Monte Carlo studies of 3d $\mathcal{N}=6$ superconformal Chern-Simons gauge theory via localization method

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Message:

By definition,

$^\forall$Supersymmetric theory has fermions which make heavy computational costs

Localization method reduces

General 3d $\mathcal{N}=2$ SUSY theory in a BPS sector

(=supersymmetric sector)

↓

A matrix model without fermions

3d $\mathcal{N}=6$ superconformal Chern-Simons theory (="ABJM" theory)

[Aharony-Bergman-Jafferis-Maldacena '08]

$$
\begin{aligned}
\mathcal{L} \;=\; & k\mathrm{Tr}\Big[\frac{1}{2}\epsilon^{\mu\nu\rho}\Big(-A_\mu\partial_\nu A_\rho - \frac{2}{3}A_\mu A_\nu A_\rho + \tilde{A}_\mu\partial_\nu\tilde{A}_\rho + \frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\Big) \\
& -|D_\mu\phi_\alpha|^2 + i\bar{\psi}^\alpha\Gamma^\mu D_\mu\psi_\alpha \\
& -i\epsilon^{\alpha\beta\gamma\delta}\phi_\alpha\bar{\psi}_\beta\phi_\gamma\bar{\psi}_\delta - i\bar{\psi}_\beta\phi_\alpha\bar{\phi}^\alpha\psi^\beta + 2i\bar{\psi}_\alpha\phi_\beta\bar{\phi}^\alpha\psi^\beta \\
& +\frac{1}{3}\phi_\alpha\bar{\phi}^\beta\phi_\beta\bar{\phi}^\gamma\phi_\gamma\bar{\phi}^\alpha + \frac{2}{3}\phi_\beta\bar{\phi}^\alpha\phi_\gamma\bar{\phi}^\beta\phi_\alpha\bar{\phi}^\gamma - \phi_\gamma\bar{\phi}^\gamma\phi_\beta\bar{\phi}^\alpha\phi_\alpha\bar{\phi}^\beta + (\mathrm{h.c.})\Big]
\end{aligned}
$$

$$(\alpha,\beta,\gamma,\delta=1,2)$$

- $U(N)\times U(N)$ gauge group
- ϕ_α : Bi − fundamental scalar $\in (N,\bar{N})$
- ψ_α : Bi − fundamental fermison $\in (N,\bar{N})$

ABJM & AdS/CFT correspondence

$$S_{CS} \sim k \int \mathrm{Tr}\left[A \wedge dA - \frac{2}{3} i A \wedge A \wedge A\right]$$

$$k \sim \frac{1}{g_{YM}^2}$$

(k: Chern-Simons level)

CFT_3 / AdS_4

relatively easy
(∃ traditional technique)

$$\lambda = \frac{N}{k} = \text{fixed}, \quad N \gg 1$$

Type IIA superstring
on $\mathrm{AdS}_4 \times \mathrm{CP}^3$

ABJM theory
($= 3\mathrm{d}\ \mathcal{N} \geq 6$ SCFT)

(Intermediate)
Extremely hard！

relatively hard

$$k \ll N^{1/5}$$

M − theory
on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathrm{Z}_k$

Investigate the whole region by Monte Carlo simulation!

Contents

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How do we put ABJM on a computer?

～Orthodox approach（=Lattice）～

Action : $S_{\mathrm{ABJM}} = S_{\mathrm{CS}} + S_{\mathrm{Matter}}$

Difficulties in "formulation"

- It is not easy to construct CS term on a lattice [Cf. Bietenholz-Nishimura '00]
- It is generally difficult to treat SUSY on a lattice [Cf. Giedt '09]

Practical difficulties

- ∃Many fermionic degrees of freedom → Heavy computational costs
- CS term = purely imaginary → sign problem

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hopeless…

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Practical difficulties

- $\exists$ Many fermionic degrees of freedom → Heavy computational costs
- CS term = purely imaginary → sign problem

hopeless... ⇒ Apply the localization method

Localization method

Original partition function:

[Cf. Pestun '08]

$$Z = \int d\Phi \exp\left(-S[\Phi]\right),$$

where

$$QS[\Phi] = 0,\ Q : \text{a fermionic nilpotent charge},\ Q^2 = 0$$

1 parameter deformation:

$$Z(t) = \int d\Phi \exp\left(-S[\Phi] - tQV[\Phi]\right)$$

Consider t-derivative:

$$\frac{dZ(t)}{dt} = \int d\Phi\ (QV)e^{-S-tQV} = \int d\Phi\ Q(Ve^{-S-tQV}) = 0$$

$(QS = 0)$

Assuming Q is unbroken

$$\Rightarrow\ Z = \lim_{t\to+0} Z(t) = Z(t) = \lim_{t\to\infty} Z(t)$$

We can use saddle point method!!

(Cont'd) Localization method

$$Z = \lim_{t \to \infty} \int d\Phi \exp\left(-S[\Phi] - tQV[\Phi]\right)$$

Consider fluctuation around saddle points: $\Phi \to \Phi_0 + \frac{1}{\sqrt{t}}\delta\Phi$

$$Z = \sum_{\Phi_0} \exp\left(-S[\Phi_0]\right) \cdot Z_{1-\text{loop}}[\Phi_0]$$

where

$$Z_{1-\text{loop}}[\Phi_0] = \int d\delta\Phi \exp\left(-QV[\Phi_0 + \delta\Phi]|_{\text{Gaussian}}\right)$$

Cf.

For Q-invariant operator,

$$Z\langle \mathcal{O}(\Phi)\rangle = \sum_{\Phi_0} \mathcal{O}(\Phi_0) \exp\left(-S[\Phi_0]\right) \cdot Z_{1-\text{loop}}[\Phi_0]$$

Localization of ABJM theory on S^3

[Kapustin-Willet-Yaakov '09]

$$S_{\mathrm{ABJM}} = S_{\mathrm{CS}} + S_{\mathrm{Matter}}$$

$$S_{\mathcal{N}=2\ \mathrm{SYM}} = Q(*), \qquad S_{\mathrm{Matter}} = Q(*)$$

$$Z_{\mathrm{ABJM}} = Z(t) = \int d\Phi \exp\left[-S_{\mathrm{CS}}[\Phi] - t(S_{\mathcal{N}=2\ \mathrm{SYM}} + S_{\mathrm{Matter}})\right]$$

Tedious calculation

$t \to \infty$

Gauge 1-loop

$$Z_{ABJM} = \frac{1}{N!^2}\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

Matter 1-loop

CS term

(Cont'd)How do we put ABJM on a computer?

Lattice approach seems hopeless… (∵SUSY, sign problem, etc)

↓ Localization method

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N\mu}{(2\pi)^N} \frac{d^N\nu}{(2\pi)^N} \frac{\prod_{i<j} \left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2 \left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2} \exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

Sign problem

↓ Integrate out N-variables

$$Z_{ABJM} = \frac{1}{2^N N!} \int \frac{d^N x}{(2\pi k)^N} \frac{\prod_{i<j} \tanh^2\left(\frac{x_i-x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)}$$

Easy to perform simulation even by our laptop

How to calculate the free energy

$$Z(N,k) = \frac{1}{2^N N!}\int \frac{d^N x}{(2\pi k)^N} \frac{\prod_{i<j}\tanh^2\left(\frac{x_i - x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)} = \frac{1}{2^N N!}\int \frac{d^N x}{(2\pi k)^N} e^{-S(N,k)}$$

Reweighting:

$$\begin{aligned}
Z(N_1+N_2,k) &= Z(N_1,k)Z(N_2,k)\frac{Z(N_1+N_2,k)}{Z(N_1,k)Z(N_2,k)} \\
&= \frac{N_1!N_2!}{(N_1+N_2)!} Z(N_1,k)Z(N_2,k)\frac{\int d^{N_1+N_2}x\ e^{-S(N_1+N_2,k)}}{\int d^{N_1+N_2}x\ e^{-S(N_1,k)-S(N_2,k)}} \\
&= \frac{N_1!N_2!}{(N_1+N_2)!} Z(N_1,k)Z(N_2,k)\left\langle e^{-S(N_1+N_2,k)+S(N_1,k)+S(N_2,k)}\right\rangle_{N_1,N_2} \\
&= \frac{N_1!N_2!}{(N_1+N_2)!} Z(N_1,k)Z(N_2,k)\left\langle \prod_{i=1}^{N_1}\prod_{J=N_1+1}^{N}\tanh^2\left(\frac{x_i - x_J}{2k}\right)\right\rangle_{N_1,N_2}
\end{aligned}$$

VEV under the action:

$$S(N_1,k)+S(N_2,k)$$

Note: $Z(1,k) = \frac{1}{2}\int \frac{dx}{2\pi k}\frac{1}{2\cosh\left(\frac{x}{2}\right)} = \frac{1}{4k}$

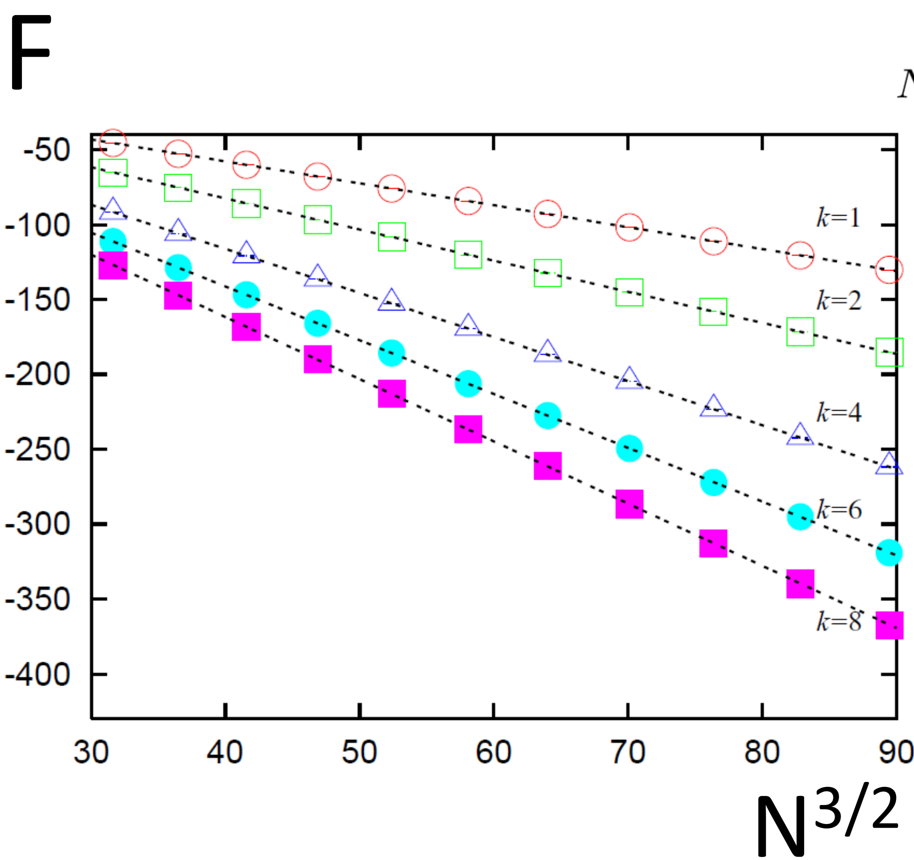
(A part of)

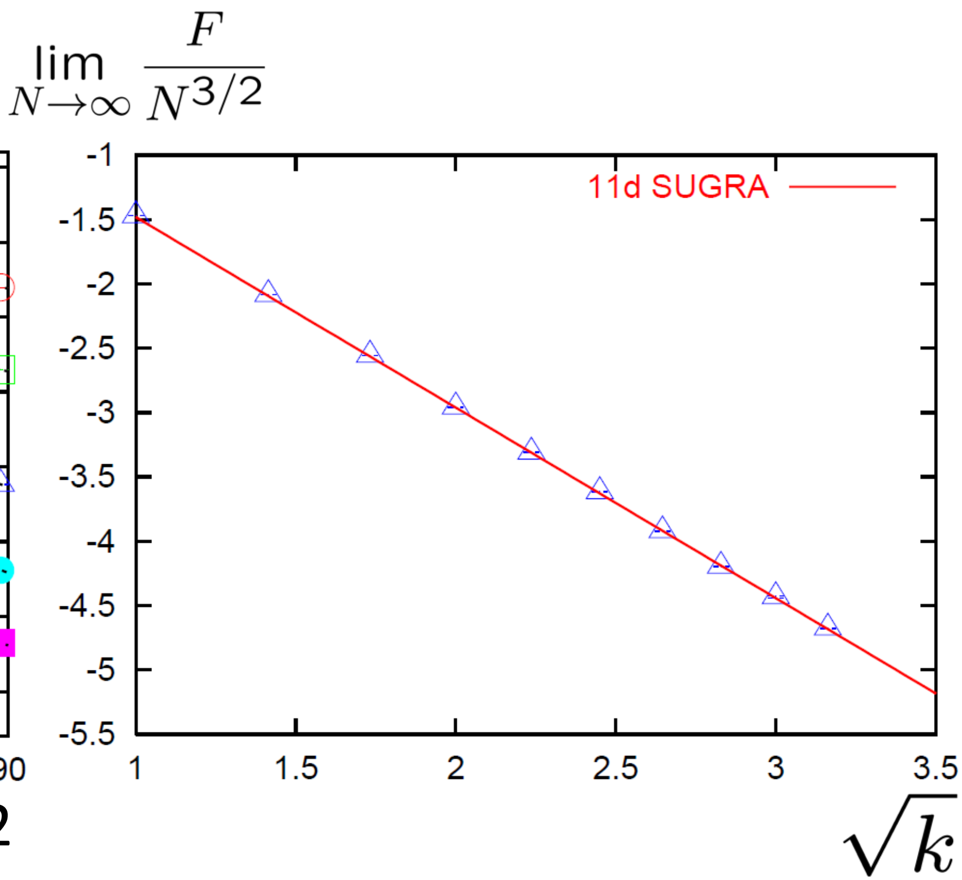
Result

3/2 power low in 11d SUGRA limit

$(k = \text{fixed}, \ \ N \to \infty)$ [Cf. Drukker-Marino-Putrov '10, Herzog-Klebanov-Pufu-Tesileanu '10]

11d classical SUGRA: $F = -\frac{\pi\sqrt{2}}{3}\sqrt{k}N^{\frac{3}{2}}$





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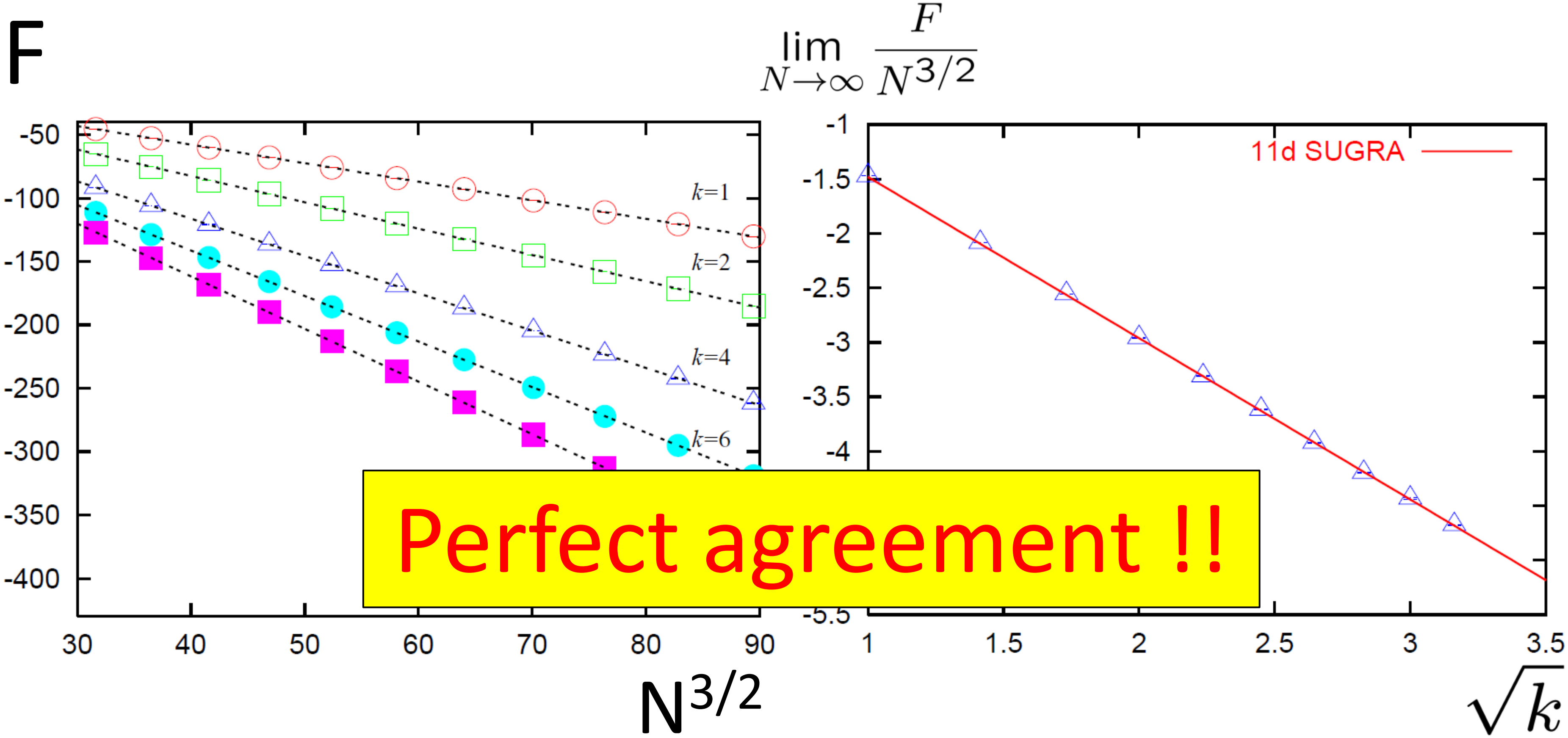


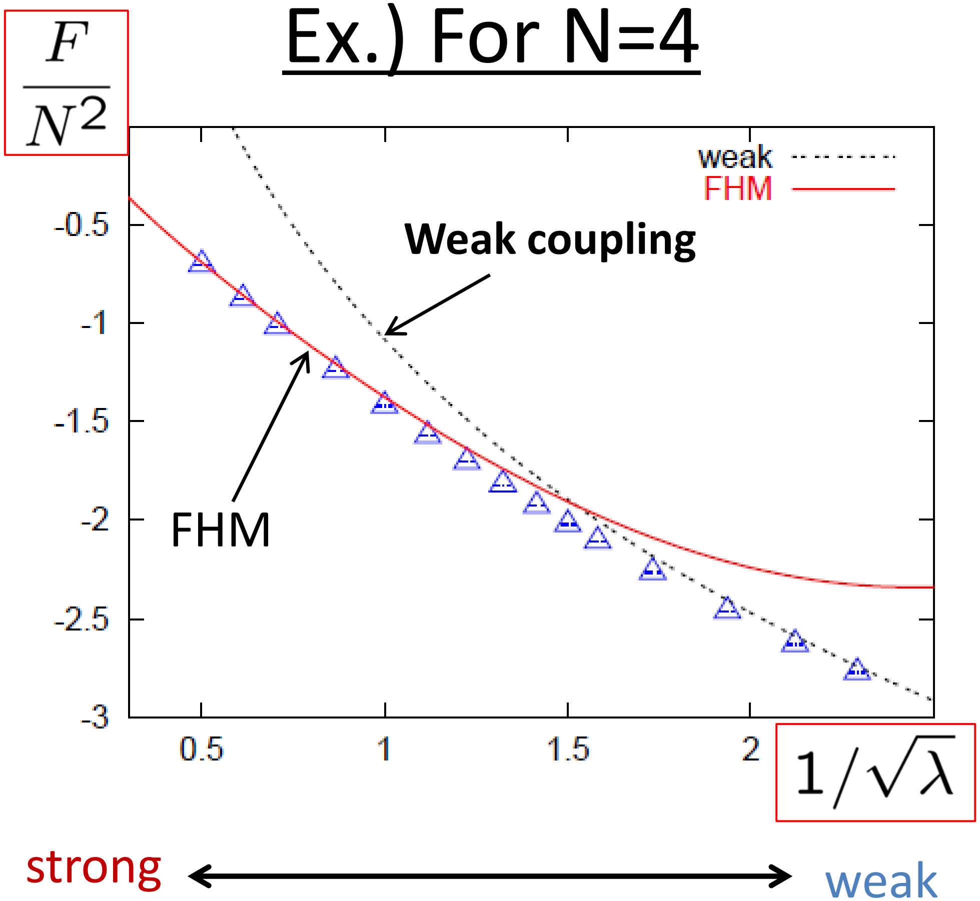
Finite N corrections

[Fuji-Hirano-Moriyama '11]

Result of summing up all order of 1/N expansion at strong coupling:

$$F_{\rm FHM} = \log\left[\frac{1}{\sqrt{2}}\left(\frac{4\pi^2 N}{\lambda}\right)^{1/3} {\rm Ai}\left[\left(\frac{\pi}{\sqrt{2}}\left(\frac{N}{\lambda}\right)^2\right)^{2/3}\left(\lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}\right)\right]\right] \quad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

Ex.) For N=4



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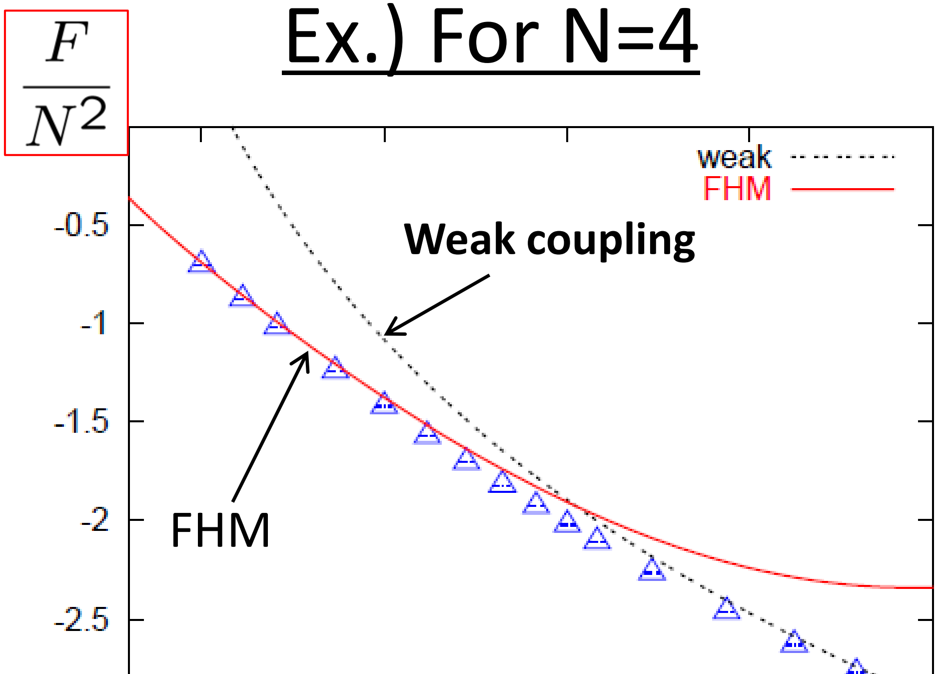

Almost agrees with FHM for strong coupling

→more precise comparison by taking difference

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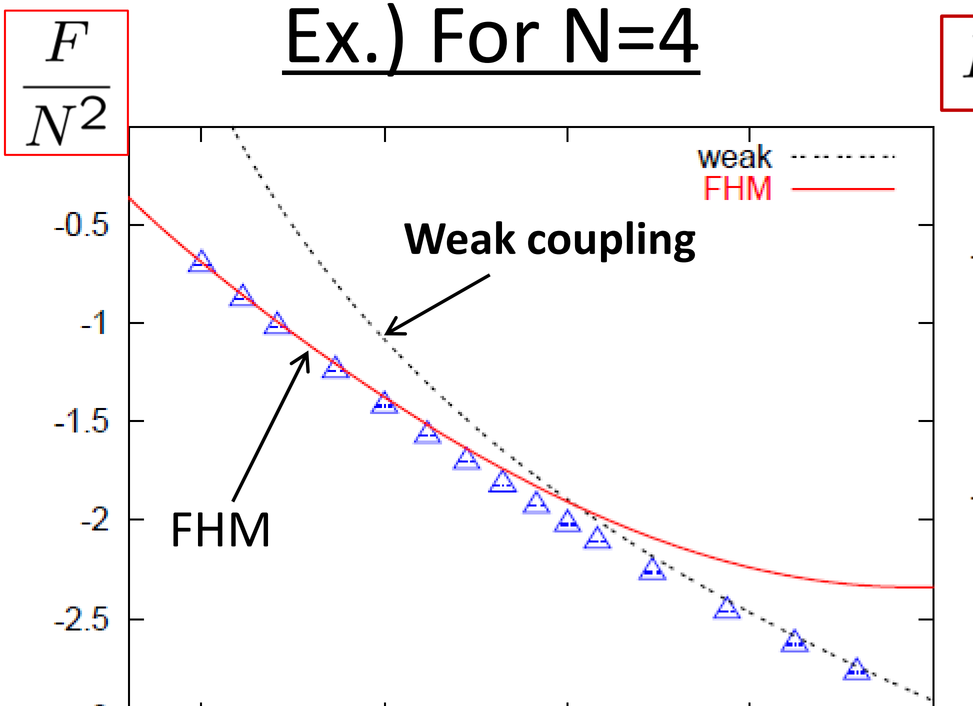
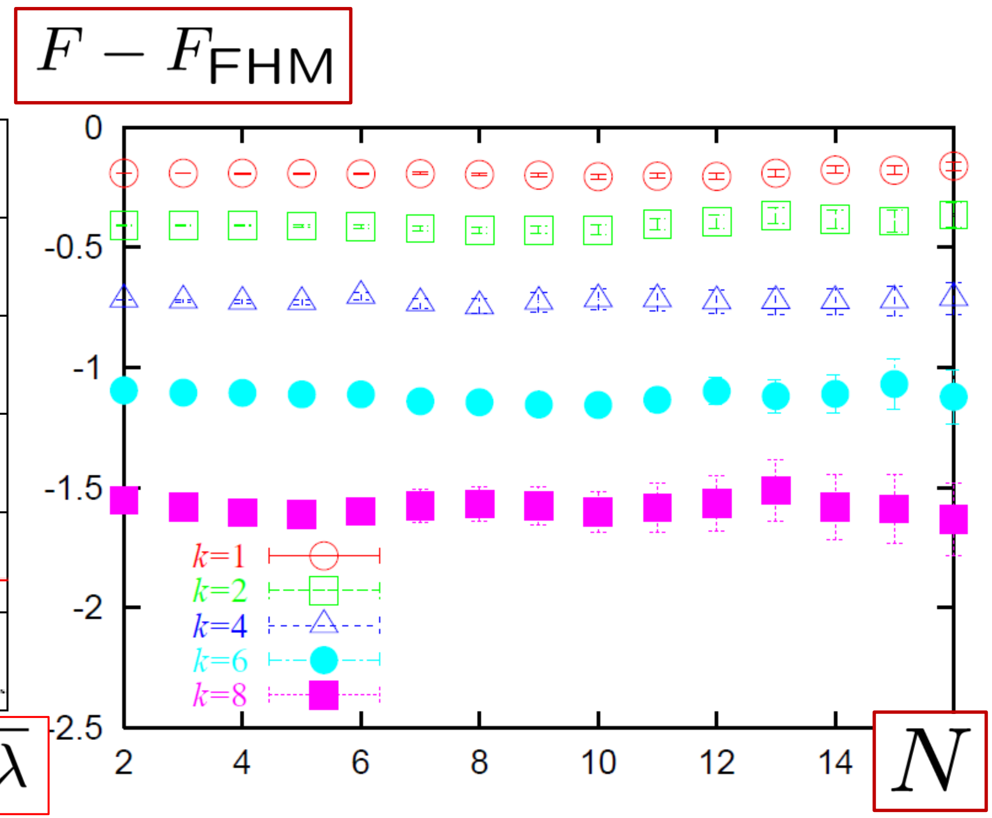


Almost agrees with FHM for strong coupling
→more precise comparison by taking difference

strong ⟷ weak



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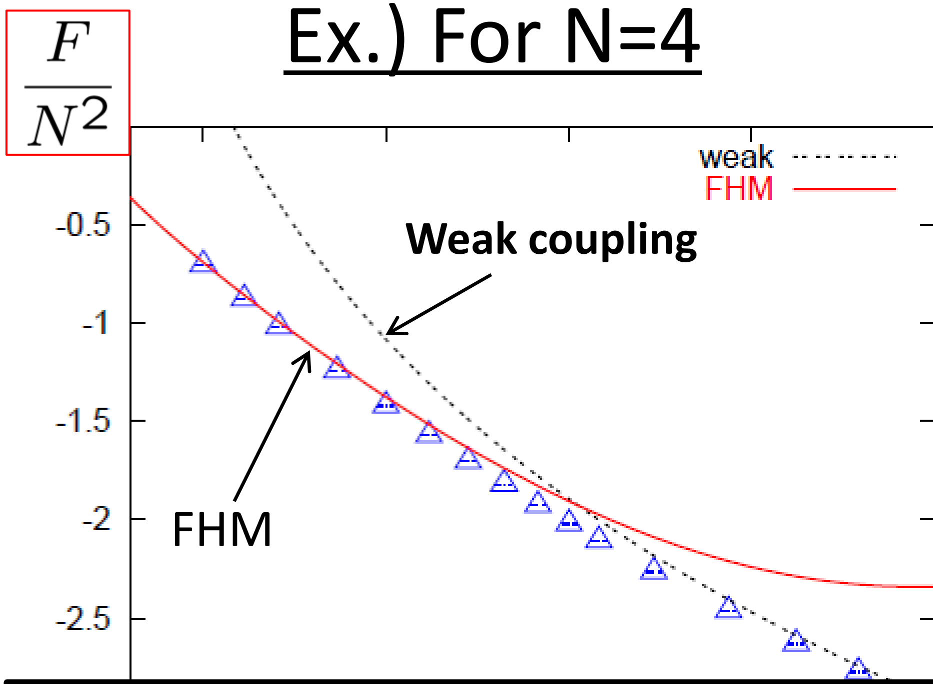
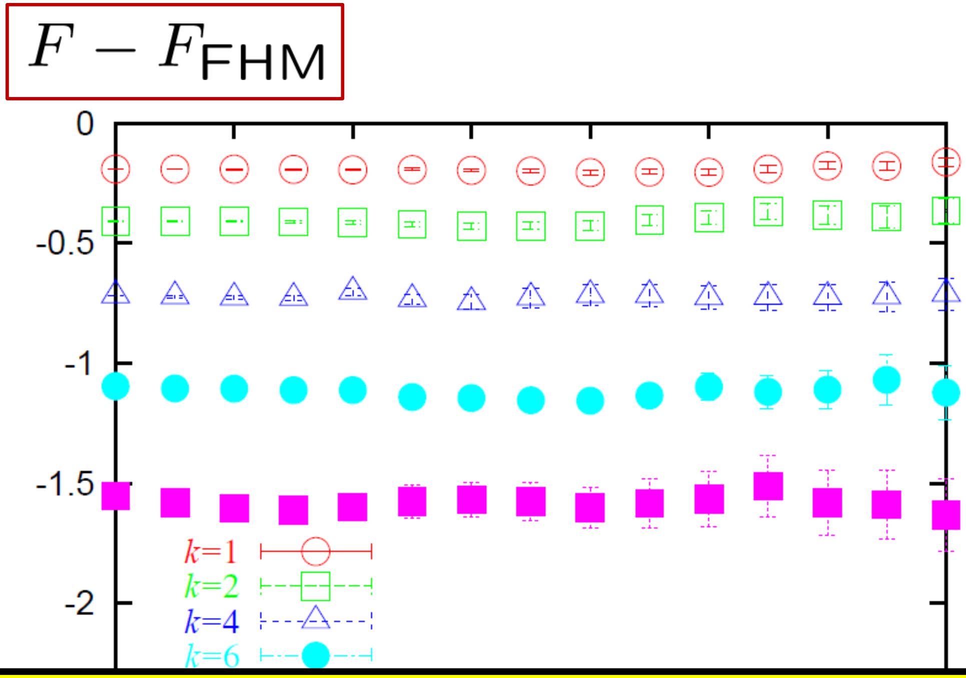




Discrepancy independent of N and dependent on k
→different from exp dumped behavior

Summary

Summary

Monte Carlo calculation of the Free energy
in U(N) × U(N) ABJM theory on S^3 (with keeping all symmetry)

Localization method reduces

the ABJM theory in the BPS sector

The matrix model without fermions

Very useful for first numerical study of SUSY theories!!

(And exercise of Monte Carlo simulation for your student)

Localization method is applicable to general 3d $\mathcal{N}=2$ theory

- Chern-Simons term $\longrightarrow$ $\exp\left[-4\pi^2 i \sum_i \sigma_i^2\right]$
- Vectormultiplet (Any gauge group) $\longrightarrow$ $\prod_{\alpha\in\Delta_+} \left(2\sinh\left(\pi\alpha_i\sigma_i\right)\right)^2$
- Chiralmultiplet (Any representation with R-charge q) $\longrightarrow$ $\prod_{\rho\in R} f\left(i - iq - \rho_i\sigma_i\right)$

$$\left(f(z) = \exp\left[iz\log\left(1-e^{2\pi z}\right) + \frac{i}{2}\left(-\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2(e^{2\pi z})\right) - \frac{i\pi}{12}\right] \right)$$

Now you can try to various problems with good cost-performance!!

Thank you!

Appendix

Developments on ABJM Free energy

- June 2008: ABJM was born. [Aharony-Bergman-Jafferis-Maldacena]
- July 2010: Planar limit for strong coupling $\left(\lambda = \frac{N}{k} = \text{fixed},\ \lambda \gg 1\ \text{N} \gg 1\right)$ [Drukker-Marino-Putrov]

$$F_{\text{DMP}} = -\frac{\pi\sqrt{2}}{3}\frac{(\lambda - 1/24)^{3/2}}{\lambda^2}N^2 \quad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

$\lambda \gg 1$ ⇒

$$F_{\text{DMP}} \simeq -\frac{\pi\sqrt{2}}{3}\frac{N^2}{\sqrt{\lambda}}$$

Agrees with SUGRA's result!!

※**CP³ has nontrivial 2-cycle** $(R^2_{CP^3} = 4\pi\sqrt{2\lambda})$ [Cf. Cagnazzo-Sorokin-Wulff '09]

⇒ $$\exp\left[-2\pi\sqrt{2\lambda}\right] = \exp\left[-\frac{1}{2\pi}(\pi R^2_{CP^3})\right] = \exp\left[-\frac{1}{2\pi\alpha'}\text{Area}(CP^1)\right]\Big|_{\alpha'=1}$$

∼string wrapped on CP¹⊂ CP³ = **worldsheet instanton** ?

- November 2010: Calculation for k=fixed, N→∞ [Herzog-Klebanov-Pufu-Tesileanu]

$$F = -\frac{\pi\sqrt{2k}}{3}N^{3/2} + o(N^{3/2})$$

Formally same
(※ λ=N/k)

（Cont'd）Development on ABJM free energy

- June 2011 : Summing up all genus around planar limit for strong λ

[Fuji-Hirano-Moriyama]

$$F_{\mathrm{FHM}} = \log\left[\frac{1}{\sqrt{2}}\left(\frac{4\pi^2 N}{\lambda}\right)^{1/3} \mathrm{Ai}\left[\left(\frac{\pi}{\sqrt{2}}\left(\frac{N}{\lambda}\right)^2\right)^{2/3}\left(\lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}\right)\right]\right] \quad \text{up to } O(e^{-2\pi\sqrt{2\lambda}})$$

Formally same

- October 2011 : Exact calculation for N=2 [Okuyama]
- October 2011 : Calculation for k<<1, k<<N [Marino-Putrov]

$$F_{\mathrm{Fermi}} = \log\left[\frac{1}{\sqrt{2}}\left(4\pi^2 k\right)^{1/3} \mathrm{Ai}\left[\left(\frac{\pi k^2}{\sqrt{2}}\right)^{2/3}\left(\frac{N}{k} - \frac{1}{24} - \frac{1}{3k^2}\right)\right]\right] + A(k) - \frac{1}{2}\log 2$$

up to $O(e^{-2\pi\sqrt{2N/k}}), O(e^{-\pi\sqrt{2kN}})$

where $$A(k) = \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320} \quad \text{up to } O(k^5)$$

Correction to Airy function
→How about for large k??

- February 2012 : Numerical simulation in the whole region(=this talk)

[Hanada-M.H.-Honma-Nishimura-Shiba-Yoshida]

At least up to instanton effect, **for all k,**

Free energy is a smooth function of k !!

Localization of ABJM theory on S^3

[Kapustin-Willet-Yaakov '09]
[Cf. Jafferis '10, Hama-Hosomichi-Lee '10]

$$S_{\text{ABJM}} = S_{\text{CS}} + S_{\text{Matter}}$$

In $\mathcal{N}=2$ language,

$$\text{ABJM} \supset \text{vectormultiplet} \times 2 + \text{Bi}-\text{fundamental chiralmultiplet} \times 2$$
$$+ \text{Anti}-\text{bi}-\text{fundamental chiralmultiplet} \times 2$$

$$\mathcal{N}=2 \text{ Vectormultiplet} \supset \left(A_\mu, \sigma, D, \lambda, \bar{\lambda}\right)$$
$$\mathcal{N}=2 \text{ Chiralmultiplet} \supset \left(\phi, \bar{\phi}, F, \bar{F}, \psi, \bar{\psi}\right)$$

Let δ_ϵ and $\delta_{\bar{\epsilon}}$ be SUSY trans. generated by $\epsilon, \bar{\epsilon}$.

$\left(\delta_\epsilon^2 = 0 = \delta_{\bar{\epsilon}}^2\right)$

$$\delta_\epsilon \delta_{\bar{\epsilon}} \text{Tr}\left[\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\right] = S_{\mathcal{N}=2 \text{ SYM}}$$
$$\delta_\epsilon \delta_{\bar{\epsilon}} \text{Tr}\left[\bar{\psi}\psi - 2i\bar{\phi}(\sigma^{(1)} - \sigma^{(2)})\phi - \bar{\phi}\phi\right] = S_{\mathcal{N}=2 \text{ chiralmult.}}$$

(Cont'd) Localization of ABJM theory on S[3]

$$\delta_\epsilon \delta_{\bar{\epsilon}} \mathrm{Tr}\left[\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\right] = S_{\mathcal{N}=2\ \mathrm{SYM}}$$

$$\delta_\epsilon \delta_{\bar{\epsilon}} \mathrm{Tr}\left[\bar{\psi}\psi - 2i\bar{\phi}(\sigma^{(1)} - \sigma^{(2)})\phi - \bar{\phi}\phi\right] = S_{\mathcal{N}=2\ \mathrm{chiralmult.}}$$

$$Z_{\mathrm{ABJM}} = Z(t) = \int d\Phi \exp\left[-S_{\mathrm{CS}}[\Phi] - t(S_{\mathcal{N}=2\ \mathrm{SYM}} + S_{\mathrm{Matter}})\right]$$

$t \to \infty$

Saddle point:

$$\sigma^{(1)} = \mathrm{const.},\ \sigma^{(2)} = \mathrm{const.},\ D^{(1)} = -\sigma^{(1)},\ D^{(2)} = -\sigma^{(2)},\ \text{Other fields} = 0$$

Taking the diagonal gauge as $\sigma^{(1)} = \mathrm{diag}(\mu_i)$, $\sigma^{(2)} = \mathrm{diag}(\nu_i)$,

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N\mu}{(2\pi)^N} \frac{d^N\nu}{(2\pi)^N} \frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2 \left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2} \exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2 - \nu_i^2)\right]$$

Gauge 1-loop

Matter 1-loop

CS term

Simplification of ABJM matrix model

[Kapustin-Willett-Yaakov '10, Okuyama '11, Marino-Putrov '11]

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N\mu}{(2\pi)^N} \frac{d^N\nu}{(2\pi)^N} \frac{\prod_{i<j} \left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2 \left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2} \exp\left[\frac{ik}{4\pi}\sum_{i=1}^N (\mu_i^2-\nu_i^2)\right]$$

Cauchy identity:

$$u_i = e^{\mu_i}, \quad v_i = e^{\nu_i}$$

$$\frac{\prod_{i<j}(u_i-u_j)(v_i-v_j)}{\prod_{i,j}(u_i+v_j)} = \sum_\sigma (-1)^\sigma \prod_i \frac{1}{u_i+v_{\sigma(i)}} \Leftrightarrow \frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]} = \sum_\sigma (-1)^\sigma \prod_i \frac{1}{2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)}$$

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N} \prod_i \frac{1}{\left[2\cosh\left(\frac{\mu_i-\nu_i}{2}\right)\right]\left[2\cosh\left(\frac{\mu_i-\nu_{\sigma(i)}}{2}\right)\right]} \exp\left[\frac{ik}{4\pi}\sum_{i=1}^N(\mu_i^2-\nu_i^2)\right]$$

Fourier trans.: $$\frac{1}{2\cosh p} = \frac{1}{\pi}\int dx \frac{e^{\frac{2i}{\pi}px}}{2\cosh x}$$

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma(-1)^\sigma \frac{1}{\pi^{2N}} \int d^Nx d^Ny \frac{1}{\prod_i 2\cosh x_i \cdot 2\cosh y_i}$$
$$\int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N} \exp\left[\frac{i}{\pi}\sum_i(\mu_i-\nu_i)x_i + \frac{i}{\pi}\sum_i(\mu_i y_i - \nu_i y_{\sigma(i)}) + \frac{ik}{4\pi}\sum_{i=1}^N(\mu_i^2-\nu_i^2)\right]$$

(Cont'd) Simplification of ABJM matrix model

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \frac{1}{\pi^{2N}} \int d^N x d^N y \frac{1}{\prod_i 2\cosh x_i \cdot 2\cosh y_i} \int \frac{d^N\mu}{(2\pi)^N}\frac{d^N\nu}{(2\pi)^N} \exp\left[\frac{i}{\pi}\sum_i (\mu_i - \nu_i)x_i + \frac{i}{\pi}\sum_i (\mu_i y_i - \nu_i y_{\sigma(i)}) + \frac{ik}{4\pi}\sum_{i=1}^N (\mu_i^2 - \nu_i^2)\right]$$

Gaussian integration

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \frac{1}{k^N \pi^{2N}} \int d^N x d^N y \frac{1}{\prod_i 2\cosh x_i \cdot 2\cosh y_i} e^{-\frac{2i}{k\pi}\sum_{i=1}^N x_i (y_i - y_{\sigma(i)})}$$

Fourier trans.: $\frac{1}{2\cosh p} = \frac{1}{\pi}\int dx \frac{e^{\frac{2i}{\pi}px}}{2\cosh x}$

$$Z_{\mathrm{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \int \frac{d^N x}{(2\pi k)^N} \frac{1}{\prod_i 2\cosh\left(\frac{x_i}{2}\right) \cdot 2\cosh\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}$$

Cauchy id.: $\sum_\sigma (-1)^\sigma \prod_i \frac{1}{2\cosh\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)} = \frac{1}{2^N}\prod_{i<j}\tanh^2\left(\frac{x_i - x_j}{2k}\right)$

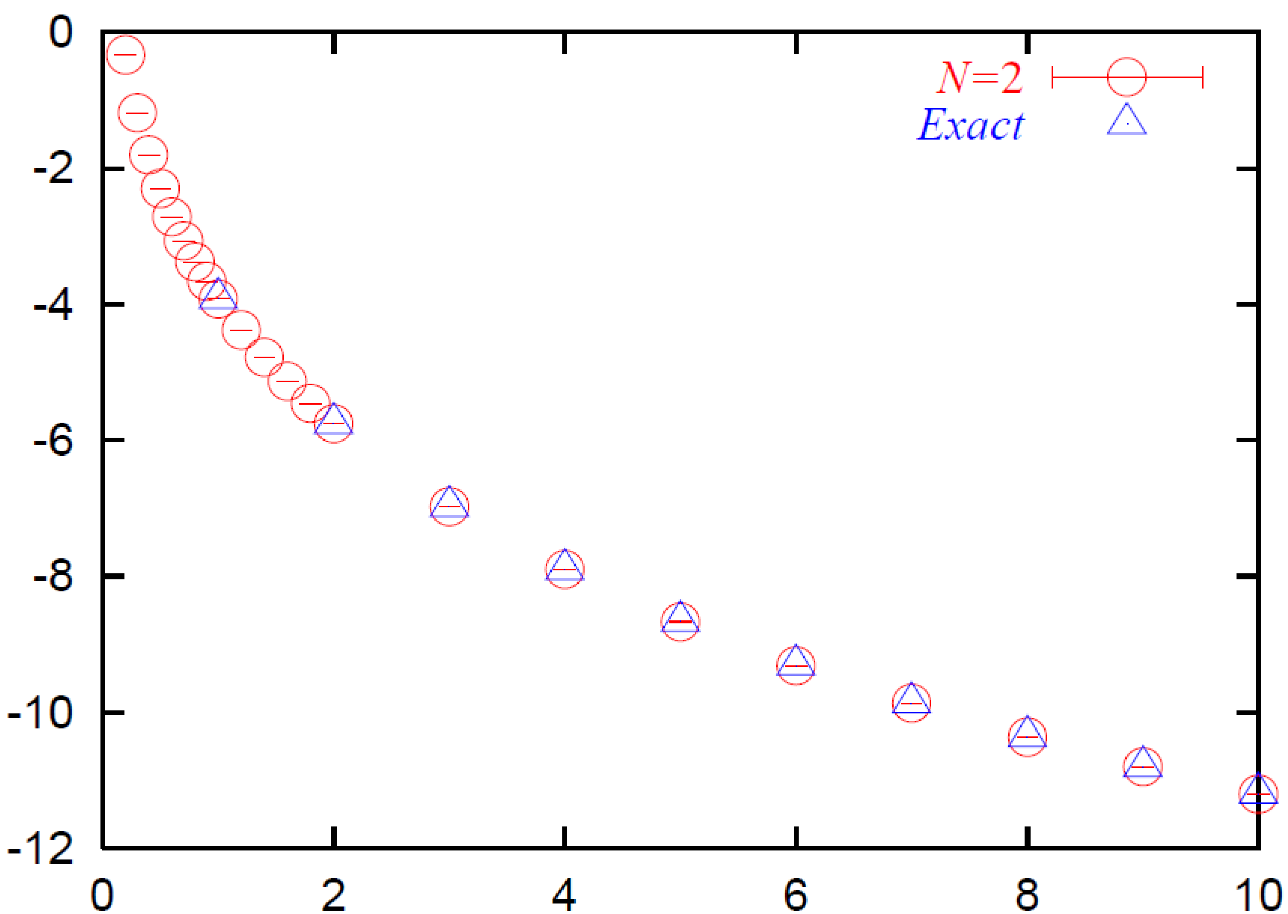
$$Z(N,k) = \frac{1}{2^N N!}\int \frac{d^N x}{(2\pi k)^N} \frac{\prod_{i<j}\tanh^2\left(\frac{x_i - x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)}$$

Warming up: Free energy for N=2

There is the exact result for N=2: [Okuyama '11]

$$
\begin{cases}
Z_{ABJM} = \frac{1}{16}\left[\frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2}-\frac{s}{k}\right)\tan^2\frac{\pi s}{k} + \frac{(-1)^{\frac{k-1}{2}}}{\pi}\right] & \text{for odd } k \\
Z_{ABJM} = \frac{1}{16}\left[\frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2}-\frac{s}{k}\right)^2\tan^2\frac{\pi s}{k}\right] & \text{for even } k
\end{cases}
$$

F (free energy)



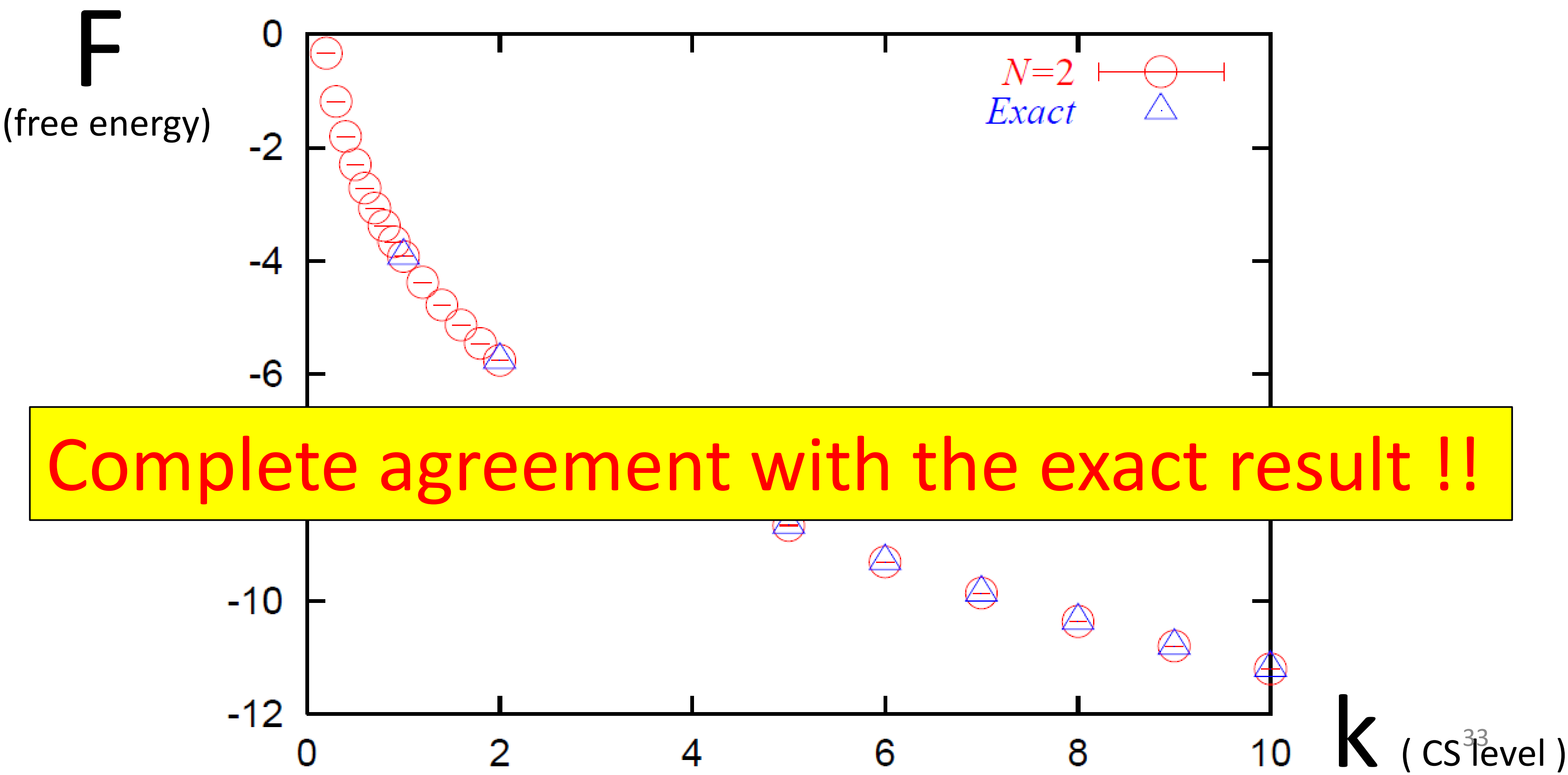
k (CS level)

Warming up: Free energy for N=2

There is the exact result for N=2: [Okuyama '11]

$$Z_{ABJM} = \frac{1}{16}\left[\frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2}-\frac{s}{k}\right)\tan^2\frac{\pi s}{k} + \frac{(-1)^{\frac{k-1}{2}}}{\pi}\right] \quad \text{for odd k}$$

$$Z_{ABJM} = \frac{1}{16}\left[\frac{1}{k}\sum_{s=1}^{k-1}(-1)^{s-1}\left(\frac{1}{2}-\frac{s}{k}\right)^2\tan^2\frac{\pi s}{k}\right] \quad \text{for even k}$$



Complete agreement with the exact result !!

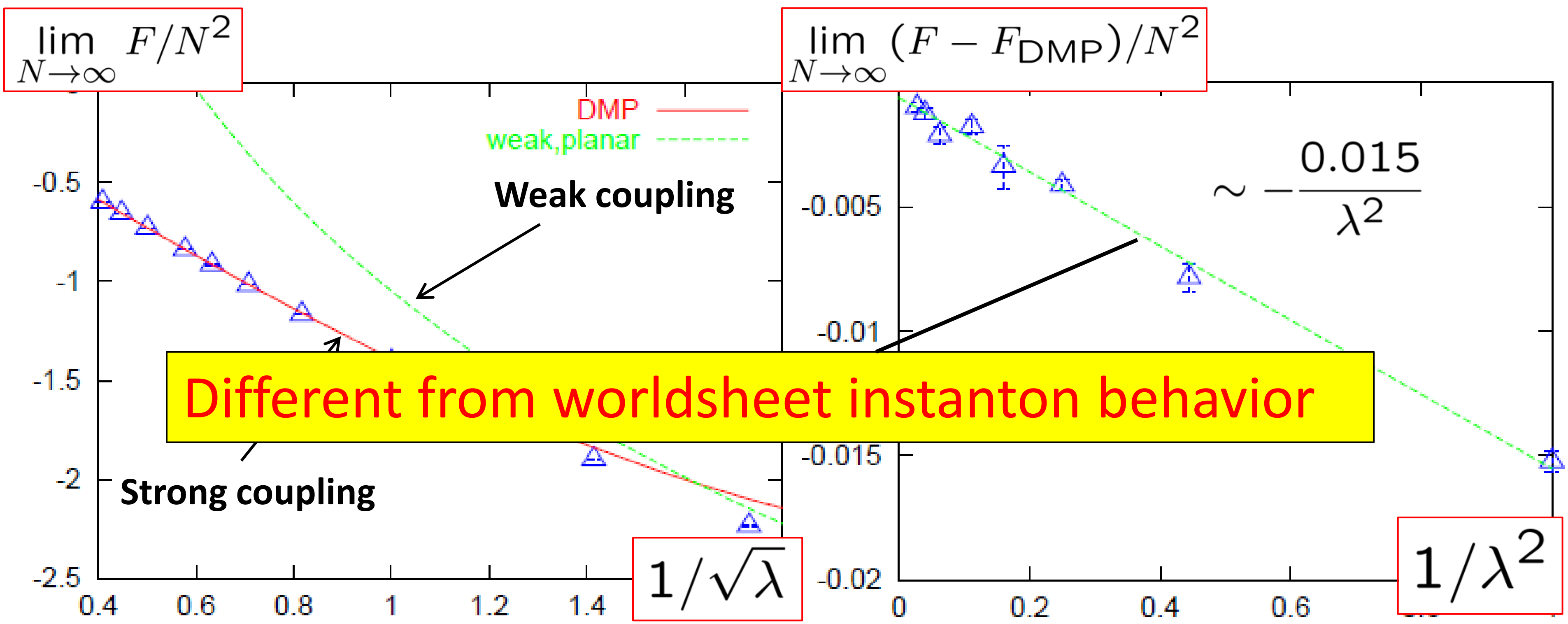
Result for Planar limit

$\left(\lambda = \frac{N}{k} = \text{fixed},\ N \to \infty\right)$

[Drukker-Marino-Putrov '10]

- Weak couling : $F_{\text{weak,planar}} = N^2\left(\log 2\pi\lambda - \frac{3}{2} - 2\log 2\right)$ up to $O(\lambda^2)$
- Strong coupling : $F_{\text{DMP}} = -\frac{\pi\sqrt{2}}{3}\frac{(\lambda - 1/24)^{3/2}}{\lambda^2}N^2$ up to $O(e^{-2\pi\sqrt{2\lambda}})$

Worldsheet instanton

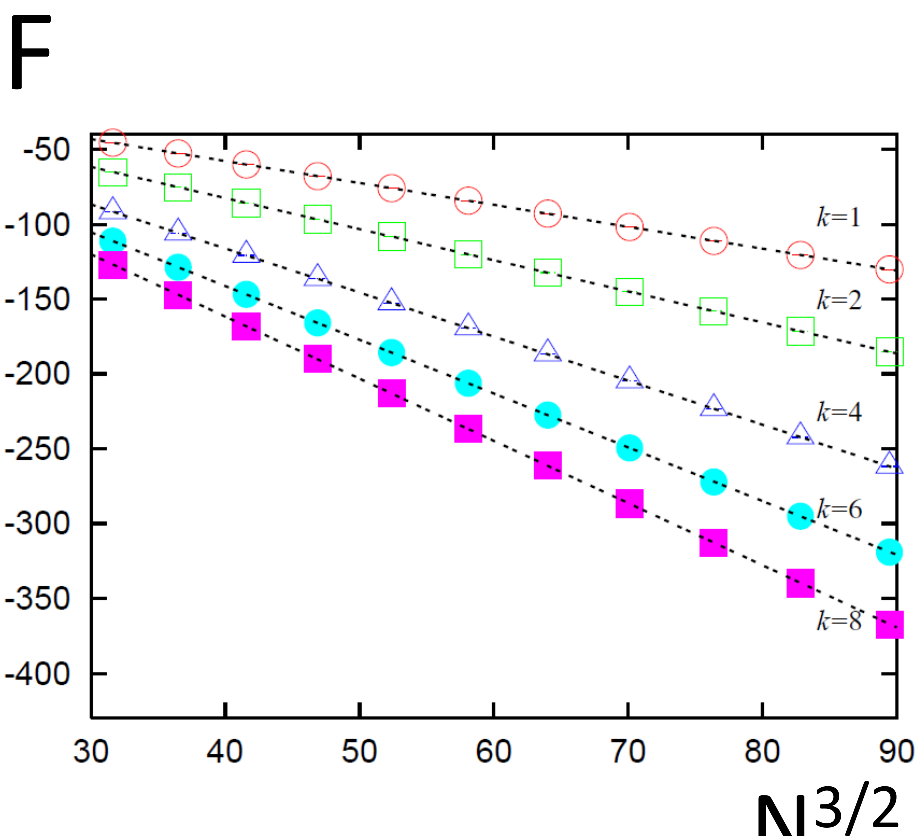


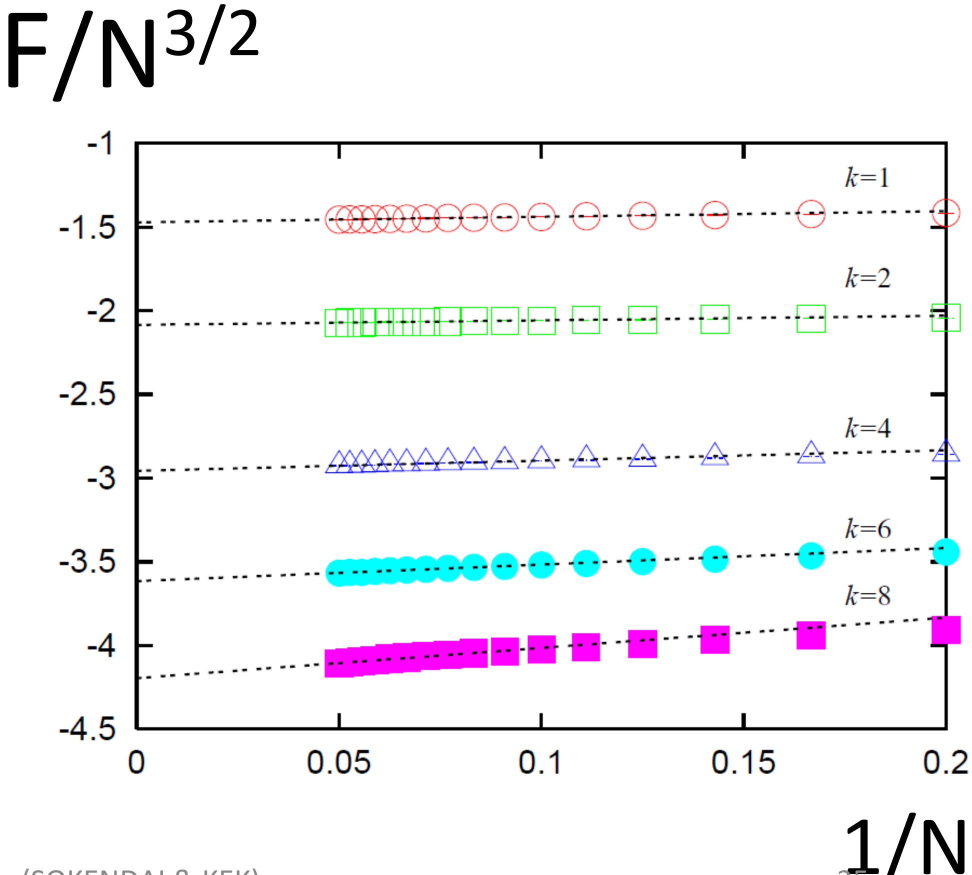
strong ⟷ weak　　strong ⟷ weak

3/2 power low in 11d SUGRA limit

[Drukker-Marino-Putrov '10, Herzog-Klebanov-Pufu-Tesileanu '10]

11d classical SUGRA: $F = -\frac{\pi\sqrt{2}}{3}\sqrt{k}N^{\frac{3}{2}}$





Comparison in the whole region

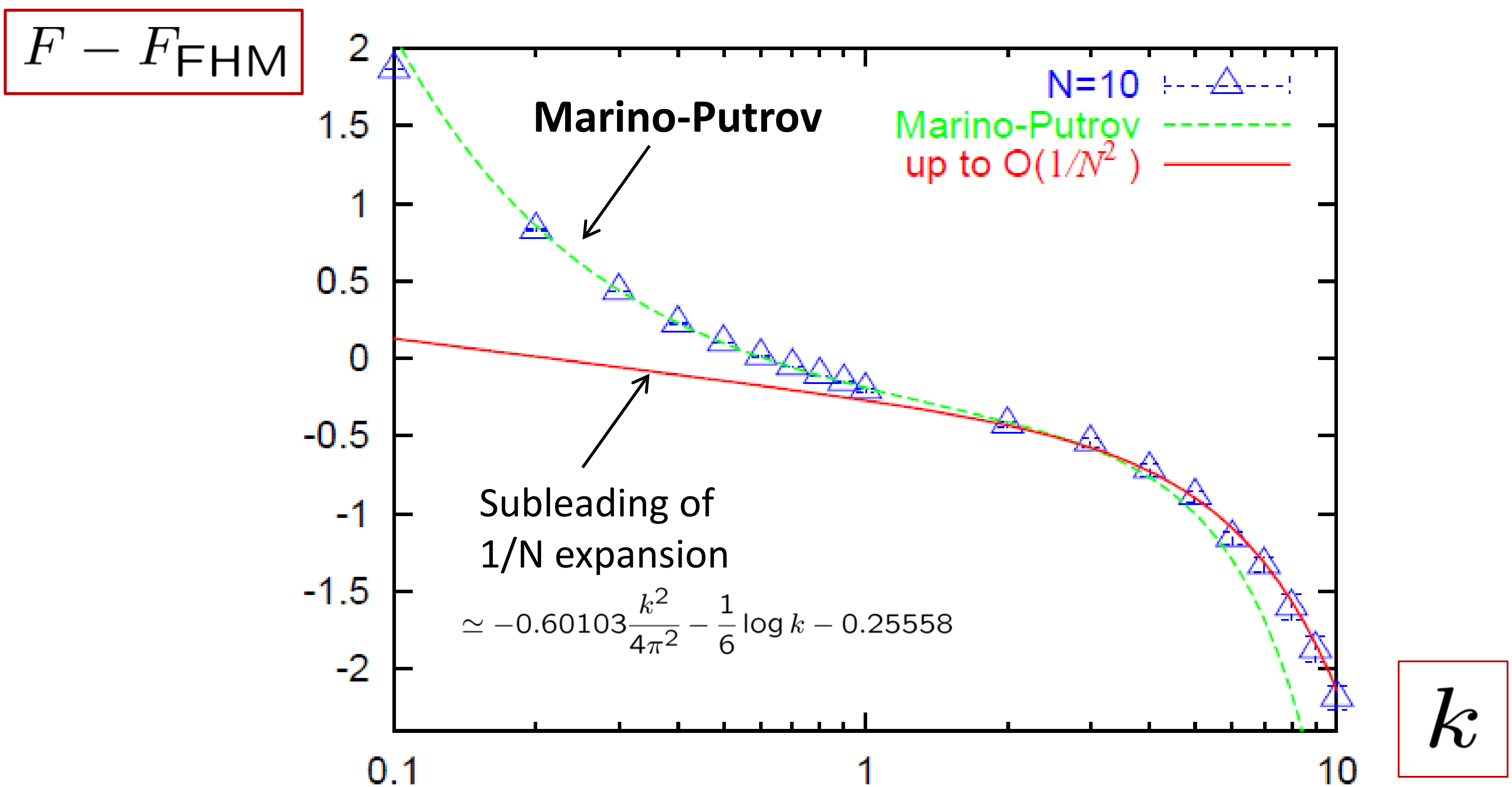
Calculation for k<<1, k<<N

[Marino-Putrov '12]

$$F_{\mathrm{MP}} = \log\left[\frac{1}{\sqrt{2}}\left(4\pi^2 k\right)^{1/3} \mathrm{Ai}\left[\left(\frac{\pi k^2}{\sqrt{2}}\right)^{2/3}\left(\frac{N}{k} - \frac{1}{24} - \frac{1}{3k^2}\right)\right]\right] + A(k) - \frac{1}{2}\log 2$$

up to $O(e^{-2\pi\sqrt{2N/k}}), O(e^{-\pi\sqrt{2kN}})$

$$A(k) = \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320} \qquad \text{up to } O(k^5)$$



Fermi gas approach

[Marino-Putrov '11]

$$Z(N,k) = \frac{1}{2^N N!}\int \frac{d^N x}{(2\pi k)^N}\frac{\prod_{i<j}\tanh^2\left(\frac{x_i-x_j}{2k}\right)}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)}$$

Cauchy id.: $\sum_\sigma (-1)^\sigma \prod_i \frac{1}{2\cosh\left(\frac{x_i-x_{\sigma(i)}}{2k}\right)} = \frac{1}{2^N}\prod_{i<j}\tanh^2\left(\frac{x_i-x_j}{2k}\right)$

$$Z_{\text{ABJM}} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \int \frac{d^N x}{(2\pi k)^N}\frac{1}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)\cdot 2\cosh\left(\frac{x_i-x_{\sigma(i)}}{2k}\right)} = \frac{1}{N!}\sum_\sigma (-1)^\sigma \int d^N x \prod_{i=1}^N \rho(x_i, x_{\sigma(i)})$$

Regard as a Fermi gas system

Result:

$$F_{\text{Fermi}} = \log\left[\frac{1}{\sqrt{2}}(4\pi^2 k)^{1/3}\,\text{Ai}\left[\left(\frac{\pi k^2}{\sqrt{2}}\right)^{2/3}\left(\frac{N}{k}-\frac{1}{24}-\frac{1}{3k^2}\right)\right]\right] + A(k) - \frac{1}{2}\log 2$$

up to $O(e^{-2\pi\sqrt{2N/k}}), O(e^{-\pi\sqrt{2kN}})$

where $A(k) = \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320}$ up to $O(k^5)$

Our result says that this remains even for large k??

Origin of Discrepancy for the Planar limit (without MC)

[Marino-Putrov '10]

$$Z_{ABJM} = \frac{1}{N!^2} \int \frac{d^N\mu}{(2\pi)^N} \frac{d^N\nu}{(2\pi)^N} \frac{\prod_{i<j} \left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2 \left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j} \left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2} \exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}(\mu_i^2-\nu_i^2)\right]$$

Analytic continuation: $N_1 \to N, \;\; N_2 \to -N$

[Cf. Yost '91, Dijkgraaf-Vafa '03]

Lens space $L(2,1)=S^3/Z_2$ matrix model:

$$Z_{L(2,1)}(N_1,N_2) = \frac{1}{N_1!N_2!} \int \frac{d^{N_1}\mu}{(2\pi)^{N_1}} \frac{d^{N_2}\nu}{(2\pi)^{N_2}} \prod_{i<j} \left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2 \prod_{a<b} \left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2 \prod_{i,b} \left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2 e^{-\frac{ik}{4\pi}\left(\sum_i \mu_i^2+\sum_a \nu_a^2\right)}$$

※This is dual to the topological A-model on local $F_0=P^1 \times P^1$

[Aganagic-Klemm-Marino-Vafa '02]

Genus expansion:

$$F(g_s,\lambda) = \sum_{g=0}^{\infty} F_g(\lambda) g_s^{2g-2}$$

$$= -\frac{N^2}{(2\pi\lambda)^2} F_0(\lambda) + F_1(\lambda) - \frac{(2\pi\lambda)^2}{N^2} F_2(\lambda) + \cdots$$

$$g_s = \frac{2\pi i}{k}$$

(Cont'd))Origin of Discrepancy for the Planar limit (without MC)

[Drukker-Marino-Putrov '10]

The "derivative" of planar free energy is exactly found as

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G^{2,3}_{3,3}\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{matrix} \middle| -\frac{\kappa^2}{16}\right) + \frac{i\pi^2\kappa}{2} {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right) \qquad \lambda(\kappa) = \frac{\kappa}{8\pi} {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\frac{\kappa^2}{16}\right)$$

We impose the boundary condition: $F_0(0) = 0$ $\left(\text{Cf. } F_{\text{weak,planar}} = N^2\left(\log 2\pi\lambda - \frac{3}{2} - 2\log 2\right)\right)$

By using asymptotic behavior,

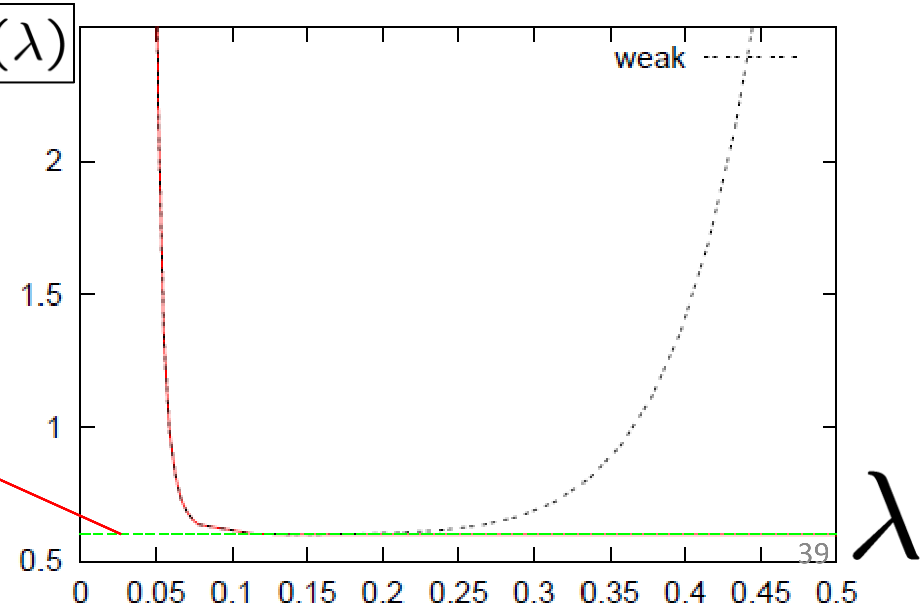
$$F_0(\lambda) = c_0 + \hat{F}_0(\lambda) + \mathrm{O}\left(e^{-8\pi\sqrt{2\hat{\lambda}}}\right) \qquad \left(\hat{\lambda} = \lambda - \frac{1}{24}\right)$$

$$\hat{F}_0(\lambda) = \frac{4\pi^3\sqrt{2}}{3}\hat{\lambda}^{3/2} - e^{-2\pi\sqrt{2\hat{\lambda}}} + e^{-4\pi\sqrt{2\hat{\lambda}}}\left(\frac{9}{8} + \frac{1}{\pi\sqrt{2\hat{\lambda}}}\right) - e^{-6\pi\sqrt{2\hat{\lambda}}}\left(\frac{82}{27} + \frac{9\sqrt{2}}{4\pi\sqrt{\hat{\lambda}}} + \frac{1}{\pi^2\hat{\lambda}} + \frac{\sqrt{2}}{12\pi^3\hat{\lambda}^{3/2}}\right)$$

Necessary for satisfying b.c. , taken as 0 for previous works

$F_0(\lambda) - \hat{F}_0(\lambda)$

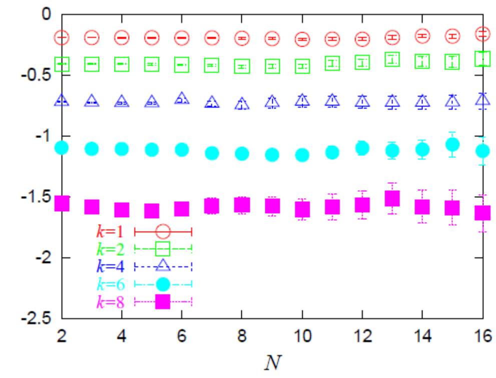
$c_0 \simeq 0.60103$



Origin of discrepancy for all genus

Discrepancy is fitted by

$$F - F_{\rm FHM} \simeq -0.60103\frac{k^2}{4\pi^2} - \frac{1}{6}\log k - 0.25558$$



This is explained by ``constant map'' contribution in language of topological string:

[Bershadsky-Cecotti-Ooguri-Vafa '93, Faber-Pandharipande '98, Marino-Pasquetti-Putrov '09]

$$F_{\rm const} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

Divergent, but Borel summable:

$$-\frac{1}{3}\int_0^\infty \frac{1}{e^{kx}-1}\left(\frac{3}{x^3} - \frac{1}{x} - \frac{3}{x\sinh^2 x}\right)$$

Comparison with discrepancy and Fermi gas

$$F_{\text{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

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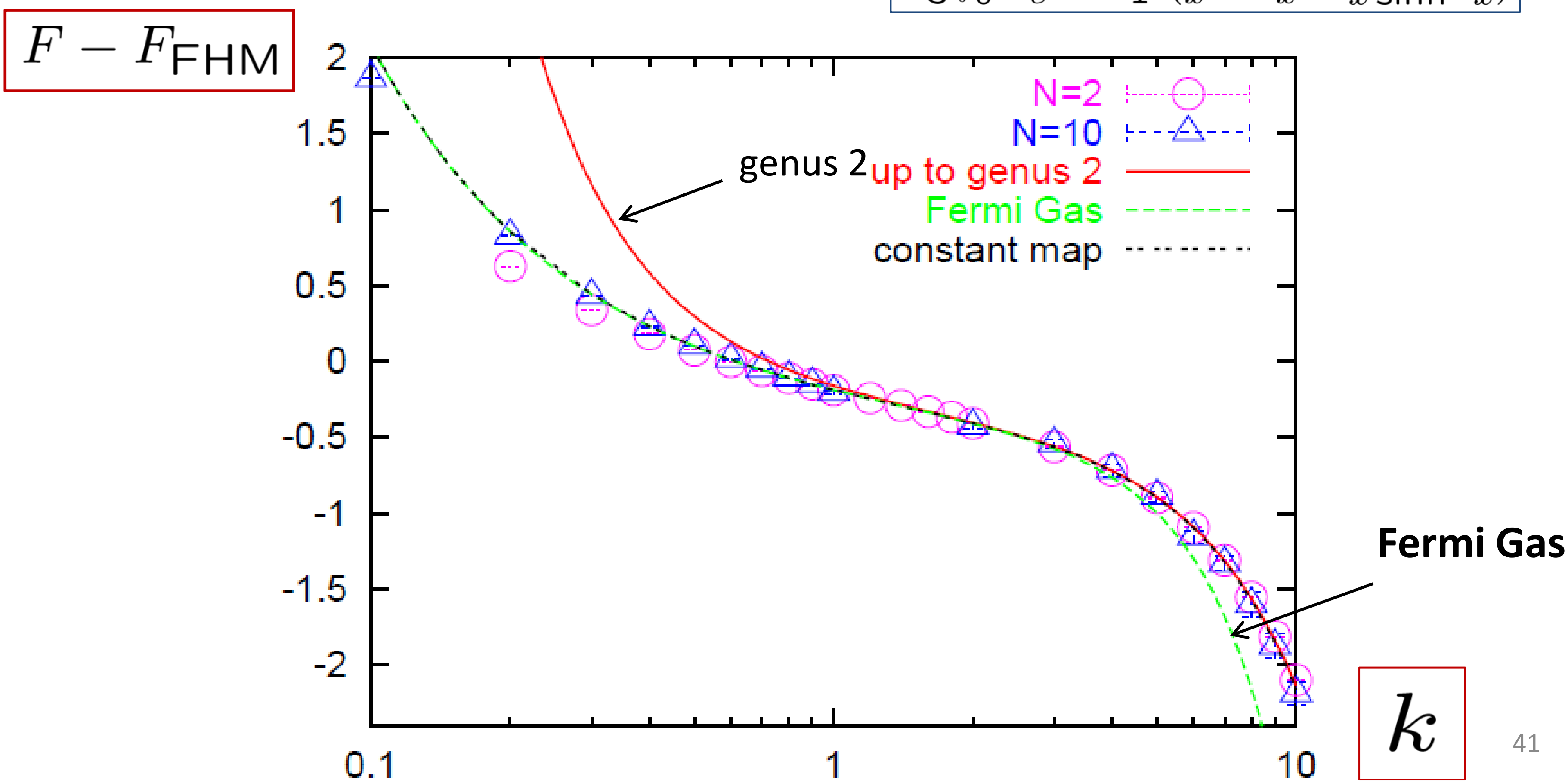


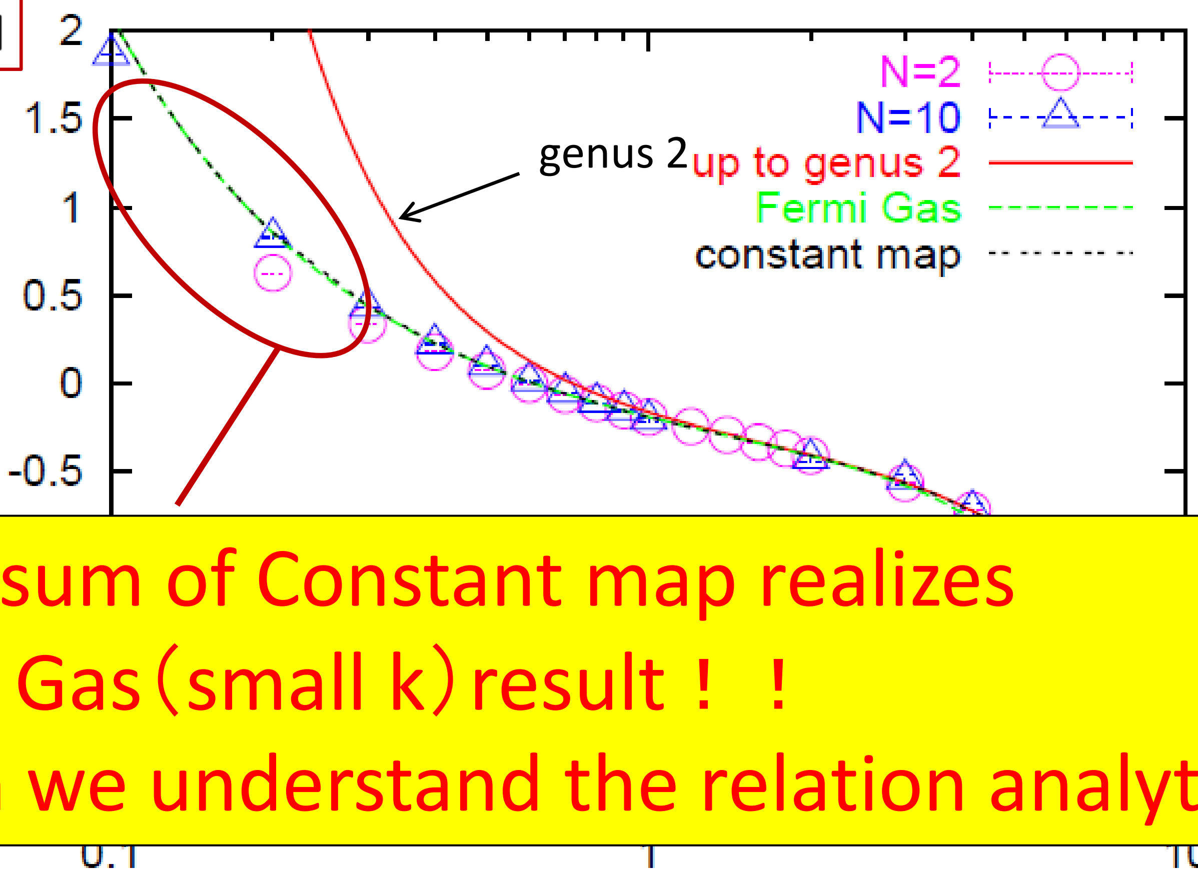
Comparison with discrepancy and Fermi gas

$$F_{\mathrm{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

Divergent, but Borel summable:

$$-\frac{1}{3}\int_0^{\infty}\frac{1}{e^{kx}-1}\left(\frac{3}{x^3}-\frac{1}{x}-\frac{3}{x\sinh^2 x}\right)$$

$F - F_{\mathrm{FHM}}$



Borel sum of Constant map realizes
Fermi Gas（small k）result！！
→Can we understand the relation analytically?

Fermi Gas from Constant map

Constant map contribution :

$$F_{\text{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) + \sum_{g=2}^{\infty}\frac{4^g B_{2g}B_{2g-2}}{4g(2g-2)(2g-2)!}\left(\frac{2\pi i}{k}\right)^{2g-2}$$

Borel sum

$$F_{\text{const}} = -\frac{\zeta(3)}{8\pi^2}k^2 - \frac{1}{6}\log k + \frac{1}{6}\log\frac{\pi}{2} + 2\zeta'(-1) - \frac{1}{3}\int_0^\infty \frac{1}{e^{kx}-1}\left(\frac{3}{x^3} - \frac{1}{x} - \frac{3}{x\sinh^2 x}\right)$$

Expand around k=0

True for all k?

$$F_{\text{const}} = -\frac{1}{2}\log 2 + \frac{2\zeta(3)}{\pi^2 k} + \sum_{n=1}^{\infty}(-1)^n\frac{B_{2n}B_{2n-2}}{(2n)!}\pi^{2n-2}k^{2n-1}$$

$$= -\frac{1}{2}\log 2 + \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320} + \frac{\pi^4 k^5}{907200} + \cdots ,$$

All order form？

Agrees with Fermi Gas result！

→Fermi Gas result is asymptotic series around k=0

Note on Level shift

[Kao-Lee-Lee '95]

$$\mathcal{L} = k\mathcal{L}_{\text{CS}} + \frac{1}{g_{\text{YM}}^2}\mathcal{L}_{\text{YM}}$$

topological mass $\sim g_{\text{YM}}^2$

Integrate out all fields except the gauge field

At 1-loop level,

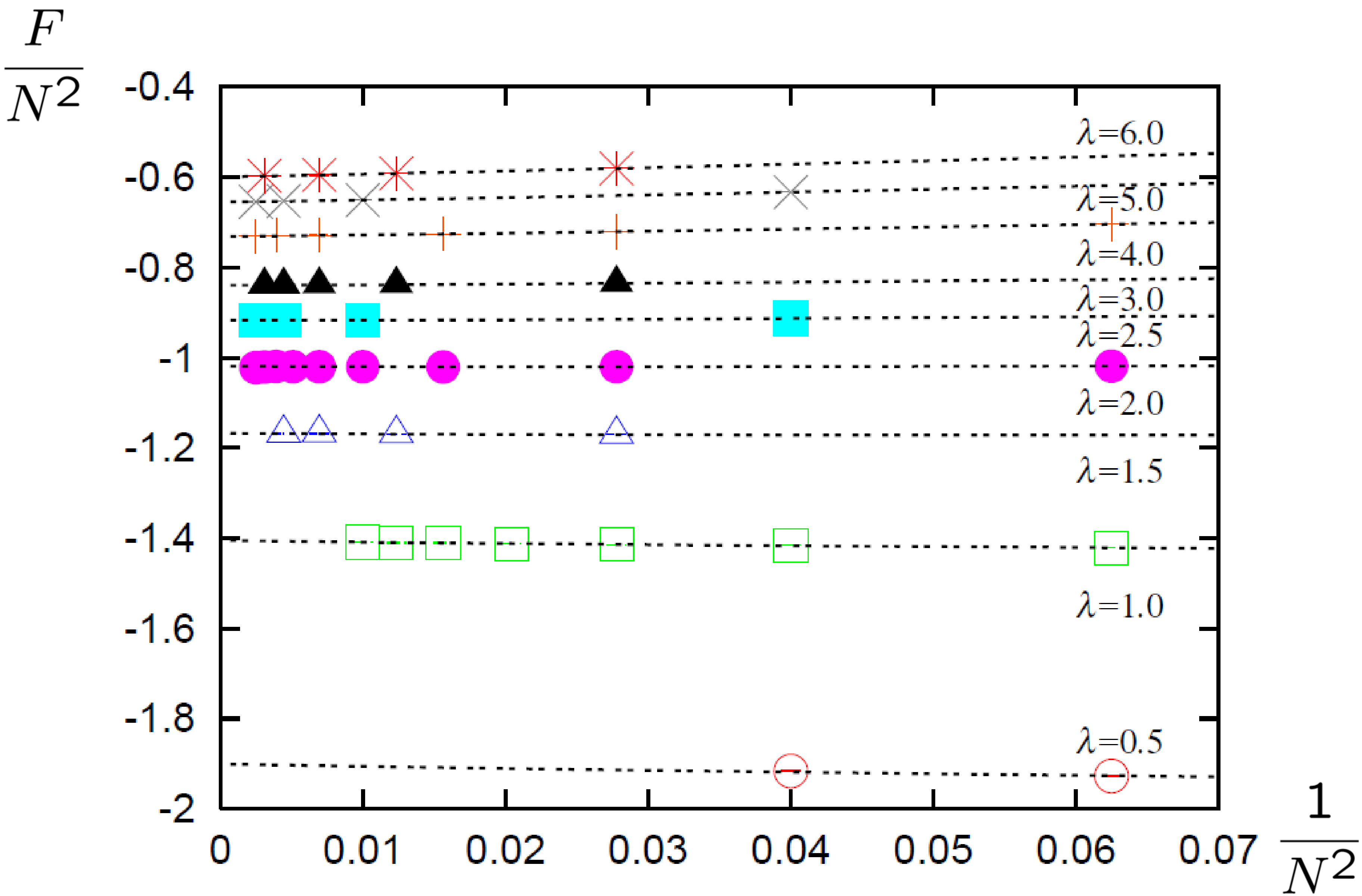
$$\mathcal{N} = 0 \text{ SUSY} \quad : \quad \delta k = N$$

$$\mathcal{N} = 1 \text{ SUSY} \quad : \quad \delta k = N/2$$

$$\mathcal{N} = 2 \text{ SUSY} \quad : \quad \delta k = 0$$

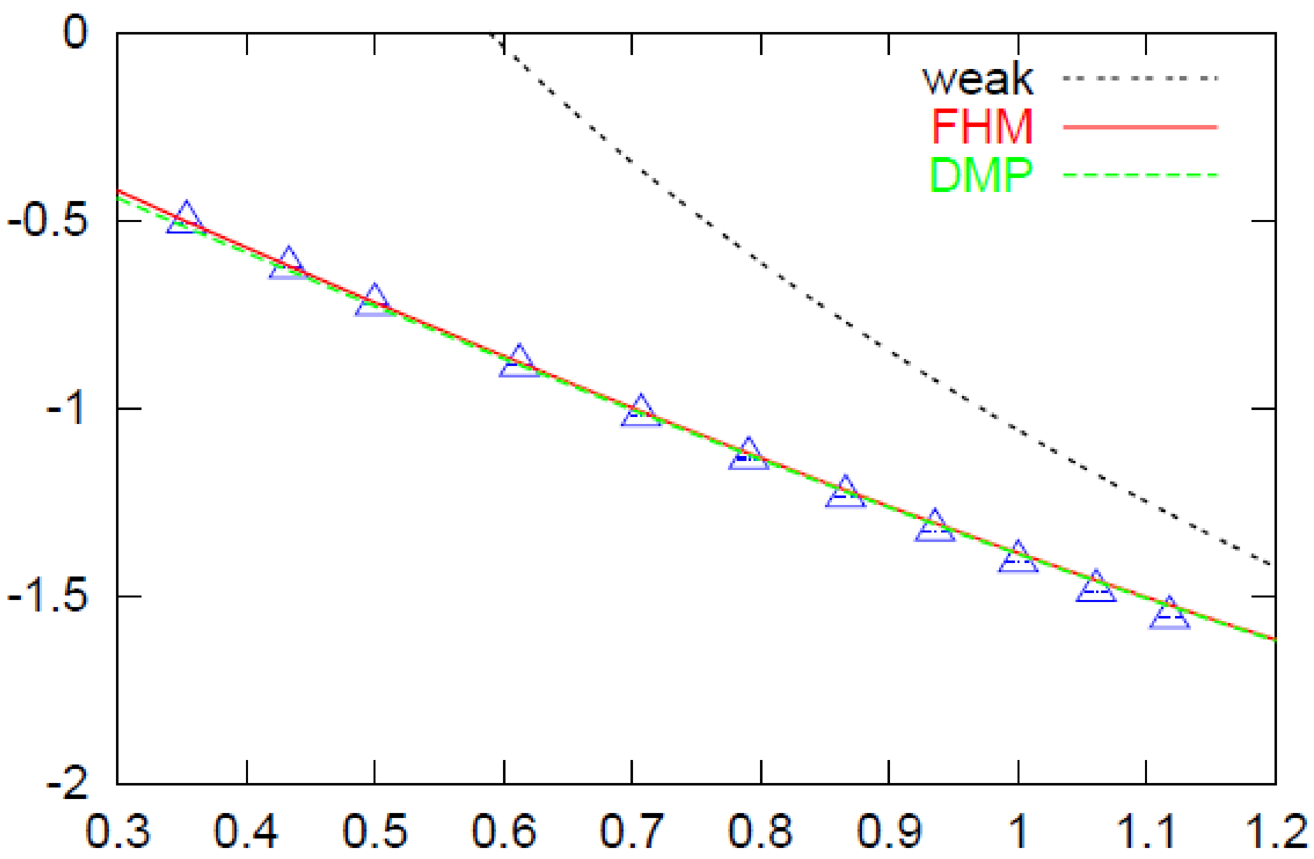
$$\mathcal{N} = 3 \text{ SUSY} \quad : \quad \delta k = 0$$

Taking planar limit



N=8

$\frac{F}{N^2}$



$1/\sqrt{\lambda}$

Higher genus

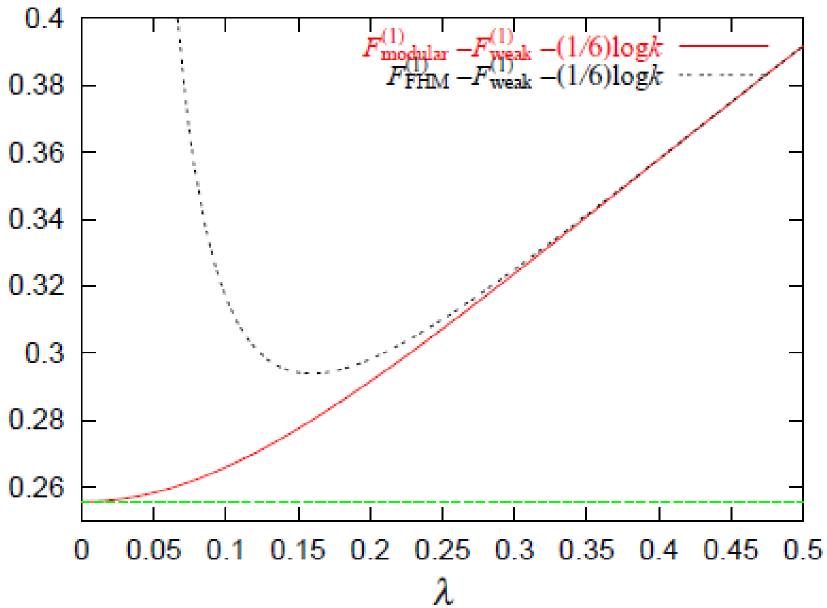
$$F^{(1)}_{\rm modular}(\lambda) = -\log \eta(\tau)$$

$$F^{(2)}_{\rm modular}(\lambda) = \frac{1}{432\vartheta_2^4\vartheta_4^8}\left(-\frac{5}{3}E_2^3 + 3\vartheta_2^4 E_2^2 - 2E_4E_2\right) + \frac{16\vartheta_2^{12} + 15\vartheta_2^8\vartheta_4^4 + 21\vartheta_2^4\vartheta_4^8 + 2\vartheta_4^{12}}{12960\vartheta_2^4\vartheta_4^8},$$

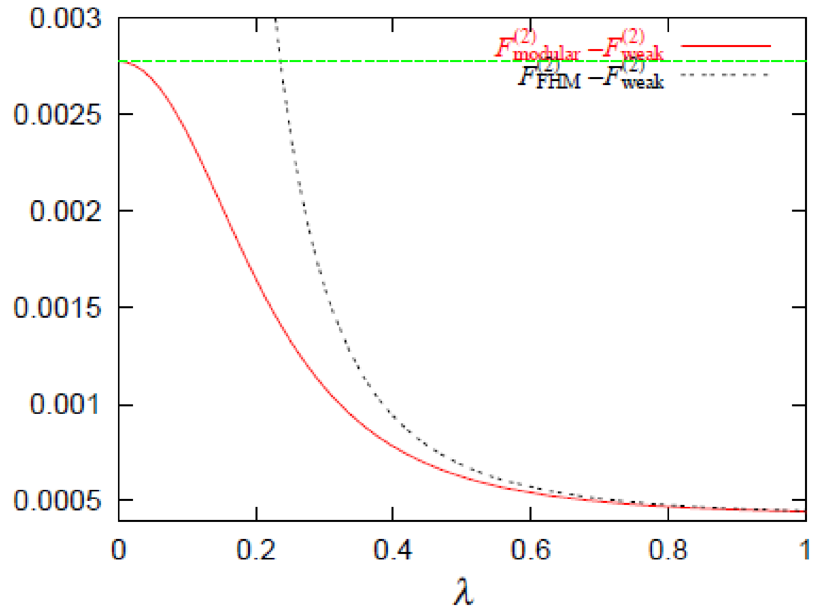
$$F^{(1)}_{\rm weak}(\lambda) = -\frac{1}{6}(\log\lambda + \log k) + 2\zeta'(-1)$$

$$F^{(2)}_{\rm weak}(\lambda) = -\frac{B_4}{16\pi^2\lambda^2}.$$

$F^{(1)}_{\rm modular} - \left(F^{(1)}_{\rm weak} + \frac{1}{6}\log k\right)$:



$F^{(2)}_{\rm modular} - F^{(2)}_{\rm weak}$



$$\Delta F^{(1)} = -\frac{1}{6}\log k - c_1, \qquad \Delta F^{(2)} = -c_2, \qquad c_1 \simeq 0.25558, \qquad c_2 \simeq 0.0027777$$

ABJ(M) matrix model and Lens space matrix model

Hermitian supermatrix: $\Phi = \begin{pmatrix} A & \Psi \\ \Psi^\dagger & C \end{pmatrix}$

$$\left(\begin{array}{c} A(C) : N_1 \times N_1 (N_2 \times N_2) \text{ “Bosonic” Hermitian matrix} \\ \Psi : N_1 \times N_2 \text{ Complex “Fermionic” matrix} \end{array} \right)$$

Supermatrix model

[Alvarez-Gaume ’91, Yost ’91,]

$$Z_s(N_1|N_2) = \int \mathcal{D}\Phi \; e^{-\frac{1}{g_s}\mathrm{Str}V(\Phi)}$$

Diagonalizing $A = \mathrm{diag}(\mu_i)$, $C = \mathrm{diag}(\nu_i)$,

$$Z_s(N_1|N_2) = \int d\mu d\nu \frac{\prod_{i<j}(\mu_i - \mu_j)^2 \prod_{a<b}(\nu_a - \nu_b)^2}{\prod_{i,a}(\mu_i - \nu_a)^2} \times e^{-\frac{1}{g_s}(\sum_i V(\mu_i) - \sum_a V(\nu_a))}$$

Bosonic matrix model

$$Z_b(N_1|N_2) = \int \mathcal{D}A\mathcal{D}C e^{-\frac{1}{g_s}(\mathrm{tr}V(A) + \mathrm{tr}V(C))}$$

Diagonalizing $A = \mathrm{diag}(\mu_i)$, $C = \mathrm{diag}(\nu_i)$,

$$Z_b(N_1|N_2) = \int d\mu d\nu \prod_{i<j}(\mu_i - \mu_j)^2 \prod_{a<b}(\nu_a - \nu_b)^2 \prod_{i,a}(\mu_i - \nu_a)^2 \times e^{-\frac{1}{g_s}(\sum_i V(\mu_i) + \sum_a V(\nu_a))}$$

[Diagramatic proof: Dijkgraaf-Vafa ’03 , Dijkgraaf-Gukov-Kazakov-Vafa ’03]

$$Z_s(N_1|N_2) = Z_b(N_1|-N_2)$$

(Cont'd)ABJ(M) matrix model and Lens space matrix model

[Marino-Putrov '10]

ABJ(M) matrix model:

$$Z_{\rm ABJ}(N_1,N_2)=\frac{1}{N_1!N_2!}\int\frac{d^{N_1}\mu}{(2\pi)^{N_1}}\frac{d^{N_2}\nu}{(2\pi)^{N_2}}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{a<b}\left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2}{\prod_{i,b}\left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2}e^{-\frac{ik}{4\pi}(\sum_i\mu_i^2-\sum_a\nu_a^2)}$$

Lens space $L(2,1)=S^3/Z_2$ matrix model:

$$Z_{L(2,1)}(N_1,N_2)=\frac{1}{N_1!N_2!}\int\frac{d^{N_1}\mu}{(2\pi)^{N_1}}\frac{d^{N_2}\nu}{(2\pi)^{N_2}}\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\prod_{a<b}\left[2\sinh\left(\frac{\nu_a-\nu_b}{2}\right)\right]^2\prod_{i,b}\left[2\cosh\left(\frac{\mu_i-\nu_b}{2}\right)\right]^2e^{-\frac{ik}{4\pi}(\sum_i\mu_i^2+\sum_a\nu_a^2)}$$

$$\boxed{Z_{\rm ABJ}(N_1,N_2)=Z_{L(2,1)}(N_1,-N_2)}$$

Lens space matrix model and topological string

[Aganagic-Klemm-Marino-Vafa '02]

Chern – Simons theory on S^3/Z_2

[Witten '92]

Low energy effective theory
of D – branes wrapping S^3/Z_2
in the topological A – model on $T^*(S^3/Z_2)$

Canonical
quantization

Large N transition
[Cf. Gopakumar-Vafa '98]

Lens space S^3/Z_2 Matrix Model

The topological A – model
on local $F_0 = P^1 \times P^1$

Mirror symmetry

The topological B – model
onthe mirror manifold