Leibniz rule, locality and supersymmetry on lattice

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Contents

Motivation

- Leibniz no-go theorem for infinite volume system on lattice

  Kato, Sakamoto and So, JHEP 0805 (2008) 057

Two important clues

Translation inv. & Locality (for infinite system)

=> Holomorphic function property

- "Locality" in finite system
- Multi-flavor system vs. Matrix representation

Diff. op $(D\varphi)_n \equiv \sum_{m=-\infty}^{\infty} D_{nm}\varphi_m, \quad D_{mn} = D(m-n)$

$$\hat{D}(w) \equiv \sum_{m-n=-\infty}^{\infty} w^{m-n} D(m-n), \quad w = e^{ia_L(p \pm i\varepsilon)},$$

$$e^{-a_L|\varepsilon|} \le |w| \le e^{a_L|\varepsilon|},$$

Holomorphic property of $\hat{D}(w)$ for infinite system

= exponential locality of $D_{mn} = D(m-n)$

For finite system, $(D\varphi)_n \equiv \sum_{m=-N/2+1}^{N/2} D_{nm}\varphi_m$

$$\hat{D}(w) \equiv \sum_{m-n=-N/2+1}^{N/2} w^{m-n} D(m-n)$$

Just an algebraic function of w

Algebraic function of w -> Irrelevant to locality

We must find alternative "language" for "locality" in finite systems.

Another motivations

- Multi(finite)-flavor system ⟶ no-go theorem on Leibniz rule
- Multi(infinite)-flavor system ⟶ Leibniz rule holds

What difference between them?

K-S-S, 2008

1 Definition of local lattice theories

Lattice space $\quad a_L = 1 \qquad \text{size} = N$

Lattice fields $\quad \varphi_n \qquad \psi_{\alpha,n} \qquad U_{n,\mu}$

Lattice action $\quad S(\varphi_n, \psi_n, U_{n,\mu})$

Product rule $\quad (\varphi \times \eta)_n \equiv \sum_{m\ell}^{\infty} C_{nm\ell} \varphi_m \eta_\ell$

Difference operator

$$(D\varphi)_n \equiv \sum_m^{\infty} D_{nm} \varphi_m \quad , \quad \left(D1\right)_n = 0$$

Locality on Field products and difference operators

- Translation invariance

Product rule $\qquad C_{nm\ell} = C_{n+k\ m+k\ \ell+k} = C(n-\ell, m-\ell)$

Difference operator $\qquad D_{n\ m} = D_{n+k\ m+k} = D(n-m)$

w-representation $\qquad w \approx e^{ipa_L} = e^{ip},\ z \approx e^{iqa_L} = e^{iq}$

Infinite system $\qquad \hat{C}(w,z) \equiv \sum_{m,n=-\infty}^{\infty} w^m z^n C(n,m)$

$$\hat{D}(w) \equiv \sum_{m=-\infty}^{\infty} w^m D(m) \quad , \quad \hat{D}(1) = 0$$

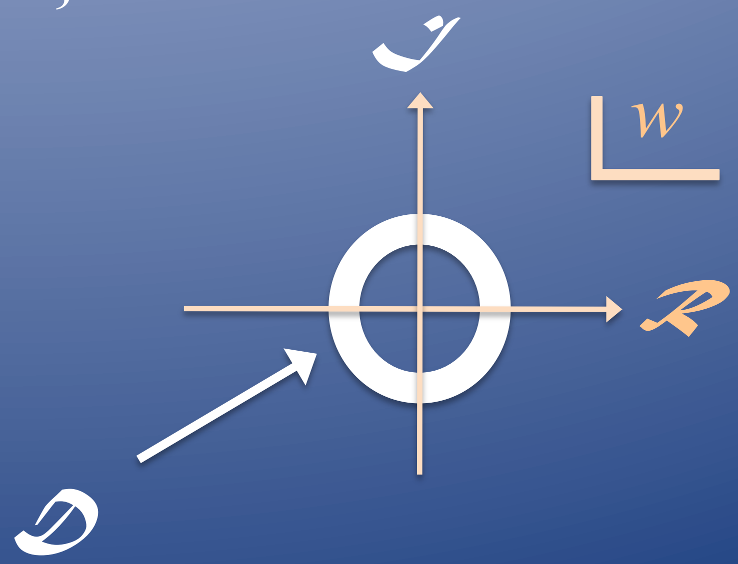
A lemma for infinite system

Locality = holomorphism on w-rep.

holomorphic domain

$$\mathcal{D} = \{ w \,|\, 1-\varepsilon < |w| < 1+\varepsilon, \varepsilon > 0 \}$$

Annulus region ,
NOT circle
in a complex plane



$$D(-n)=\frac{1}{2\pi i}\oint_{|w|=1\pm\frac{\varepsilon}{2}} dw\, w^{-1+n}\hat{D}(w),$$ Indep. of $\pm\frac{\varepsilon}{2}$

- $\hat{D}(w)$ holom. at $w\in\mathscr{D}$ $\Rightarrow |D(-n)|\le\frac{1}{2\pi}\left|\oint_{|w|=1\pm\frac{\varepsilon}{2}} dw\,\hat{D}(w)\right| e^{\mp\frac{\varepsilon(n-1)}{2}}$
- $|D(n)|\le Ke^{-\frac{\varepsilon|n|}{2}}$ (for large $|n|$) $\Rightarrow \hat{D}(w)$ holom.

$\because$)

infinite series $\sum_{n=-\infty}^{\infty} w^n D(n)$ uniformly converge at $w\in\mathscr{D}$

2 Leibniz rule in infinite and finite systems

- For infinite system,

Assume translation inv., locality, Leibniz rule
and nontrivial product

→ no-go theorem K-S-S (2008)

Leibniz rule by using translation inv. and locality

$$\hat{C}(w,z)(\hat{D}(wz)-\hat{D}(w)-\hat{D}(z))=0 \quad , \quad \hat{D}\left(1\right)=0$$

→ Nontrivial product $\hat{C}(w,z)\neq 0 \;\Rightarrow\; \hat{D}(wz)-\hat{D}(w)-\hat{D}(z)=0$

Unique solution $\hat{D}(w)=K\log w\approx iKp$

SLAC type, not holomorphic in $w\in\mathcal{D}$ q.e.d

- For finite system,
  we cannot use the holomorphic property of complex functions.
- Instead, we use discrete property of an ortho-complete set in finite size(N).

$$\sum_{a}^{N} \omega_N^{(n+m)a} = N\delta_{n+m,0}, \quad \sum_{n}^{N} \omega_N^{(a+b)n} = N\delta_{a+b,0}$$

- A key point

N-dep. (size dep.) for finite system

N-dep. is an important criterion for considered operators.

Or N as Infrared cut-off for infinite system

A lemma for finite system

"Locality" $=$ exponential damping, $\propto e^{-\varepsilon|n|}$

$1 \ll |n| \ll N$

$=$ "*N*-absolute bound"
instead of holomorphism

"*N*-absolute bound" for diff. op and product rule

$$\max_a \left\{ \left| \hat{D}_{a+i\varepsilon N} \right| \right\} = O(N^0), \quad \max_{a,b} \left\{ \left| \hat{C}_{a+i\varepsilon N, b+i\eta N} \right| \right\} = O(N^0)$$

for large N and finite $\varepsilon > 0,\ \eta > 0$

$$\hat{D}_a \equiv \sum_m^N w_a^m D(m) = \sum_m^N \omega_N^{am} D(m) \quad , \quad \hat{D}_0 = 0$$

$$\hat{D}_{a+i\varepsilon N} \equiv \sum_m^N \omega_N^{(a+i\varepsilon N)m} D(m)$$

Complex Extension

$$\hat{C}_{ab} \equiv \sum_{m,n}^N w_a^m z_b^n C(n,m) = \sum_{m,n}^N \omega_N^{am+bn} C(n,m)$$

$$\hat{C}_{a+i\varepsilon N, b+i\eta N} \equiv \sum_{m,n}^N \omega_N^{(a+i\varepsilon N)m+(b+i\eta N)n} C(n,m)$$

Complex Extension

$$w_a \equiv e^{\frac{2\pi i a}{N}} = \omega_N^a, \; z_b \equiv e^{\frac{2\pi i b}{N}} = \omega_N^b, \; \omega_N \equiv e^{\frac{2\pi i}{N}}, \; \omega_N^N = 1$$

"locality" = exponential damping, **1<< n << N**

$$\sum_{a}^{N}\omega_N^{(n+m)a} = N\delta_{n+m,0}, \quad \sum_{n}^{N}\omega_N^{(a+b)n} = N\delta_{a+b,0}$$

$$\hat{D}_{a+i\varepsilon N} \equiv \sum_{m}^{N}\omega_N^{m(a+i\varepsilon N)} D_m$$

Identity

$$\frac{1}{N}\sum_{a}^{N}\omega_N^{n(a+i\varepsilon N)}\hat{D}_{a+i\varepsilon N} = \frac{1}{N}\sum_{a}^{N}\omega_N^{an}\hat{D}_a = D(-n)$$

a la Cauchy's Integral Theorem

$$\left|D(-n)\right| \le \frac{1}{N}\sum_{a}^{N}\left|\hat{D}_{a+i\varepsilon N}\right| e^{-2\pi\varepsilon n} \le \max_a\left\{\left|\hat{D}_{a+i\varepsilon N}\right|\right\} e^{-2\pi\varepsilon n}$$

The sufficient conditions for "locality"

$$\max_a\left\{\left|\hat{D}_{a+i\varepsilon N}\right|\right\} = O(N^0), \quad \max_{a,b}\left\{\left|\hat{C}_{a+i\varepsilon N, b+i\eta N}\right|\right\} = O(N^0)$$

Necessary Condition: conversely, if

$$\left|D(-n)\right| \le K e^{-\kappa|n|} \quad, \kappa > 0, \text{ for } 1 << |n| << N$$

Then for $\kappa \pm 2\pi\varepsilon > 0$

$$|\hat{D}_{a+i\varepsilon N}| = \left|\sum_m^N \omega_N^{m(a+i\varepsilon N)} D(m)\right| \le \sum_m^N e^{-2\pi\varepsilon m} \left|D(m)\right| \le K \sum_m^N e^{-(\kappa \pm 2\pi\varepsilon)|m|} = O\left(N^0\right)$$

Similarly, $\left|C(-m,-n)\right| \le K e^{-\kappa|m| - \lambda|n|} \quad, \quad \kappa > 0, \; \lambda > 0$

for $\kappa \pm 2\pi\varepsilon > 0, \lambda \pm 2\pi\eta > 0$

$$\left|\hat{C}_{a+i\varepsilon N, b+i\eta N}\right| = \left|\sum_{m,n}^N \omega_N^{m(a+i\varepsilon N)+n(b+i\eta N)} C(m,n)\right| \le \sum_{m,n}^N e^{-2\pi(\varepsilon m+\eta n)} \left|C(m,n)\right| \le K \sum_{m,n}^N e^{-(\kappa\pm 2\pi\varepsilon)|m| - (\lambda \pm 2\pi\eta)|n|} = O\left(N^0\right)$$

No-go theorem for Leibniz rule in finite system

- A theorem:

Translation inv., "locality", Leibniz rule and nontrivial product $\hat{C}_{a,b}$ cannot be simultaneously satisfied on finite lattice.

Proof)

Leibniz rule by using only translation inv.

$$\hat{C}_{a,b}(\hat{D}_{a+b} - \hat{D}_{a} - \hat{D}_{b}) = 0$$

- For any a and b, if $\hat{C}_{a,b} \neq 0$, then

$$\hat{D}_{a+b} - \hat{D}_a - \hat{D}_b = 0$$

The general solution

$$\hat{D}_a = \frac{a}{b}\hat{D}_b \propto a$$

This is SLAC-type owing to $\omega_N^a = e^{\frac{2\pi i a}{N}} = e^{ip} \Rightarrow \quad a \propto p$

And "locality" excludes this SLAC solution

$\because)$

$$\frac{1}{N}\sum_a^N \left| \hat{D}_{a+i\varepsilon N} \right| = \frac{1}{N}\sum_a^N \left| a + i\varepsilon N \right| \left| \frac{\hat{D}_b}{b} \right| = O\left(N^1\right) > O(N^0)$$

For Large N,

$\hat{C}_{a,b}$: nontrivial $\Leftrightarrow \hat{C}_{a,b} \neq 0$ for almost a,b

$\Leftrightarrow \#\{(a,b)\,|\,\hat{C}_{a,b} \neq 0\}/N^2 \to 1$

$\hat{C}_{a,b}$: trivial $\Leftrightarrow \#\{(a,b)\,|\,\hat{C}_{a,b} \neq 0\}/N^2 \to 0$

or $\#\{(a,b)\,|\,\hat{C}_{a,b} \neq 0\}/N^2 \to A, 0 < A < 1$

If $\hat{C}_{a,b}$ is nontrivial,

then $D(n)$ is SLAC-type and "nonlocal".

q.e.d.

What happens if $\hat{C}_{a,b}$ is trivial or nonlocal?

eg. $\hat{C}_{a,b} = K\delta_{a+b,1}$ case, $\frac{1}{N^2}\sum_{a,b}^{N}\left|\hat{C}_{a+i\varepsilon N,b+i\eta N}\right| = O\left(\frac{|K|}{N}\right)$

$$C(m,n) = \frac{1}{N^2}\sum_{a,b}^{N}\omega_N^{-am-bn}\hat{C}_{a,b} = \frac{K}{N^2}\sum_{a}^{N}\omega_N^{-am-(1-a)n} = \frac{K}{N}\delta_{m,n}\omega_N^{-n}$$

$$C_{lmn} = C(l-m, l-n) = \frac{K}{N}\delta_{m,n}\omega_N^{-(l-n)}$$

"nonlocal" product rule ($K = O(N^1)$)
or trivial product rule ($K = O(N^0)$)

Leibniz rule $\rightarrow \hat{D}_1 = \hat{D}_a + \hat{D}_{1-a}$ "local" diff. op.!

Special solution, $\hat{D}_0 = 0,\ \hat{D}_1,\ \hat{D}_2 = \hat{D}_{N-1} = \hat{D}_3 = \hat{D}_{N-2} = \cdots = \frac{1}{2}\hat{D}_1$

If we take such a "nonlocal" product rule,

we can construct "nonlocal" supersymmetric theory.

Lattice action (D=1+0)

$$S=\frac{1}{2}D\phi\cdot D\phi+i\bar{\psi}\cdot D\psi+F\cdot F+\frac{ig}{2}F\cdot\left(\phi\times\phi\right)+ig\phi\cdot\left(\bar{\psi}\times\psi\right)$$

$$\phi\cdot\chi\equiv\sum_{n}^{N}\phi_n\chi_n,\quad\left(\phi\times\chi\right)_n\equiv\sum_{m,l}^{N}C_{nml}\phi_m\chi_l$$

SUSY transf.

$$\delta\phi=\varepsilon\bar{\psi}+\psi\bar{\varepsilon}$$

$$\delta\psi=\varepsilon\left(iD\phi+F\right),\ \delta\bar{\psi}=\bar{\varepsilon}\left(-iD\phi+F\right),$$

$$\delta F=-i\varepsilon D\bar{\psi}-i\bar{\varepsilon}D\psi$$

Action inv.

$$\delta S=0$$

3 Multi-flavor systems and matrix representation

- Fields

$$\varphi_n^a \qquad \psi_n^a$$

- Two kinds of flavor
- Type-A : no kinetic terms (auxiliary fields?)
- Type-B : non-interacting fields

$$\left(\phi\times\chi\right)_l^a \equiv \sum_{m,n,b,c}^{N,N_f} C_{lmn}^{abc}\phi_m^b\chi_n^c,\ \left(D\phi\right)_l^a \equiv \sum_{m,b}^{N,N_f} D_{lm}^{ab}$$

Translation inv.

$$C_{lmn}^{abc} = C^{abc}(m-l,n-l),\ D_{lm}^{ab} = D^{ab}(m-l)$$

w-representation

$$\hat{C}_{LM}^{abc} \equiv \sum_{m,n}^{N}\omega_N^{Lm+Mn}C^{abc}(m,n),\ \hat{D}_L^{ab} \equiv \sum_{m}^{N}\omega_N^{Lm}D^{ab}(m)$$

Flavor matrix form

$$\left(\hat{C}_{LM}^{b}\right)_{ac} \equiv \hat{C}_{LM}^{abc},\ \left(\hat{D}_L\right)_{ab} \equiv \hat{D}_L^{ab}$$

Leibniz rule by flavor matrix form

$$\left(\hat{D}_{L+M}\hat{C}_{L,M}^{a}\right)_{bc} = \sum_d \hat{C}_{L,M}^{bdc}\hat{D}_L^{da} + \left(\hat{C}_{L,M}^{a}\hat{D}_M\right)_{bc}$$

Flavor decomposition by field redefinition

$$\phi_m^a \to \phi_m'^a = \sum_{n,b}^{N} U_{mn}^{ab} \phi_n^b$$

$$D \to UDU^{-1} = D_{diag} + E_+$$

Jordan's standard form

$$\hat{D}_{diag}^{ab} = \hat{\Delta}_L^a \delta_{ab}, \quad \hat{E}_+^{ab} = \varepsilon_L^a \delta_{a,b+1}, \quad \hat{\Delta}_0^a = 0, \quad \varepsilon_L^a = 0,1$$

a,b : flavor indices

L,M : w-rep.(discrete momenta)

Leibniz rule by flavor matrix form

$$\left(\hat{\Delta}_{L+M}^a - \hat{\Delta}_L^b - \hat{\Delta}_M^c\right)\hat{C}_{L,M}^{abc} = \hat{R}_{L,M}^{abc}(\varepsilon)$$

$$\hat{R}_{L,M}^{abc}(\varepsilon) \equiv -\varepsilon_{L+M}^a \hat{C}_{L,M}^{a-1bc} + \varepsilon_L^{b+1} \hat{C}_{L,M}^{ab+1c} + \varepsilon_M^{c+1} \hat{C}_{L,M}^{abc+1}$$

For finite flavor, $\hat{R}_{L,M}^{abc}(\varepsilon) = 0 \Rightarrow \left(\hat{\Delta}_{L+M}^a - \hat{\Delta}_L^b - \hat{\Delta}_M^c\right)\hat{C}_{L,M}^{abc} = 0$

$$\left(\hat{\Delta}^{a}_{L+M} - \hat{\Delta}^{b}_{L} - \hat{\Delta}^{c}_{M}\right)\hat{C}^{abc}_{L,M} \equiv \hat{D}^{abc}_{L,M}\hat{C}^{abc}_{L,M} = 0$$

$$\rightarrow \quad \hat{D}^{abc}_{L,M} = 0 \quad \text{or} \quad \hat{C}^{abc}_{L,M} = 0$$

$\hat{D}^{abc}_{L,M} = 0$ case $\rightarrow \hat{\Delta}^{a}(w) = 0$ at infinite volume limit

For finite flavor and infinite volume system,

type A trivial difference op. $\hat{\Delta}^{a}(w) = 0$

type B trivial field product $\hat{C}^{abc}_{L,M} = 0$

N x N matrix rep. for fields, $\Phi_{mn},\ \Psi_{mn}$

Product of matrix rep. $\left(\Phi\Psi\right)_{mn} \Rightarrow C^{abc}_{lmn} \propto \delta_{a,c+b}\delta_{n-l,b}\delta_{l-m,c}$

Diff. operator of matrix rep. $[d, \Phi\ ]$ by some matrix d

$$\rightarrow\ [d,\ \ ] \propto D \Rightarrow \hat{D}$$

$$D \rightarrow UDU^{-1} = D_{diag} + E_{+}$$

Jordan's standard form by $N_f \times N_f = N \times N$ matrix U

Leibniz rule by flavor matrix form

$$\left(\hat{\Delta}^{a}_{L+M} - \hat{\Delta}^{b}_{L} - \hat{\Delta}^{c}_{M}\right)\hat{C}^{abc}_{L,M} = \hat{R}^{abc}_{L,M}(\varepsilon)$$

$$\begin{cases} N = N_f \\ \hat{R}^{abc}_{L,M}(\varepsilon) \neq 0 \end{cases}$$

Origin of "non-locality" and Leibniz rule

no separation bet. D and C

4 Summary

- Various no-go theorems on "Leibniz rule on lattice"
  finite size system, multi-flavor system
- In infrared cut-off(N) system, we find term. "N-absolute bound" as "locality".
- Three cases keeping Leibniz rule in the system

1. "Local" nontrivial $C_{lmn} \Rightarrow D_{mn}$ is SLAC-type("nonlocal").
2. D_{mn} is "local" $\Rightarrow C_{lmn}$ is "nonlocal" or trivial.
   New possibility supersymmetry application,
   Dondi-Nicolai,(1977) Kawamoto et al, (2012)
3. D_{mn} is SLAC-type("nonlocal") $\Rightarrow C_{lmn}$ is arbitrary.

Multi-flavor system(MF) vs. Matrix representation(MR)

	# component	Leibniz rule	locality	separation
MF	N_f x N	x	local	$D \Leftrightarrow C$
MF	N_f x N	holds	"non-locality"	$D \Leftrightarrow C$
MR	N x $N = N_f$ x N_f	holds	"non-locality" by N infinity	no