Fisher's Zeros and Conformality in Lattice Models

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Content of the talk

- Fisher's zeros and Finite Size Scaling
- Lattice models considered:
  1. $U(1)$ 3 and 4D
  2. $SU(2)$ 4D with $\beta_{Adjoint}$
  3. $SU(3)$ 4D $N_f = 4$ and 12
- Effect of boundary conditions (in $O(2)$ sigma model)
- Conclusions

References

- Complex RG flows: A. Denbleyker, Daping Du, Yuzhi Liu, YM, and Haiyuan Zou, PRL 104 25160 (2010).
- $O(N)$ models: (with Haiyuan Zou), Phys. Rev. D83 056009 (2011).
- Hierarchical model: (with Yuzhi Liu), Phys. Rev. D83 096008 (2011).
- $U(1)$ pure gauge theory: A. Bazavov, B.A. Berg, Daping Du , YM Phys. Rev. D85 056010 (2012).
- $SU(2)$: A. Denbleyker, Daping Du, YM, in progress (see also arXiv:1112.2734, POS Lattice 2011 and Daping Du Ph. D. thesis).

Fisher's zeros and Finite Size Scaling

Decomposition of the partition function (Niemeijer and van Leeuwen 76)

$$\begin{aligned} Z &= Z_{sing.} \mathrm{e}^{G_{bounded}} \\ Z_{sing.} &= \mathrm{e}^{-L^D f_{sing.}} \end{aligned}$$

RG transformation: the lattice spacing a increases by a scale factor b

$$\begin{aligned} a &\rightarrow ba \\ L &\rightarrow L/b \\ f_{sing.} &\rightarrow b^D f_{sing.} \\ Z_{sing.} &\rightarrow Z_{sing.} \end{aligned}$$

Important Conclusion (Itzykson et al. 83)

The zeros of the partition functions are RG invariant

Fisher's zeros: zeros of the partition function in the complex β plane

$Z_{sing.}$ in terms of scaling variables

We consider discrete RG transformations

Example

$b = 2$, for a sigma model on D-dimensional cubic lattice: 2^D fields are replaced by one blocked field

Lattice size (in a units)

$$L \rightarrow L/b$$

Scaling variables (e. g. $u = \beta - \beta_c + \dots$):

$$u_i \rightarrow \lambda_i u_i$$

Relevant variables: $\lambda_i = b^{1/\nu_i}$; Irrelevant variables: $\lambda_j = b^{-\omega_j}$

RG invariance of $Z_{sing.}$

$$Z_{sing.} = Q(\{u_i L^{1/\nu_i}\}, \{u_j L^{-\omega_j}\})$$

Zeros for one relevant variable

For a single relevant variable $u \simeq \beta - \beta_c$, we have $Z_{sing} = Q(uL^{1/\nu})$.

The complex equation $Z = 0$ can be written as two real equations for two real variables and generic solutions are isolated points.

$$Z = 0 \Rightarrow uL^{1/\nu} = w_r \text{ with } r = 1, 2, \ldots$$

This implies the approximate form for the zeros:

$$\beta_r(L) \simeq \beta_c + w_r L^{-1/\nu}$$

There are many examples, where these discrete solutions follow approximate lines or lay inside cusps. In the infinite volume limit, the set of zeros may (or may not) separate the complex plane into two or more regions.

For a first order transition: $\nu \to 1/D$.

Example: 4D $U(1)$ (with Bazavov, Berg and Du)

The complex zeros appear at the intersections of ReZ=0 and ImZ=0. Results obtained by integrating a reweighted density of states calculated with multicanonical methods (arxiv 1202.2109, PRD 85)

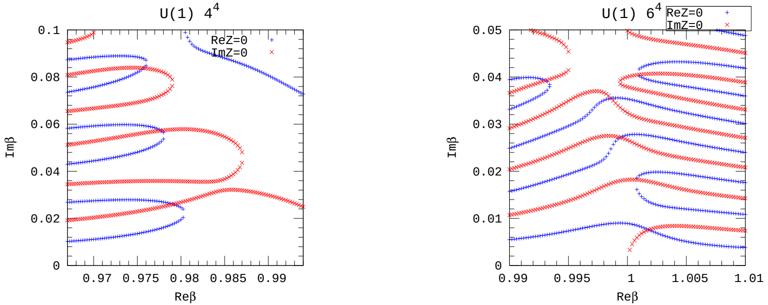
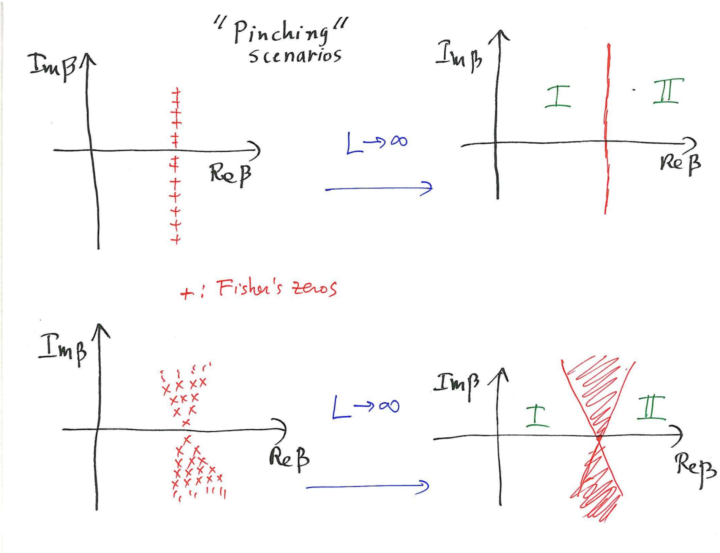
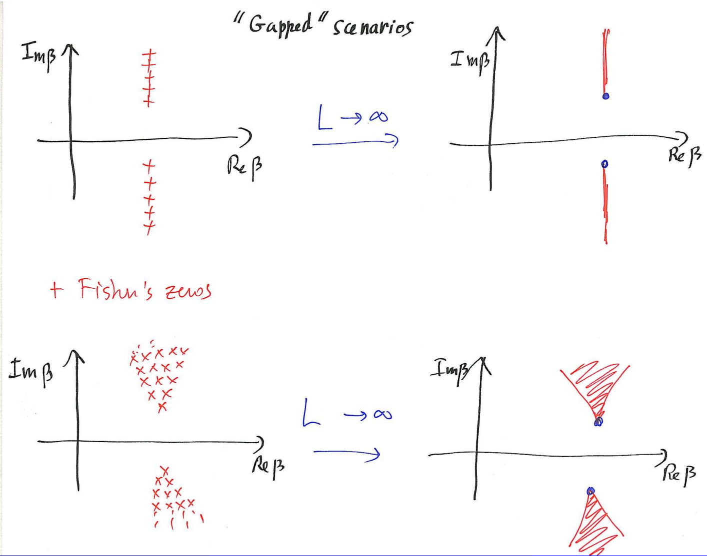


Figure: Zeros of the Re (+, blue) and Im (x, red) part of Z for $U(1)$ using the density of states for 4^4 and 6^4 lattices.

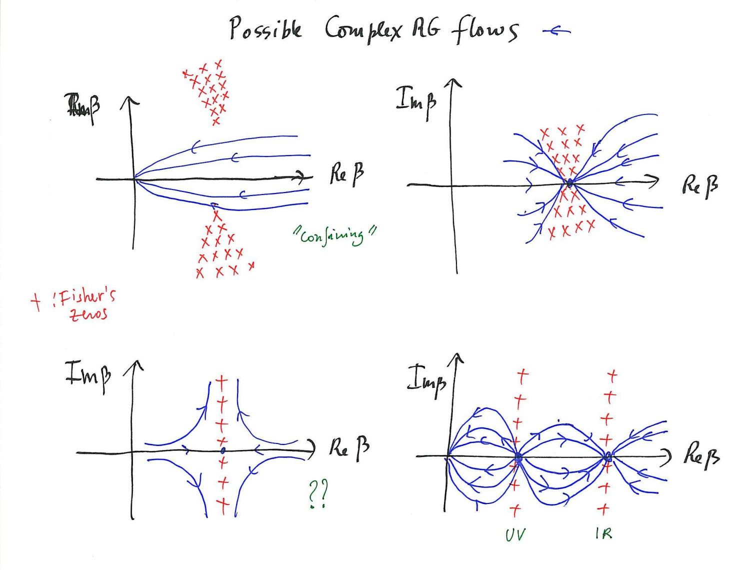
In some cases the zeros pinch the real axis



In some other cases, they leave a gap along the real axis



Possible complex RG flows (artistic rendering)



“Confining" flows: 2D $O(N)$ models in the large-N limit

RG flows go directly from weak coupling to strong coupling (mass gap).

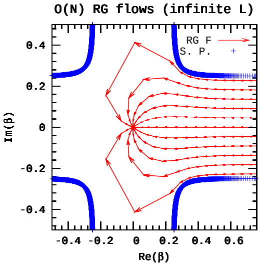


Figure: Infinite L RG flows (arrows). The blending blue crosses are the β images of two lines of points located very close above and below the $[-8, 0]$ cut of $\beta(M^2)$ in the M^2 plane. Fisher's zeros stay outside of the blue lines (PRD 80 054020).

$U(1)$ on L^4: the average plaquette distribution has a double peak distribution with equal heights at a pseudo-critical β_S. For small L, the distance between the peaks slowly decreases with the volume. (PRD 85 with Bazavov, Berg and Daping Du).

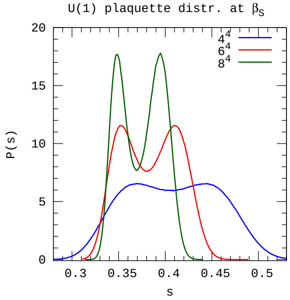
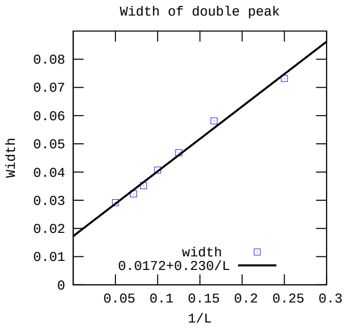
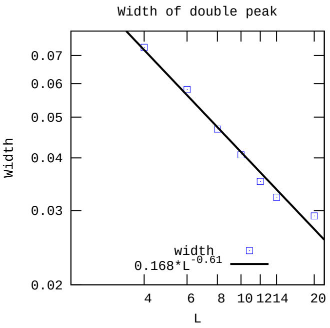


Figure: Average plaquette distribution for $U(1)$ at β_S for L= 4, 6 and 8.

4D $U(1)$: first or second order?

In the infinite volume limit, the width of the double peak distribution of the average plaquette goes to a nonzero limit (latent heat) for a first order phase transition and to zero as an inverse power of L for a second order transition. Better statistics for the large volumes are necessary to discriminate between the two scenarios.





Double peak does not always mean first order

A simple example (the average plaquette is denoted S/L^D)

$$Z(\beta) = \int dSn(S)\mathrm{e}^{-\beta S}$$

$$n(S)\mathrm{e}^{-\beta_S S} \propto (\mathrm{e}^{-(1/2\sigma^2)(S-S_1)^2} + \mathrm{e}^{-(1/2\sigma^2)(S-S_2)^2})$$

With $n(S)$ the density of state and β_S such that the two peaks have equal height. The zeros are located at $\beta_r = \beta_S + i2\pi(2r+1)/(S_2 - S_1)$.

If $(S_2 - S_1) \propto L^D$, we have a first order phase transition (latent heat) and Im$\beta_1 \propto L^{-D}$.

However, if $(S_2 - S_1) \propto L^{D-\zeta}$, then the width of the double peak in the average plaquette goes to zero at infinite volume and Im$\beta_1 \propto L^{-1/\nu}$ with $\nu = 1/(D - \zeta)$.

Fits of Im$\beta_1(L)$ (L^{-4} : first order, $L^{-1/\nu}$: second order)

Fits of $y =$ Imaginary part of lowest zero. First order hypothesis:

$$y = \frac{a_1}{L^4}\left(1 + \frac{a_2}{L} + \frac{a_3}{L^2}\right), \quad Q = 0.43; (\mathrm{all}7L)\ . \tag{1}$$

Using *only* the data from the $L = 4, 6, 8$ lattices, the 2-parameter fit

$$y = a_1 L^{a_2} \quad Q = 0.39$$

gives the exponent $a_2 = -3.082\,(35)$ instead of -4. However, the fit

$$y = \frac{a_1}{L^{3.08}}\left(1 + \frac{a_2}{L} + \frac{a_3}{L^2}\right),$$

with the seven data points leads to $Q < 10^{-8}$. A four parameter fit as in Eq. (1) but with the leading exponent fitted, gives 4.121(74) for this exponent with Q=0.72. These results seem to favor the first order possibility. However, they should be checked with higher statistics data for the larger volumes.

Fits of $\mathrm{Im}\beta_1(L)$ (L^{-4} : first order, $L^{-1/\nu}$: second order)

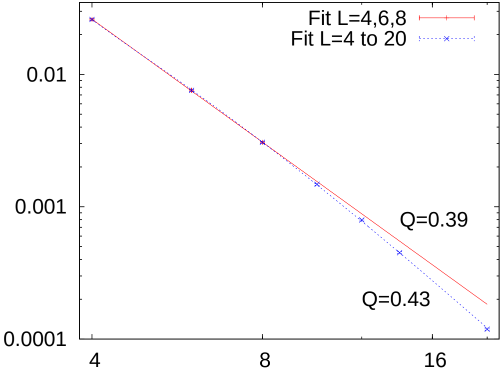


Figure: Fits of $\mathrm{Im}\beta_1(L)$ on a log-log scale.

3D $U(1)$: no zeros near the real axis (with Alan DenBleyker)

3D $U(1)$ is confining. There is a gap in the spectrum and the zeros.

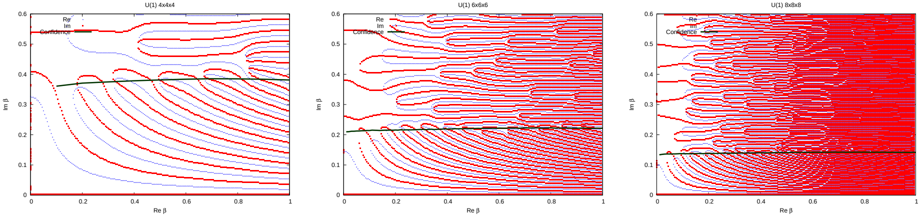


Figure: Fisher's zeros for $U(1)$ on L^3 lattices (L=4, 6 and 8 from left to right) The zeros of the real (imaginary) part are represented by the blue (red) curves and the region of confidence is below the green line (zeros near or above this line are not reliable).

$SU(2)$ with $\beta_{Adjoint}$ (with A. Denbleyker and Daping Du)

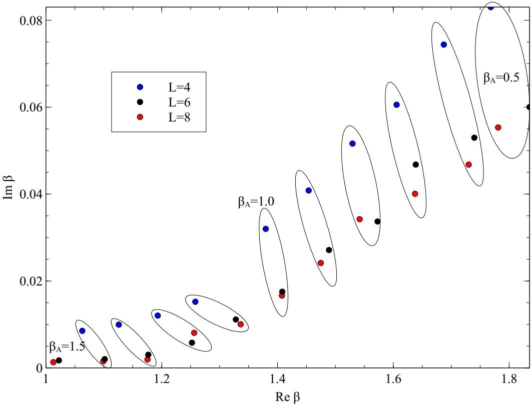
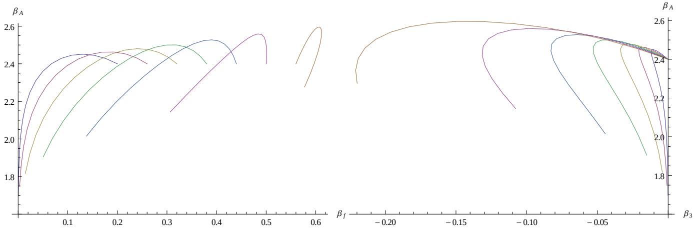


Figure: Lowest zeros for $\beta_{Adjoint}$ = 0.5, 0.6, ..., 1.5. The robustness of these results are discussed in Daping's Du thesis.

$SU(2)$ with $\beta_{Adjoint}, \beta_{3/2}, \dots$ (with Judah Unmuth-Yockey)

In the Migdal-Kadanoff approximation, RG flows can go around phase boundaries (not shown).



2-lattice matching using Migdal-Kadanoff (with Alan Denbleyker and Judah Unmuth)

Is the MK approximation reliable? The $2Rx2R$ Wilson loops for a $(2L)^4$ lattice and the RxR Wilson loop on a L^D lattice with effective couplings obtained by the MK recursion (see Bitar et al. 83 and Toussaint et al. 82). The matching is not very accurate (Does Cheng Tomboulis arxiv1206.3616, Friday talk, improvement help?)

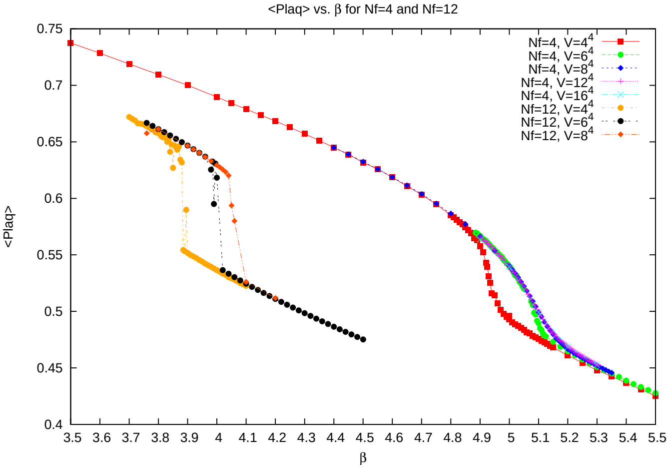
Volume	b	β_F	β_A	$\beta_{3/2}$	β_2	P_{size}	$\langle P \rangle$	σ
8^4		2.40000	0.00000			$2x2$	0.7766	0.00672
4^4	2	0.955274	-0.0496152	0.003759328	-0.000310275	1x1	0.7710	0.01226
8^4		2.40000	0.00000			$4x4$	0.9009	0.09007
4^4	2	0.955274	-0.0496152	0.003759328	-0.000310275	$2x2$	0.9973	0.01283
8^4		4.80000	0.00000			$2x2$	0.4016	0.00369
4^4	2	4.47578	-0.728286	0.188086	0.055336	1x1	0.2225	0.00655
8^4		4.80000	0.00000			$4x4$	0.5670	0.12841
4^4	2	4.47578	-0.728286	0.188086	0.055336	$2x2$	0.5144	0.01799

$SU(3)$ with $N_f = 4$ and 12 (Yuzhi Liu and Don Sinclair)

- We use standard Wilson gauge and naive staggered fermion action with Hybrid Monte Carlo(HMC) algorithm.
- Focus on $N_f = 4$ and $N_f = 12$ with relatively small symmetric lattices.
- Bare quark mass is set to be $m_q = 0.02$ for now.
- Configurations used to calculate Fisher's zeros:

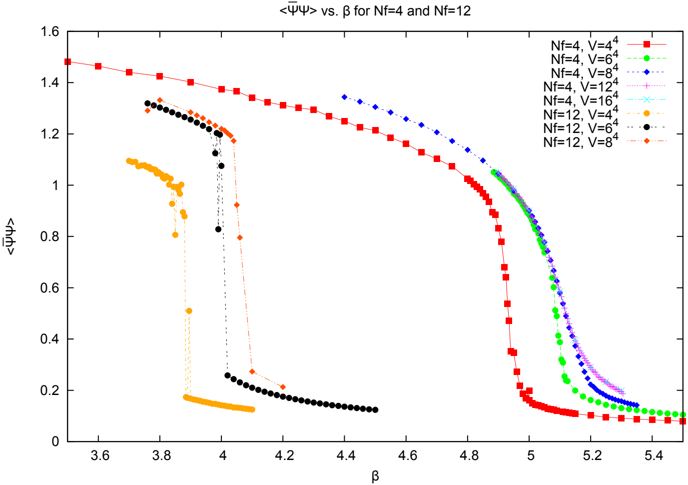
N_f	Volume	Num. of β used	Num. of config. per β
4	4^4	21	25,000
4	6^4	35	8,000
4	8^4	36	25,000
4	12^4	21	25,000
4	16^4	5	2,500
12	4^4	31	50,000
12	6^4	41	50,000
12	8^4	15	8,000

$SU(3)$ with $N_f = 4$ and 12 (with Yuzhi Liu and Don Sinclair)



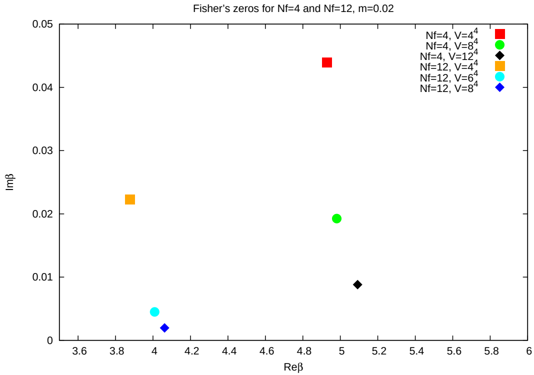
- Average plaquette for $N_f = 4$ and $N_f = 12$ at different volumes.

$SU(3)$ with $N_f = 4$ and 12 (with Yuzhi Liu and Don Sinclair)



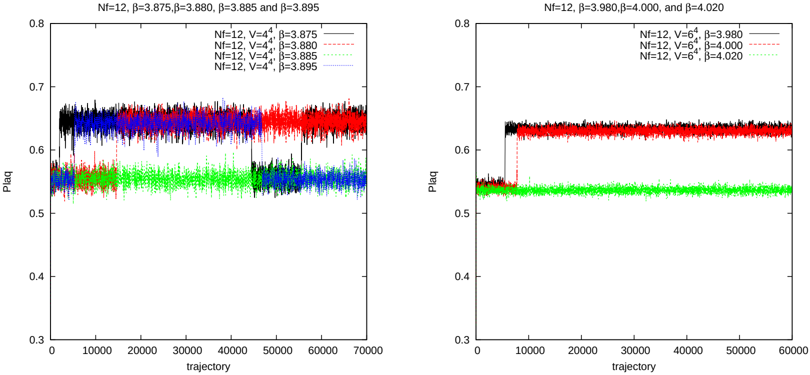
- $< \bar{\Psi}\Psi >$ for $N_f = 4$ and $N_f = 12$ at different volumes.

$SU(3)$ with $N_f = 4$ and 12 (with Yuzhi Liu and Don Sinclair)



- Fisher's zeros for $N_f = 4$ and $N_f = 12$ at different volumes.

$SU(3)$ with $N_f = 4$ and 12 (with Yuzhi Liu and Don Sinclair)



- Example signal for a strong first order phase transition.
- Finite temperature?
- Larger volumes and different masses are needed for FSS

1D O(2) with $L = 4, 8, 16, 32$ (with Haiyuan Zou)

The zeros are very different for open (o.b.c) and periodic boundary conditions (p.b.c):

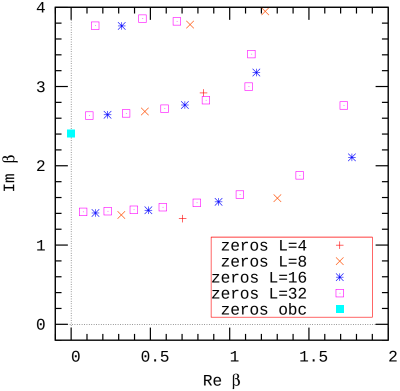


Figure: Zeros of partition function (p.b.c) with different volumes and zeros of partition function (o.b.c)

MK complex flows (Haiyuan Zou)

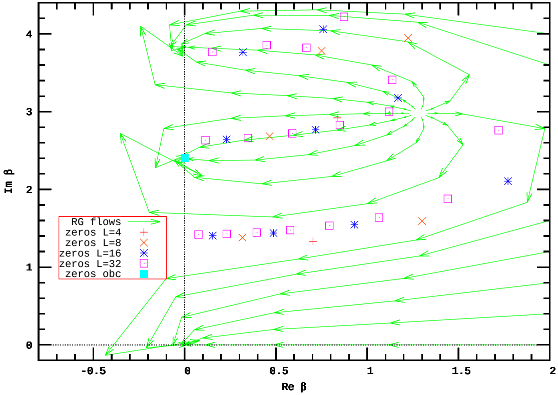


Figure: MK complex flows for the system (o.b.c) and zeros from the two different boundary conditions

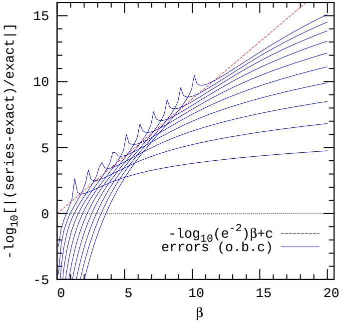
Effect of boundary conditions in 1D $O(2)$ (with Haiyuan Zou)

At finite volume, the nonperturbative parts of the average energy are very different for open and periodic boundary conditions

$$
\begin{aligned}
|(E - E_{PT})/E| &\propto \mathrm{e}^{-2\beta}(\textit{open b.c.}) \\
&\propto \mathrm{e}^{-\beta E_v}(\textit{periodic b.c.})
\end{aligned}
$$

where E_v is the energy of the periodic solution of the classical equation of motion with winding number 1.

Effect of boundary conditions in 1D $O(2)$ (with Haiyuan Zou)



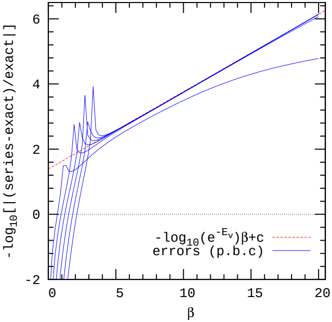


Figure: o.b.c(Left): Errors of the average energy series with order 2,4,...,20; p.b.c($L = 36$)(Right): Errors of the average energy series with order 2,4,...,12.

Comparison of Hadamard series (with Haiyuan Zou)

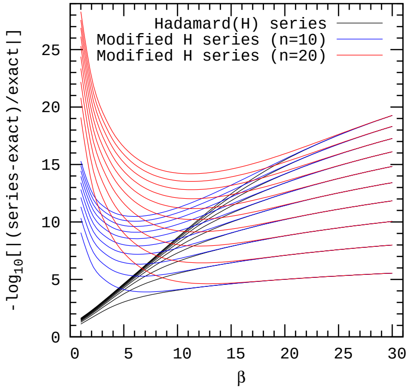


Figure: Errors of different series with order 2,4,...,20.Black:Hadamard; Blue:modified Hadamard($n = 10$); Red: modified Hadamard($n = 20$)

Conclusions

- Much progress has been been made in finding reliable ways to locate the zeros of various models
- A consistent picture of confinement in terms of complex RG flow is emerging
- Much work remains to be done for multiflavor LGT
- Better analytical approaches (based on improved RG or weak coupling expansions) are needed
- FSS of zeros is simple, however subleading corrections are important (at least for unimproved actions)
- Plans: monitor the effects of improvement on the zeros by turning on improvement adiabatically
- Thanks!

(Modified) Hadamard expansions

Hadamard expansion can make the asymptotic series converge. And the modified Hadamard expansion makes the series converge faster. (R.B.Paris, Proc.R.Soc.Lond.A (2001))

The Usual Asymptotic Expansions:

$$e^{-x}I_0(x) = \frac{1}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{a_k}{(2x)^k} \tag{2}$$

Hadamard Expansions:

$$e^{-x}I_0(x) = \frac{1}{\sqrt{2\pi x}} \sum_{k=0}^{\infty} \frac{a_k}{(2x)^k} P(k+\frac{1}{2}, 2x) \tag{3}$$

The Modified Hadamard Expansions:

$$e^{-x}I_0(x) = \frac{1}{\sqrt{2\pi x}} \{\sum_{k=0}^{M-1} \frac{a_k}{(2x)^k} P(k+\frac{1}{2}, 2x) + T_{M,n}(x)\} \tag{4}$$