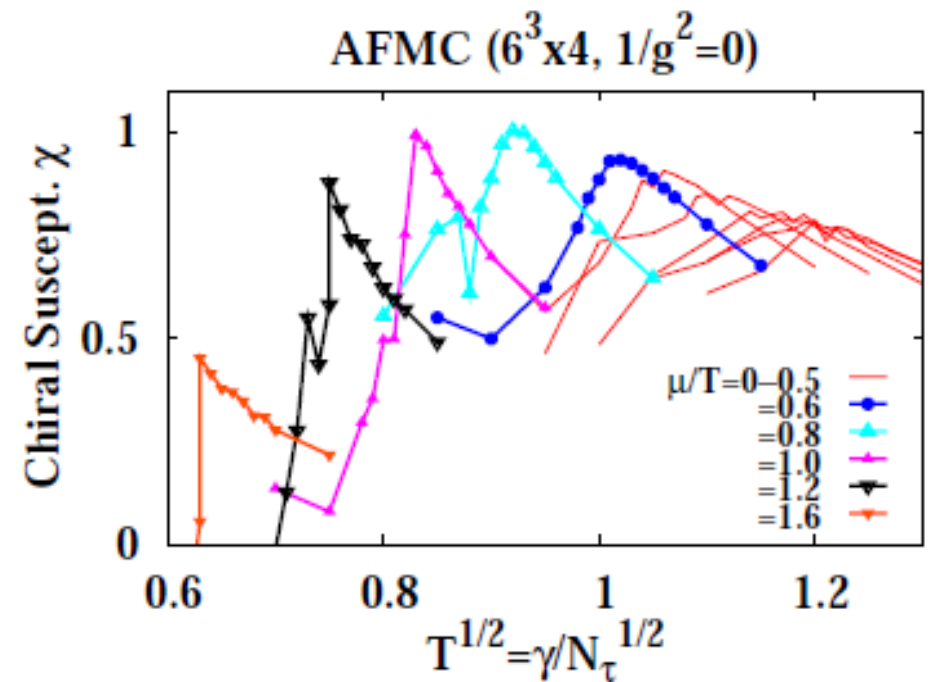
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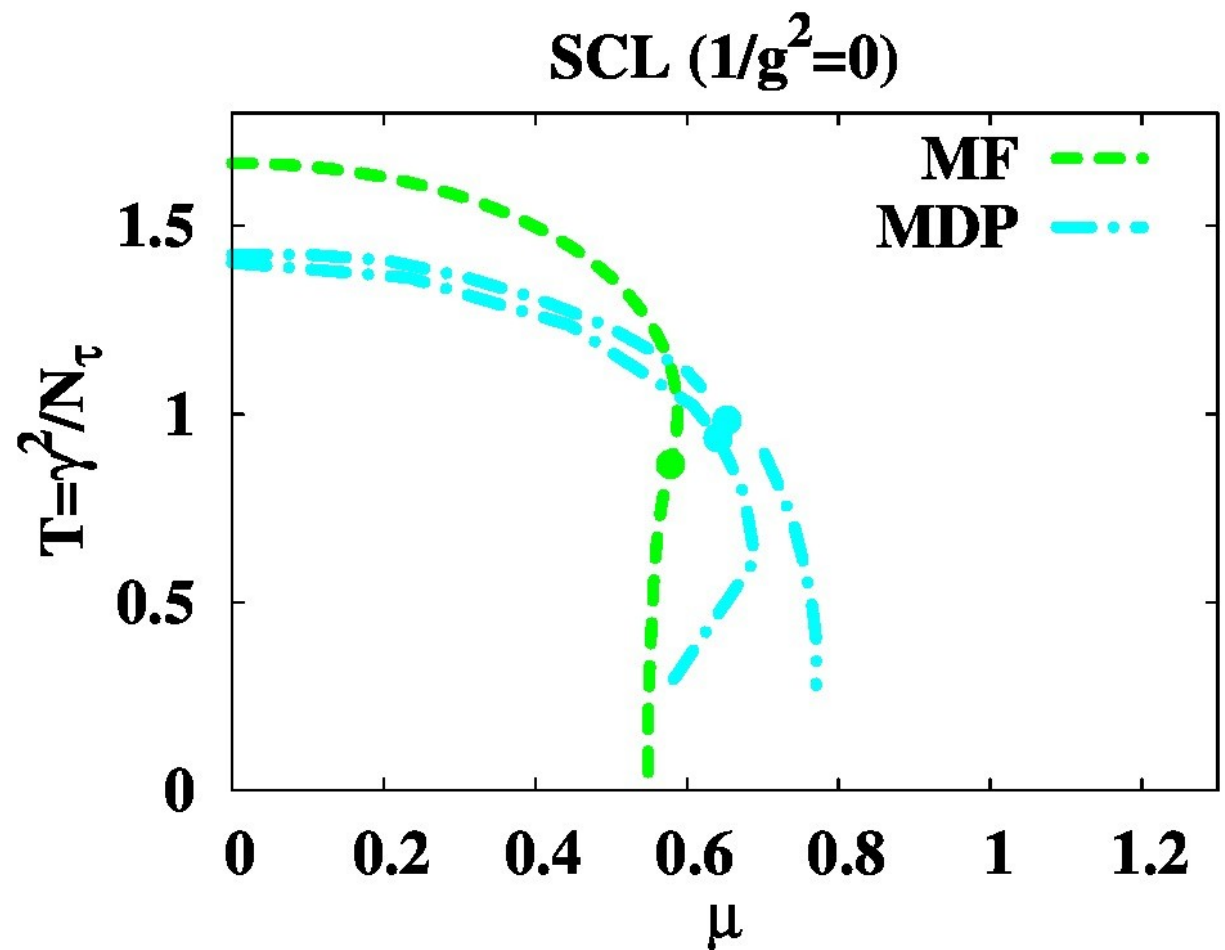
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CP region
→ σ distribution



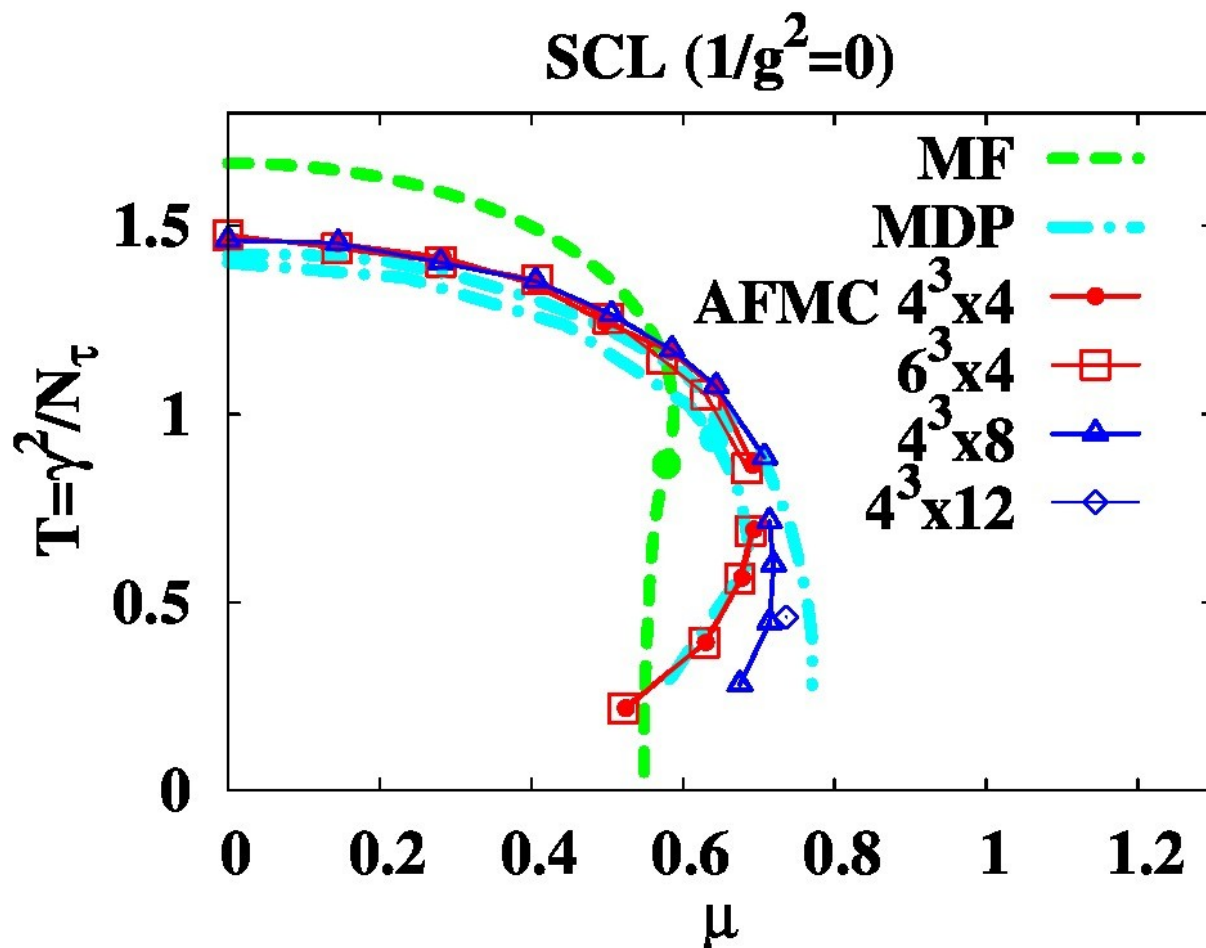
Phase Diagram



Phase Diagram

- AFMC predicts smaller T_c ($\mu=0$),
and extended Nambu-Goldstone phase at finite μ
compared with mean field results.
- AFMC results are almost consistent with MDP results.
de Forcrand, Fromm ('10),
de Forcrand, Unger ('11)

- $N_\tau=4$ results
~ MDP ($N_\tau=4$)
- $N_\tau=\infty$ Extrapolation
~ Continuous Time
MDP

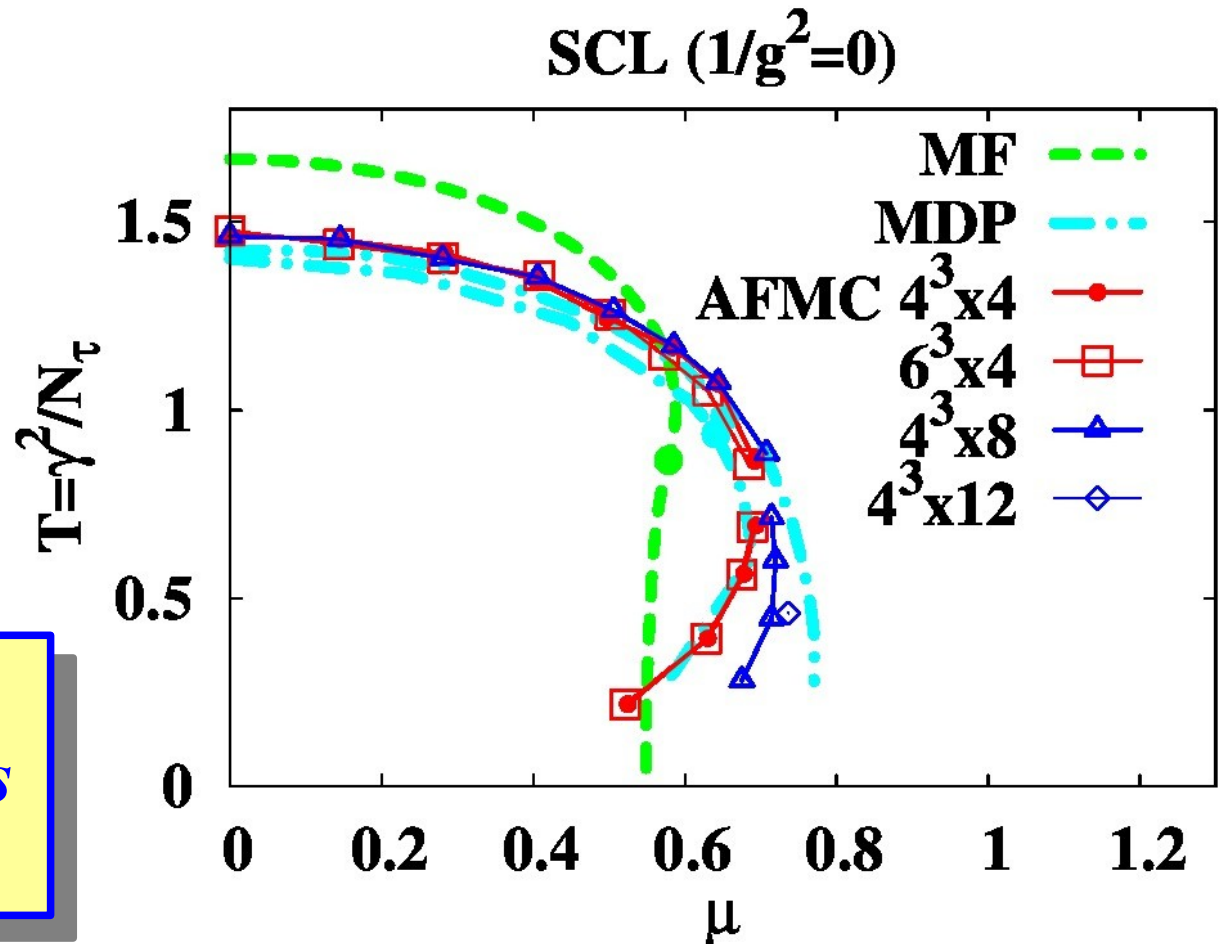


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MDP

*AFMC can be
an alternative to discuss
finite density LQCD !*



Summary

- **Strong Coupling Lattice QCD has been useful in these 40 years.**
Misumi (Tue), Kimura (Tue), Nakano (Wed), Unger (Tue, Thu)
- **We have proposed an auxiliary field MC method (AFMC), which simulates the effective action at strong coupling exactly.**
 - LO in strong coupling ($1/g^0$) and $1/d$ ($1/d^0$) expansion.
 - Phase boundary is moderately modified from MF results by fluctuations, if $T = \gamma^2/N_\tau$ scaling is adopted, as in MDP.
 - Sign problem is mild in small lattice ($\langle \cos \theta \rangle \sim (0.9-1)$ for 4^4), due to the phase cancellation coming from nearest neighbor interaction.
 - Sign problem is less severe at larger μ (except for transition region).
 - Extension to NLO SC-LQCD is straightforward.
Note: NLO & NNLO SC-LQCD with Polyakov loop effects reproduces MC results of T_c at $\mu=0$.
- **To do: Larger lattice, finite coupling, other Fermion, higher $1/d$ terms including baryonic action, chiral Polyakov coupling.**

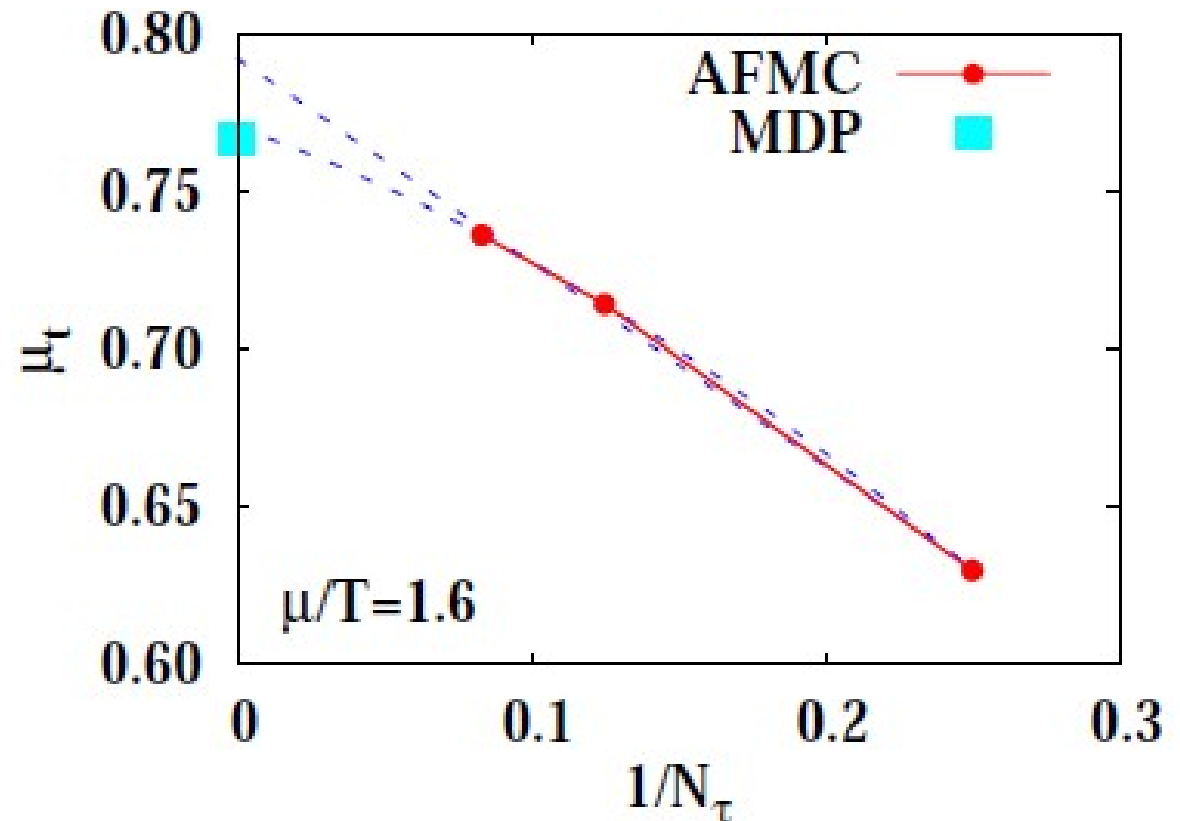
Thank you

Extrapolation to $N_\tau = \infty$ (Continuous Time)

- Extrapolation of $N_\tau=4, 8, 12$ AFMC results to ∞ agrees with Continuous time MDP results.

de Forcrand, Unger ('11)

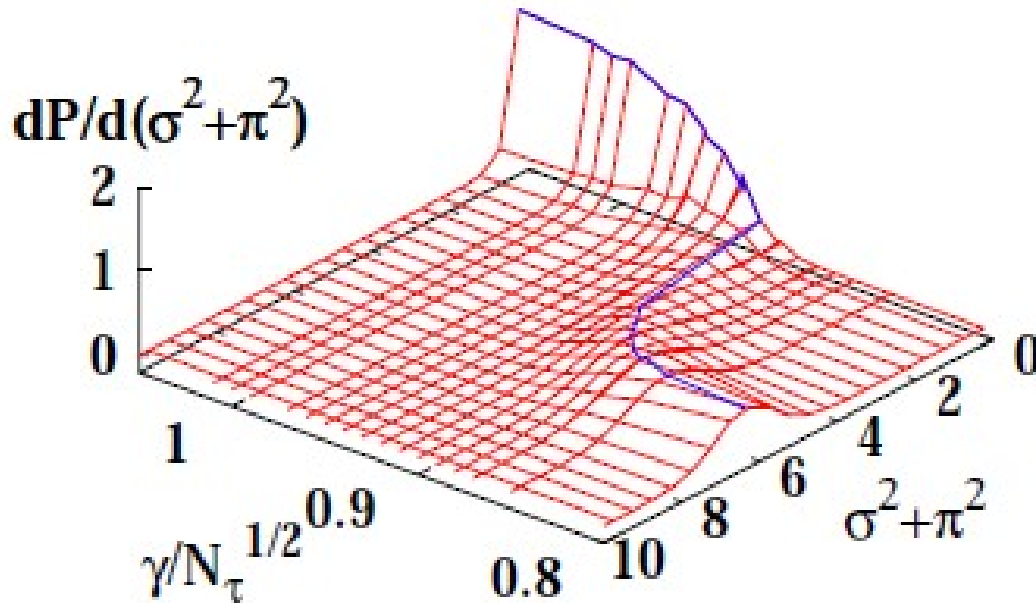
→ CT-MDP result is confirmed.



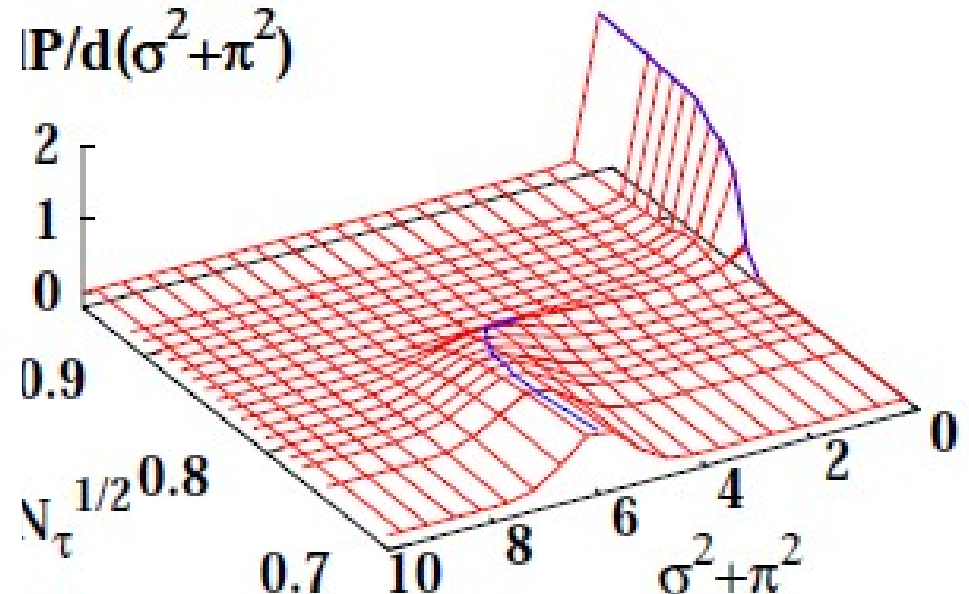
Second or First Order ?

- Probability distribution in $\sigma^2 + \pi^2$
→ Hint to distinguish 2nd (one peak) and 1st order (two peak) transition
- AFMC → CP is suggested in the region $0.8 < \mu/T < 1.0$
MDP → CP is around $\mu/T \sim 0.7$

AFMC ($1/g^2=0$, $4^3 \times 4$, $\mu/T=0.8$)



AFMC ($1/g^2=0$, $4^3 \times 4$, $\mu/T=1$)



Clausius-Clapeyron Relation

- First order phase boundary → two phases coexist

$$P_h = P_q \rightarrow dP_h = dP_q \rightarrow \frac{d\mu}{dT} = -\frac{s_q - s_h}{\rho_q - \rho_h}$$

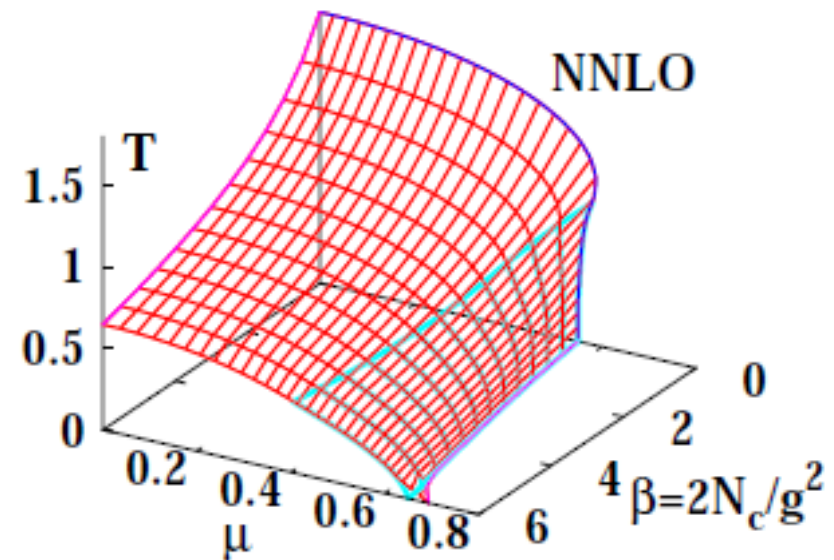
$$dP_h = \rho_h d\mu + s_h dT, \quad dP_q = \rho_q d\mu + s_q dT$$

- Continuum theory

→ Quark matter has larger entropy and density ($d\mu/dT < 0$)

- Strong coupling lattice

- ◆ SCL: Quark density is larger than half-filling, and “Quark hole” carries entropy → $d\mu/dT > 0$
- ◆ NLO, NNLO → $d\mu/dT < 0$



AO, Miura, Nakano, Kawamoto ('09)

SC-LQCD with Fermions & Polyakov loop: Outline

Effective Action & Effective Potential (free energy density)

$$Z = \int D[\chi, \bar{\chi}, U_0, U_j] \exp \left[-S_{\text{LQCD}} \right]$$

Spatial link integral

$$\int DU U_{ab} U_{cd}^+ = \delta_{ad} \delta_{bc} / N$$

$$= \int D[\chi, \bar{\chi}, U_0] \exp \left[-S_{\text{eff}}[\chi, \bar{\chi}, U_0, \Phi_{\text{stat.}}] \right]$$

Bosonization & MF Approx.

SCL

NLO

NNLO

Polyakov loop

$$\approx \exp \left(-V F_{\text{eff}}(\Phi_{\text{stat}}; T, \mu) / T \right)$$

Fermion + U_0 integral

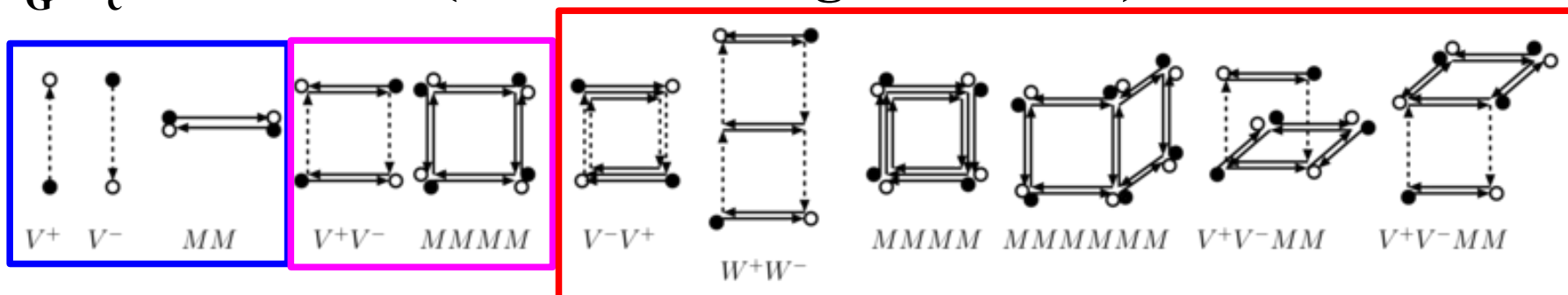
SC-LQCD with Fermions

Effective Action with finite coupling corrections

Integral of $\exp(-S_G)$ over spatial links with $\exp(-S_F)$ weight $\rightarrow S_{\text{eff}}$

$$S_{\text{eff}} = S_{\text{SCL}} - \log \langle \exp(-S_G) \rangle = S_{\text{SCL}} - \sum_{n=1} \frac{(-1)^n}{n!} \langle S_G^n \rangle_c$$

$\langle S_G^n \rangle_c$ = Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\text{eff}} = \frac{1}{2} \sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d} \sum_{x,j>0} [MM]_{j,x}$$

SCL (Kawamoto-Smit, '81)

$$+ \frac{1}{2} \frac{\beta_\tau}{2d} \sum_{x,j>0} [V^+V^- + V^-V^+]_{j,x} - \frac{1}{2} \frac{\beta_s}{d(d-1)} \sum_{x,j>0, k>0, k \neq j} [MMMM]_{jk,x}$$

NLO (Faldt-Petersson, '86)

$$- \frac{\beta_{\tau\tau}}{2d} \sum_{x,j>0} [W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)} \sum_{x,j>0, |k|>0, |l|>0, |k| \neq j, |l| \neq j, |l| \neq |k|} [MMMM]_{jk,x} [MM]_{j,x+\hat{l}}$$

$$+ \frac{\beta_{\tau s}}{8d(d-1)} \sum_{x,j>0, |k| \neq j} [V^+V^- + V^-V^+]_{j,x} \left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}} \right)$$

NNLO (Nakano, Miura, AO, '09)

SC-LQCD Eff. Pot. with Fermions & Polyakov loop

- Effective potential [free energy density, NLO + LO(Pol. loop)]

$$\mathcal{F}_{\text{eff}}(\Phi; T, \mu) \equiv -(T \log \mathcal{Z}_{\text{LQCD}})/N_s^d = \mathcal{F}_{\text{eff}}^{\chi} + \mathcal{F}_{\text{eff}}^{\text{Pol}}$$

aux. fields
w.f. ren.
zero point E.
thermal

$$\mathcal{F}_{\text{eff}}^{\chi} \simeq \left(\frac{d}{4N_c} + \beta_s \varphi_s \right) \sigma^2 + \frac{\beta_s \varphi_s^2}{2} + \frac{\beta_{\tau}}{2} (\varphi_{\tau}^2 - \omega_{\tau}^2) - N_c \log Z_{\chi} - N_c E_q - T (\log \mathcal{R}_q(T, \mu) + \log \mathcal{R}_{\bar{q}}(T, \mu))$$

$$\mathcal{R}_q(T, \mu) \equiv 1 + e^{-N_c(E_q - \tilde{\mu})/T} + N_c \left(L_{p,x} e^{-(E_q - \tilde{\mu})/T} + \bar{L}_{p,x} e^{-2(E_q - \tilde{\mu})/T} \right)$$

$$\mathcal{F}_{\text{eff}}^{\text{Pol}} \simeq -2T d N_c^2 \left(\frac{1}{g^2 N_c} \right)^{1/T} \bar{\ell}_p \ell_p - T \log \mathcal{M}_{\text{Haar}}(\ell_p, \bar{\ell}_p)$$

quad. coef.
Haar measure

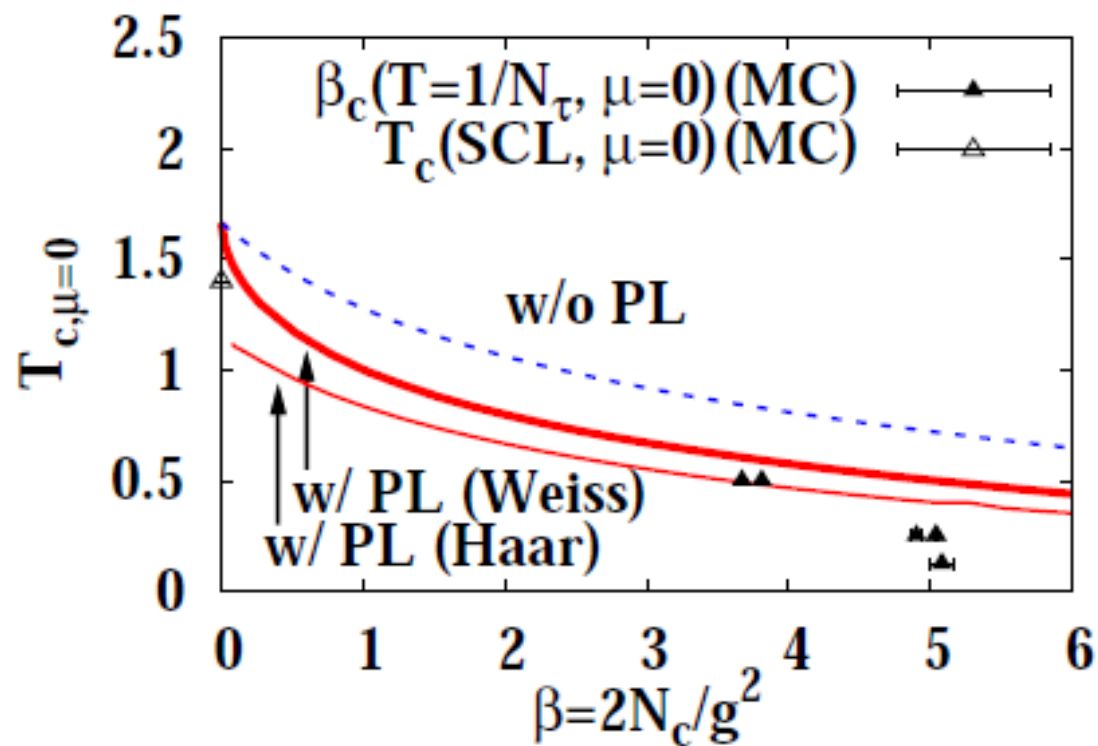
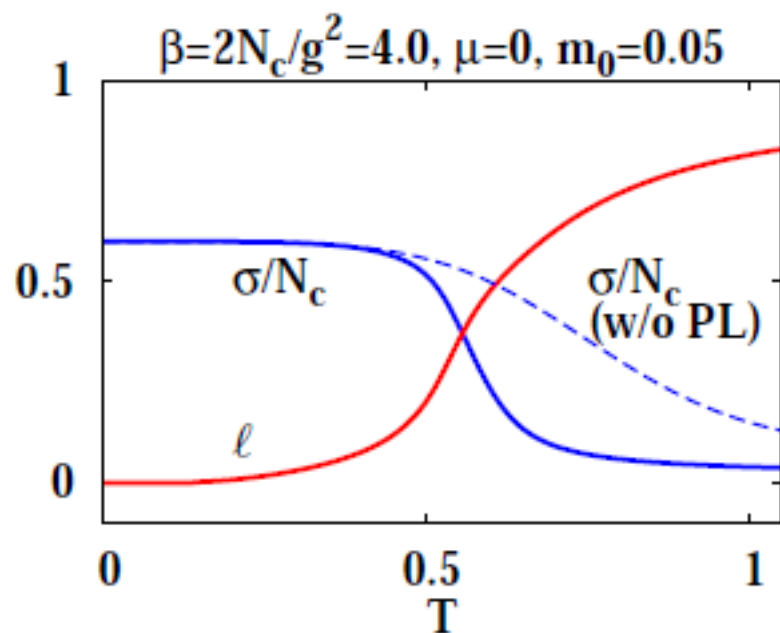
- Strong coupling lattice QCD with Polyakov loop (P-SC-LQCD)
= Polyakov loop extended Nambu-Jona-Lasino (PNJL) model
(Haar measure method, quadratic term fixed)
+ higher order terms in aux. fields
- quark momentum integral

P-SC-LQCD at $\mu=0$

T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]

- P-SC-LQCD reproduces $T_c(\mu=0)$ in the strong coupling region
($\beta = 2N_c/g^2 \leq 4$)

MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), $N_\tau=2$ (de Forcrand, private), $N_\tau=4$ (Gottlieb et al.('87), Fodor-Katz ('02)), $N_\tau=8$ (Gavai et al.('90))



Lattice Unit

Approximations in Pol. loop extended SC-LQCD

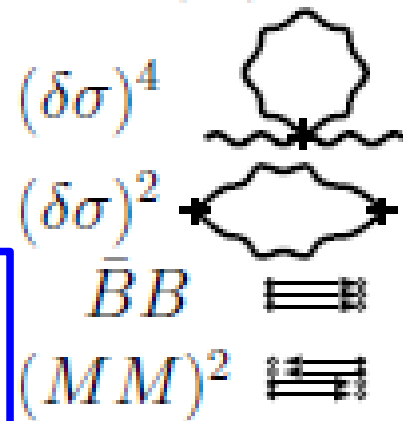
Fluctuations

(B) fluc., (C) 1/d

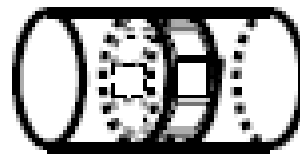
$1/g^2$ expansion
with Pol. loops

1/d
expansion

d=spatial
dim.



S_{PL} (LO)

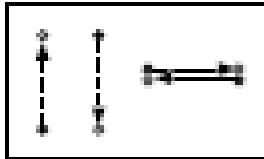


$1/g^2$
(with Pol. loop)

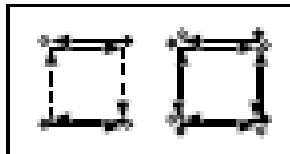
Gocksh-Ogilvie
(A) P-SC-LQCD

Nakano, Miura, AO ('11)

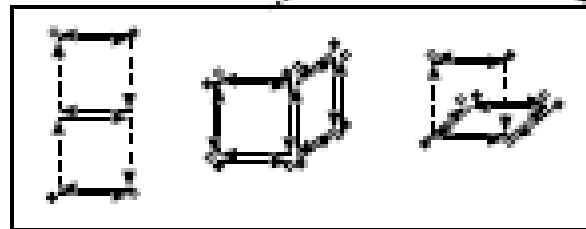
SC-LQCD



SCL



NLO



NNLO

$1/g^2$
(with quarks)

$1/g^2$ expansion
with quarks

■ Strong coupling expansion

- Fermion terms: LO($1/g^0$, SCL), NLO($1/g^2$), NNLO ($1/g^4$)
- Plaquette action: LO ($1/g^{2N\tau}$)

■ Large dimensional approximation

- $1/d$ expansion (d =spatial dim.)
→ Smaller quark # configs. are preferred.
 $\sum_j M_x M_{x+j} = O(1/d^0) \rightarrow M \propto d^{-1/2} \rightarrow \chi \propto d^{-1/4}$
- Only LO ($1/d^0$) terms are mainly evaluated.

■ (Unrooted) staggered Fermion

- $N_f=4$ in the continuum limit.

■ Mean field approximation

- Auxiliary fields are assumed to be constant.

Introduction of Auxiliary Fields

$$\begin{aligned}
 S^{(s)} &= -\frac{1}{4N_c\gamma^2} \sum_{x,j} M_x M_{x+\hat{j}} = -\frac{L^3}{4N_c\gamma^2} \sum_{\tau,\mathbf{k}} f_M(\mathbf{k}) \tilde{M}_{\mathbf{k}}(\tau) \tilde{M}_{-\mathbf{k}}(\tau) \\
 &= \frac{L^3}{4N_c\gamma^2} \sum_{\tau,\mathbf{k}, f_M(\mathbf{k})>0} f_M(\mathbf{k}) \left[\varphi_{\mathbf{k}}(\tau)^2 + \phi_{\mathbf{k}}(\tau)^2 + \varphi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} + \tilde{M}_{-\mathbf{k}}) - i\phi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} - \tilde{M}_{-\mathbf{k}}) \right. \\
 &\quad \left. + \varphi_{\bar{\mathbf{k}}}(\tau)^2 + \phi_{\bar{\mathbf{k}}}(\tau)^2 + i\varphi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} + \tilde{M}_{-\bar{\mathbf{k}}}) + \phi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} - \tilde{M}_{-\bar{\mathbf{k}}}) \right] \\
 &= \frac{\Omega}{2N_c\gamma^2} \sum_{k, f_M(\mathbf{k})>0} f_M(\mathbf{k}) [\sigma_k^* \sigma_k + \pi_k^* \pi_k] + \frac{1}{2N_c\gamma^2} \sum_x M_x [\sigma(x) + i\varepsilon(x)\pi(x)]
 \end{aligned}$$

$$\Omega = L^3 N_\tau$$

$$\sigma(x) = \sum_{k, f_M(\mathbf{k})>0} f_M(\mathbf{k}) e^{ikx} \sigma_k, \quad \pi(x) = \sum_{k, f_M(\mathbf{k})>0} f_M(\mathbf{k}) e^{ikx} \pi_k$$

$$\sigma_k = \varphi_k + i\phi_k, \quad \pi_k = \varphi_{\bar{k}} + i\phi_{\bar{k}}$$

$$V_{x,y} = \frac{1}{2} \sum_j \left(\delta_{x+\hat{j},y} + \delta_{x-\hat{j},y} \right), \quad f_M(\mathbf{k}) = \sum_j \cos k_j, \quad \bar{\mathbf{k}} = \mathbf{k} + (\pi, \pi, \pi)$$

Fermion Determinant

Falgt, Petersson, 1986

- Fermion action is separated to each spatial point and bi-linear
→ Determinant of $N_\tau \times N_c$ matrix

$$\exp(-V_{\text{eff}}/T) = \int dU_0 \begin{vmatrix} I_1 & e^\mu & 0 & \dots & e^{-\mu} U^+ \\ -e^{-\mu} & I_2 & e^\mu & \dots & 0 \\ 0 & -e^{-\mu} & I_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -e^\mu U & \dots & -e^{-\mu} & \dots & I_N \end{vmatrix} \quad \text{Nc x N}\tau$$

$$= \int dU_0 \det \left[\underbrace{X_N[\sigma] \otimes \mathbf{1}_c}_{\text{magenta}} + \underbrace{e^{-\mu/T} U^+}_{\text{blue}} + \underbrace{(-1)^{N_\tau} e^{\mu/T} U}_{\text{red}} \right] \quad \text{Nc}$$

$$= X_N^3 - 2 X_N + 2 \cosh(3 N_\tau \mu)$$

$$I_\tau/2 = [\sigma(x) + i\varepsilon(x)\pi(x)]/2 N_c \gamma^2 + m_0/\gamma$$

$$X_N = B_N + B_{N-2}(2; N-1)$$

$$B_N = I_N B_{N-1} + B_{N-2}$$

$$B_N = \begin{vmatrix} I_1 & e^\mu & 0 & \dots & 0 \\ -e^{-\mu} & I_2 & e^\mu & \dots & 0 \\ 0 & -e^{-\mu} & I_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -e^{-\mu} & \dots & I_N \end{vmatrix}$$

Constant I : $X_N = 2 \cosh(\text{arcsinh}(I/2)/T)$

Results (2): Susceptibility and Quark density

■ Weight factor $\langle \cos \theta \rangle$

$$\langle \cos \theta \rangle = Z / Z_{\text{abs}}$$

$$Z = \int D\sigma_k D\pi_k \exp(-S_{\text{eff}})$$

$$= \int D\sigma_k D\pi_k \exp(-\text{Re } S_{\text{eff}}) e^{i\theta}$$

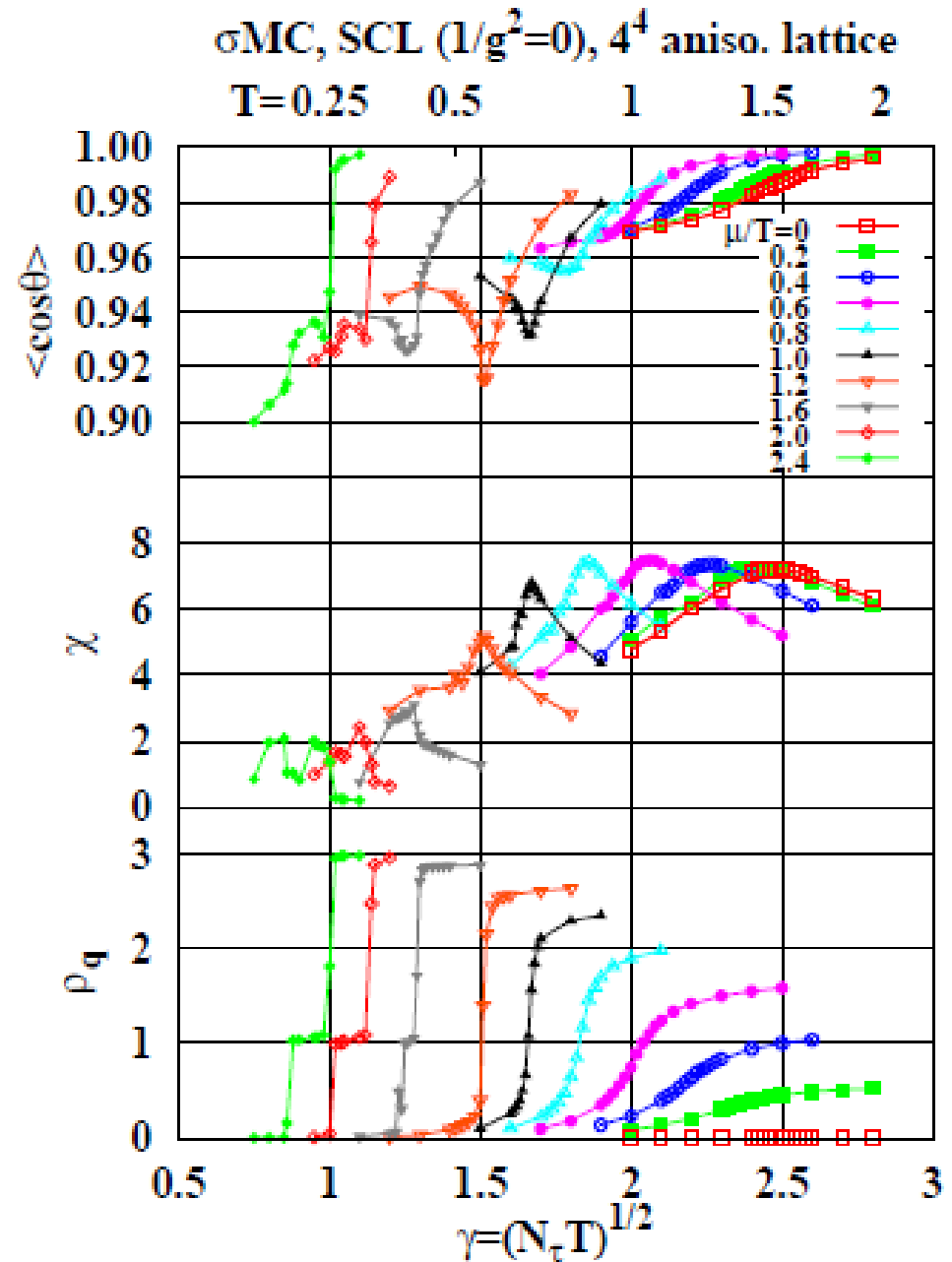
$$Z_{\text{abs}} = \int D\sigma_k D\pi_k \exp(-\text{Re } S_{\text{eff}})$$

■ Chiral susceptibility

$$\chi = -\frac{1}{L^3} \frac{\partial^2 \log Z}{\partial m_0^2}$$

■ Quark number density

$$\rho_q = -\frac{1}{L^3} \frac{\gamma^2}{N_\tau} \frac{\partial \log Z}{\partial \mu}$$



Strong Coupling Lattice QCD: Pure Gauge

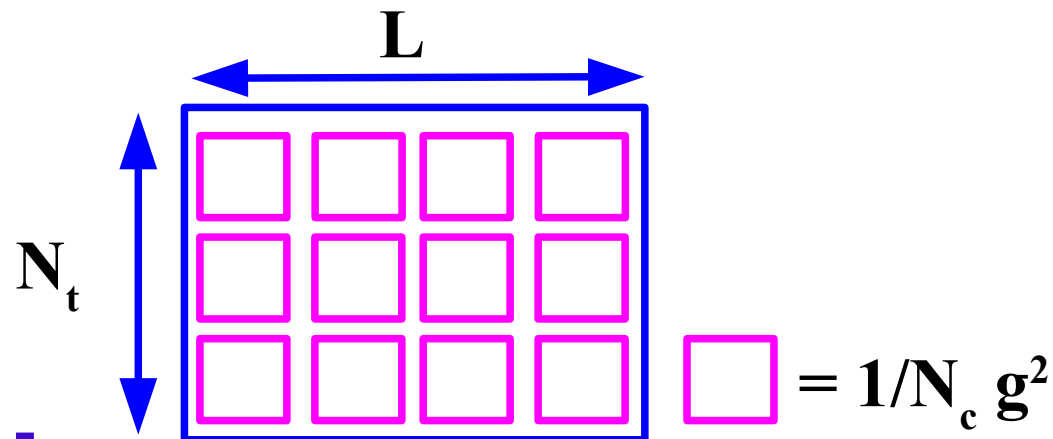
■ Quarks are confined in Strong Coupling QCD

● Strong Coupling Limit (SCL)

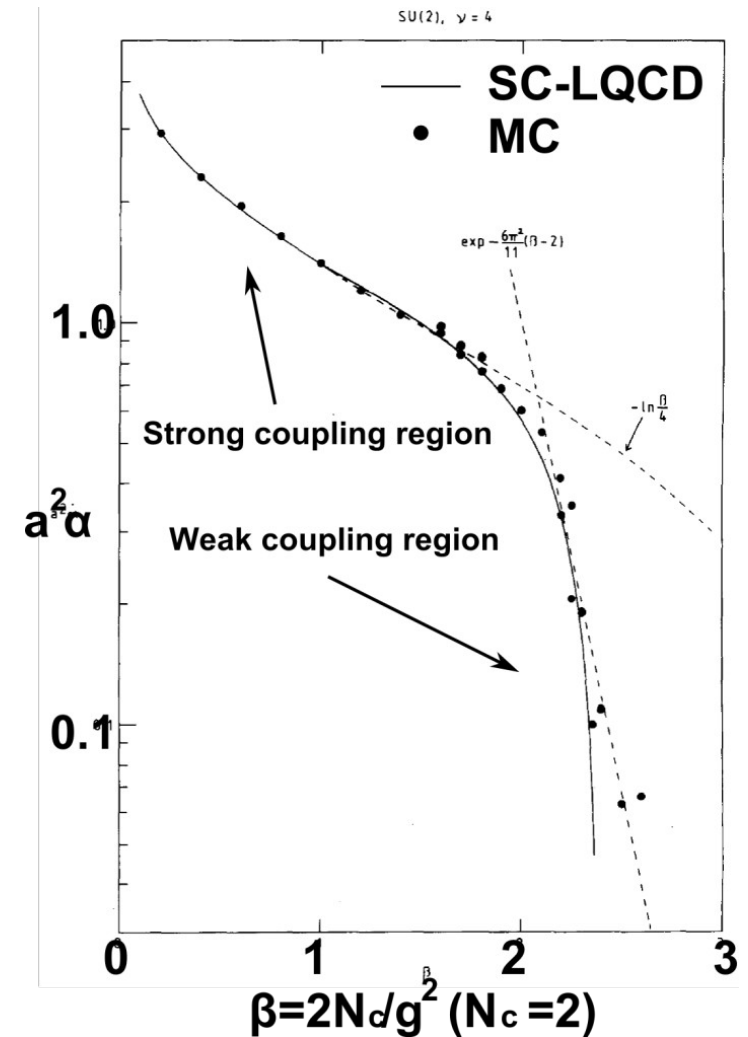
- Fill Wilson Loop with Min. # of Plaquettes
- Area Law (Wilson, 1974)

$$S_{\text{LQCD}} = -\frac{1}{g^2} \sum_{\square} \text{tr} [U_{\square} + U_{\square}^{\dagger}]$$

● Smooth Transition from SCL to pQCD in MC (Creutz, 1980; Munster 1980)



K. G. Wilson, PRD10(1974),2445
M. Creutz, PRD21(1980), 2308.
G. Munster, (1980, 1981)



Munster, '80

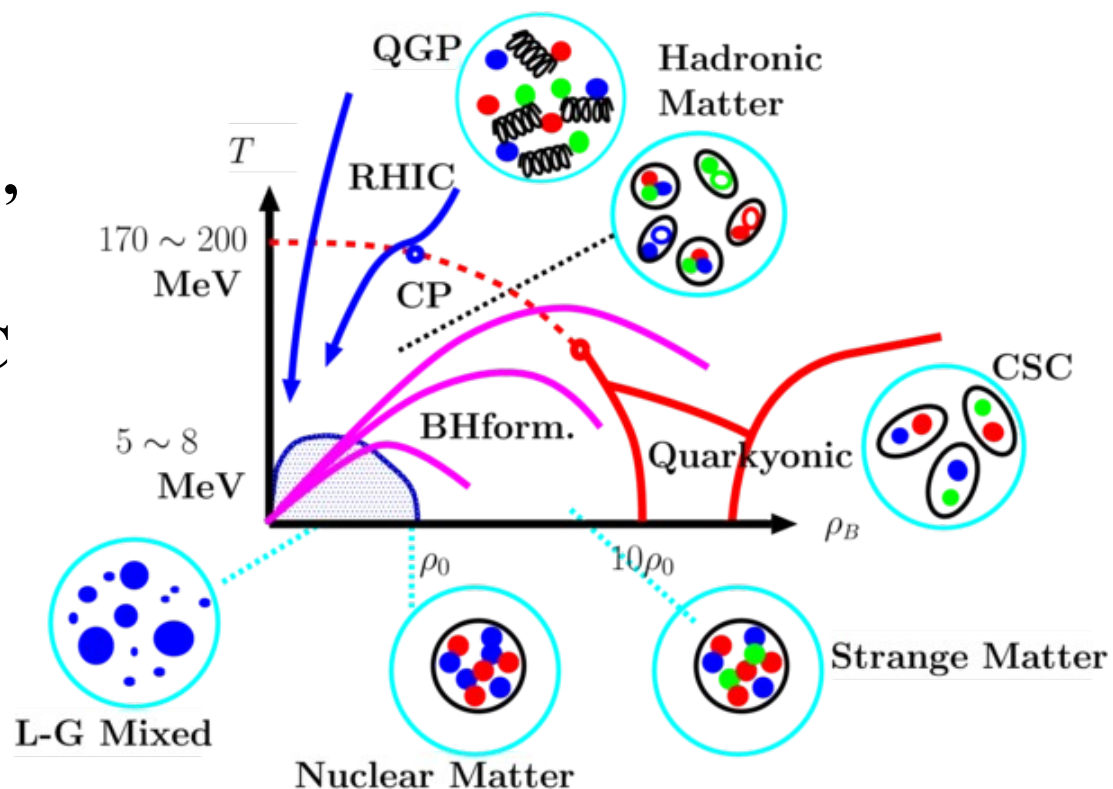
QCD Phase diagram

■ Phase transition at high T

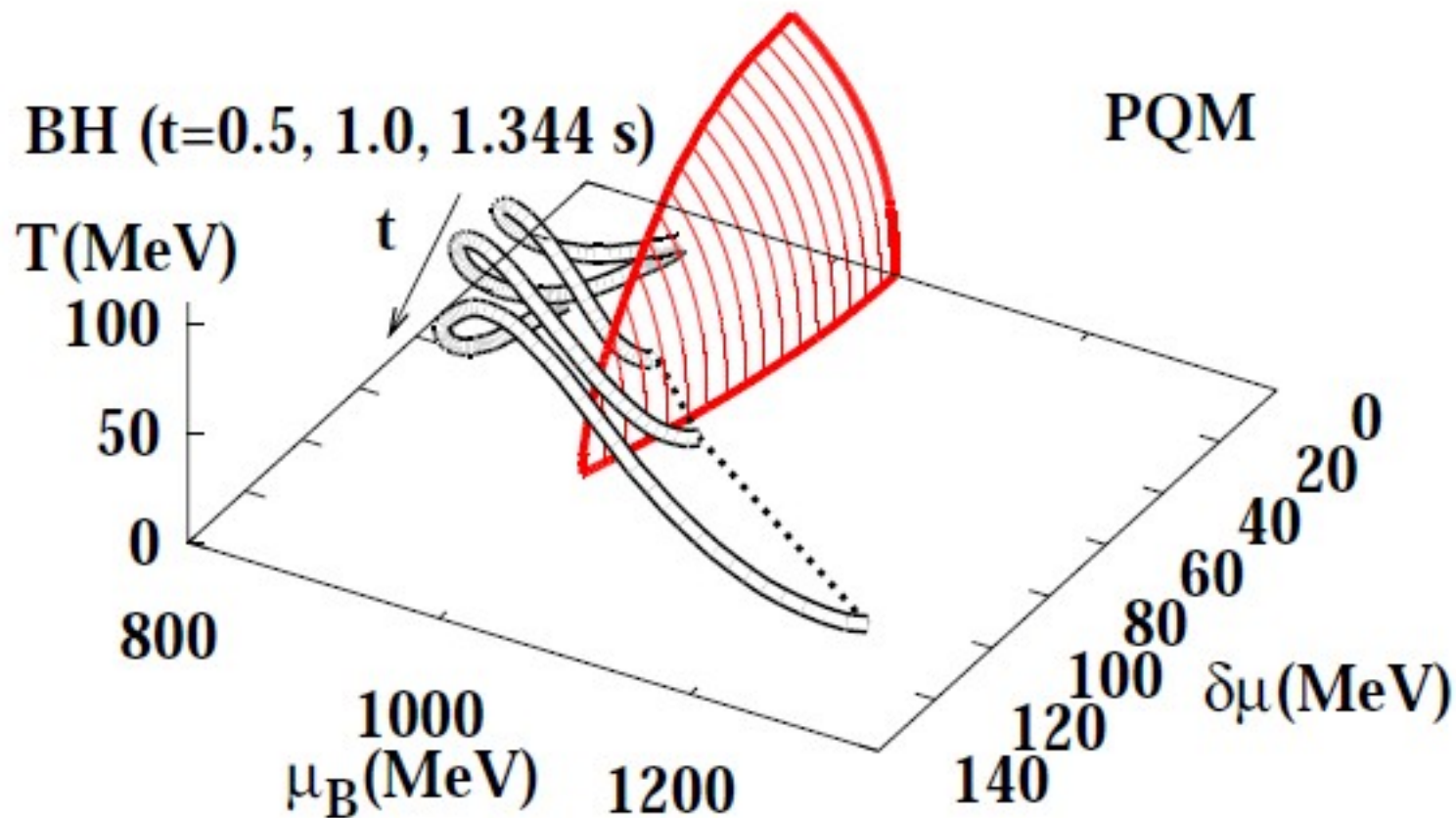
- Physics of early universe: Where do we come from ?
- RHIC, LHC, Lattice MC, pQCD,

■ High μ transition

- Physics of neutron stars:
Where do we go ?
- RHIC-BES, FAIR, J-PARC,
Astro-H, Grav. Wave, ...
- Sign problem in Lattice MC
→ Model studies,
Approximations,
Functional RG, ...



CP sweep during BH formation



*AO, H. Ueda, T. Z. Nakano, M. Ruggieri, K. Sumiyoshi,
PLB, to appear [arXiv:1102.3753 [nucl-th]]*

QCD based approaches to Cold Dense Matter

■ Effective Models

(P)NJL, (P)QM, Random Matrix, ...

E.g.: K. Fukushima, PLB 695('11)387 (PNJL+Stat.).

■ Functional (Exact, Wilsonian) RG

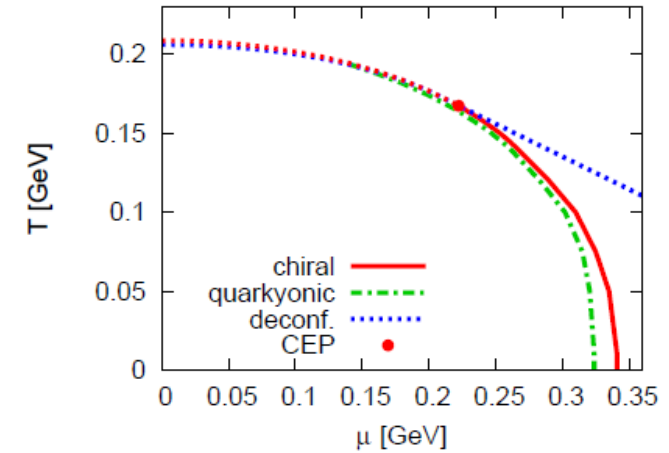
E.g.: T. K. Herbst, J. M. Pawłowski, B. J. Schaefer, PLB 696 ('11)58 (PQM-FRG).

■ Expansion / Extrapolation from $\mu=0$

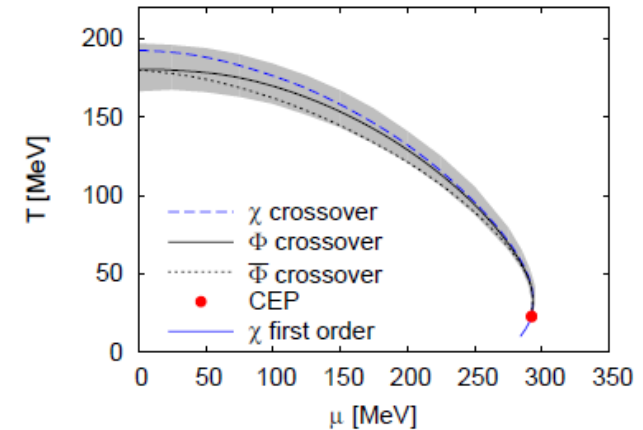
- AC, Taylor expansion, ... $\rightarrow \mu/T < 1$
- Cumulant expansion of θ dist.
(S. Ejiri, ...)

■ Strong Coupling Lattice QCD

- Mean field approaches
- Monomer-Dimer-Polymer (MDP) simulation



McLerran, Redlich, Sasaki ('09)



Herbst, Pawłowski, Schafer, ('11)

Strong Coupling Lattice QCD for finite μ

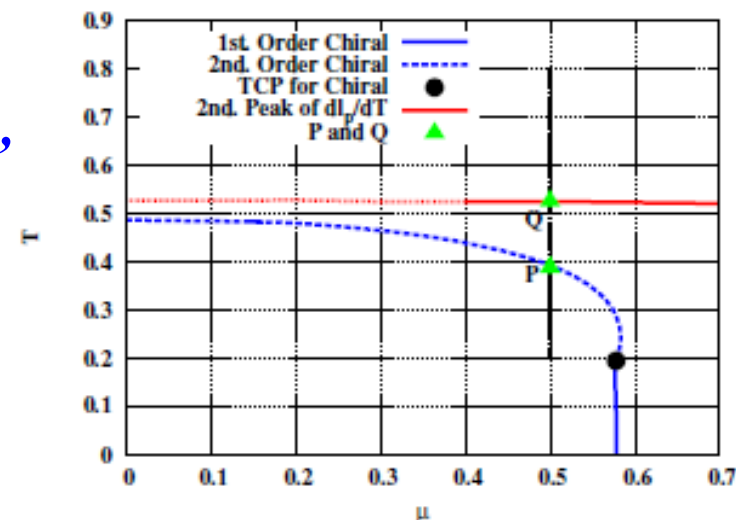
■ Mean Field approaches

Damagaard, Hochberg, Kawamoto ('85); Bilic, Karsch, Redlich ('92); Fukushima ('03); Nishida ('03); Kawamoto, Miura, AO, Ohnuma ('07).

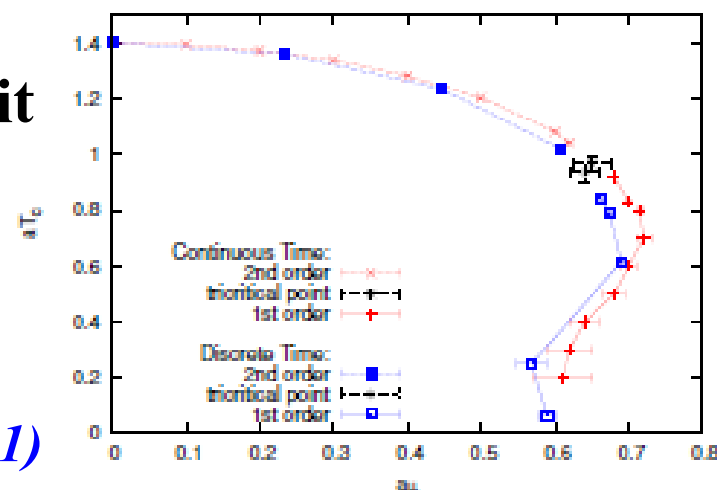
■ MDP simulation

Karsch, Mutter('89); de Forcrand, Fromm ('10); de Forcrand, Unger ('11)

- Partition function = sum of config. weights of various loops.
- Applicable only to Strong Coupling Limit ($1/g^2=0$) at present



Miura, Nakano, AO, Kawamoto, arXiv:1106.1219



de Forcrand, Unger ('11)

*Can we include both fluctuation and finite coupling effects ?
→ One of the candidates = Auxiliary field MC*