Low temperature limit of Lattice QCD

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KN, arXiv:1204.6480
XQCDJ, arXiv : 1204.1412,
KN, Nakamura, JHEP 1204 (2012) 092. EoS by Taylor & MPR
KN, Nakamura, PRD83 (2011) 114507. Imaginary chemical potential
KN, Nakamura, PRD82, 094027('10). Reduction formula

Talk on 6/28 @ Lattice 2012 Cairns, Australia, 6/24-29 2012

Introduction

- *Purpose of this talk*
  - ✓ *We derive the low-T limit of fermion determinant.*
- *Motivation*

1. *interest in low-T region of QCD phase diagram*
   - ✓ *Rich phase structure at low T*
   - ✓ *Low T & finite μ region are still challenge*
2. *A phase at high density limit*
   - ✓ *Hong, NPB582, 451 ('00). Blum, et.al.PRL76, 1019 ('96)*
3. *Such configurations may be used for e.g. reweighting*

Introduction

- *Approach : reduction formula for det Δ*
  - ✓ *Reduction formula : transformation of detΔ regarding Nt*
  - ✓ *Eigenvalues of a reduced matrix in the formula follows a Nt-scaling law*

Reduction Formula for fermion determinant

$$\det \Delta = \xi^{-N_r/2} C_0 \det(\xi + Q) \qquad \xi = e^{-\mu/T}$$

$$Q = (\alpha_1^{-1}\beta_1) \cdots (\alpha_{N_t}^{-1}\beta_{N_t}) \qquad C_0 = \prod_i \det \alpha_i$$

time

$$\begin{cases} \alpha_i = B_i r_- - 2\kappa r_+ \\ \beta_i = (B_i r_+ - 2\kappa r_-) U_4(t_i) \end{cases}$$

Bi : Spatial part of Wilson fermion matrix

r+ = (1+ g4)/2

r- = (1 – g4)/2

Gibbs, PLB 172, 53 ('86). Hasenfratz &Toussaint, NPB371, 539('92). Adams, PRL92, 162002 ('04);PRD70, 045002 ('04), Borici, PTP. Suppl. 153, 335 ('04). Alexandru &Wenger, PRD83, 034502 ('11). KN&AN, PRD82,094027 ('10).

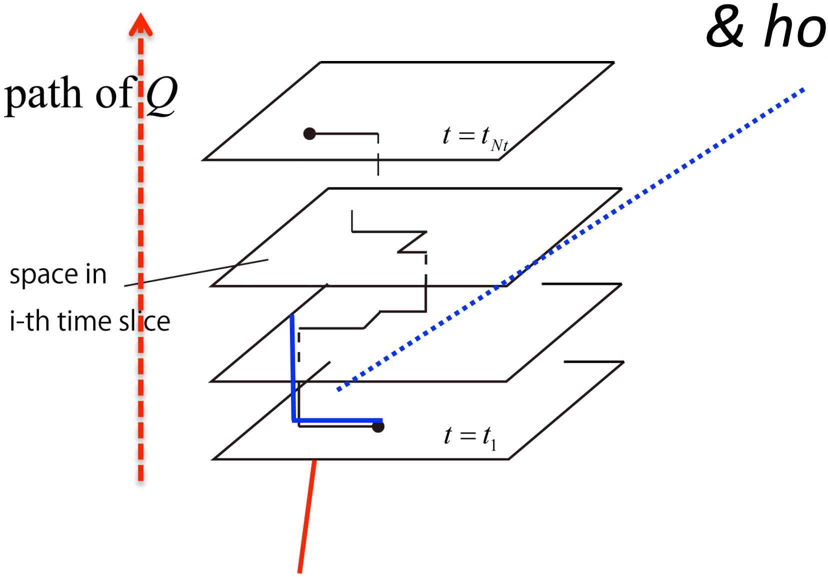
Reduction Formula for fermion determinant

- *Reduced matrix Q*
  - *rank Nr= 4 Nc x Ns^3 (= rank of Δ/Nt)*
  - *independent of μ*

$$Q = \underline{(\alpha_1^{-1}\beta_1)}\cdots(\alpha_{N_t}^{-1}\beta_{N_t})$$

spatial hop at t=t1 & hop to t=t2

Time



4NcNs^3 plane

Simulation

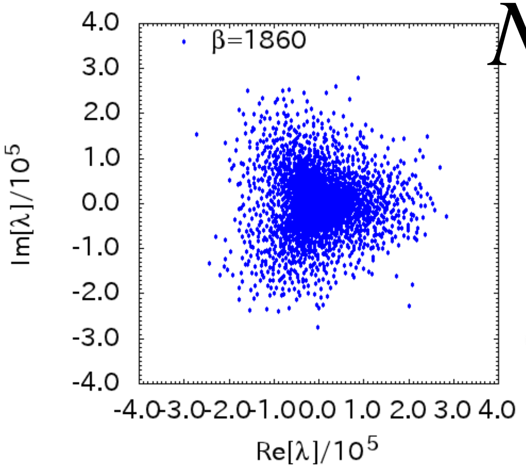
- Spectral properties of Q
  1. symmetry $\lambda \leftrightarrow 1/\lambda^*$
  2. gap $|\lambda|<1,\ \ |\lambda|>1$
  3. Nt-scaling law (new : arXiv:1204.1412)

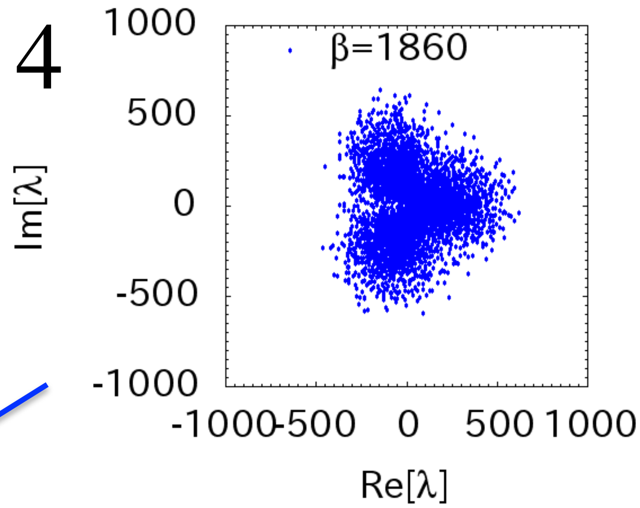
- clover-improved Wilson (Nf=2)+ RG-improved gauge
- Volume : 8^3x4, 8^3x8
- Configurations : μ=0 for β=1.86 (HMC 11K)
- quark mass : LCP with mps/mV ~ 0.8
- Eigenvalues : all ev's by LAPACK
- Details of the spectral properties : XQCDJ, 1204.1412 [hep-lat]

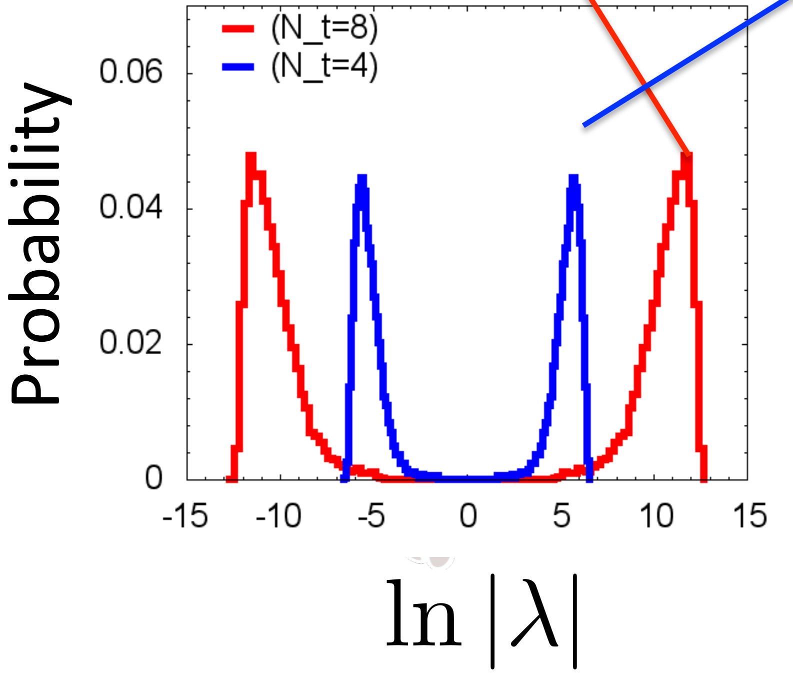
Eigenvalue distribution

$Nt = 8$

$Nt = 4$



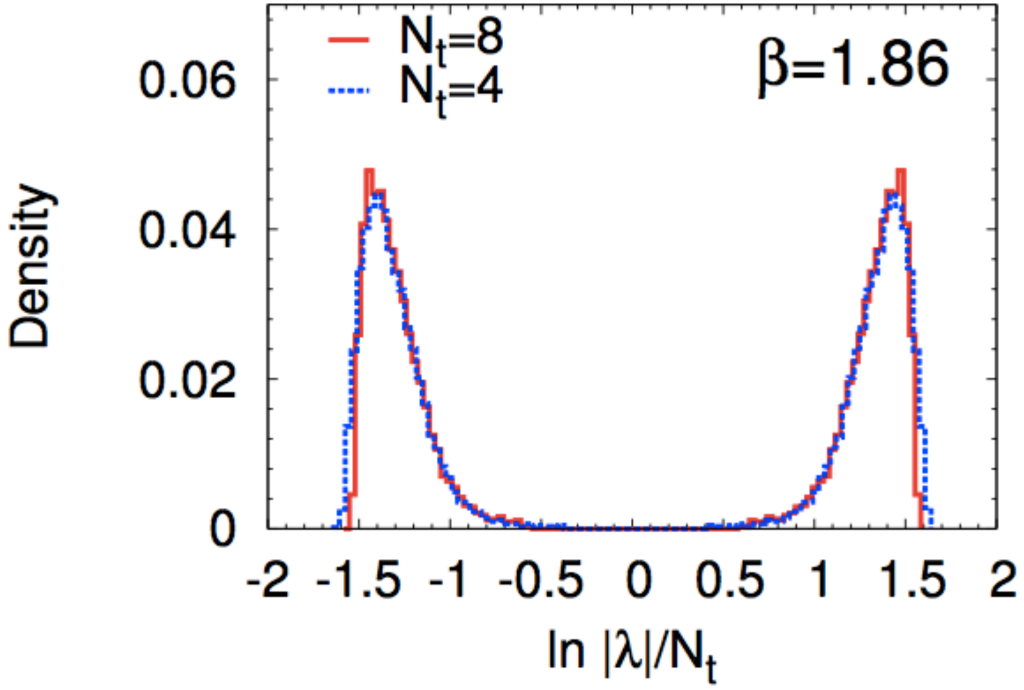




1. *symmetry* $\lambda \leftrightarrow 1/\lambda^*$
2. *gap* $|\lambda| < 1, \ \ |\lambda| > 1$
3. *Nt-scaling law*

Nt-scaling law

Histogram vs (ln|λ|)/Nt



$$|\lambda| = l^{Nt}$$

XQCDJ, arXiv:1204.1412

Is the Nt-scaling physical ?

- *Structure of the reduced matrix* $Q \sim A_1 A_2 \cdots A_{Nt}$

$$A_i = \bar{A} + \delta A_i$$

  - *In equilibrium, A-bar> δA* $Q \sim \bar{A}^{N_t} + \cancel{O(\delta A)}$

- *Similarity of Q and Polyakov line* $P \sim e^{-F/T}$

$$Q = \cdots U_4(t_1) \cdots U_4(t_2) \cdots U_4(t_{N_t})$$

- *Calculation for larger Nt is important, but has technical difficulties* $\lambda_S / \lambda_L < 10^{-16}$

Nt-dep. of det D

$$\det \Delta(\mu) = C_0 \xi^{-N_r/2} \det(Q + \xi)$$

$$|Q - \lambda I| = 0$$

$$|\lambda| = l^{Nt}$$

$$\lambda \leftrightarrow 1/\lambda^*$$

$$|1/\lambda^*| = l^{-Nt}$$

$$\lambda_n = l_n^{N_t} e^{i\theta_n}$$

$$(l > 1)$$

$$= C_0 \xi^{-N_r/2} \prod_{n=1}^{N_r/2} (l_n^{-N_t} e^{i\theta_n} + e^{-\mu/T}) \prod_{n=1}^{N_r/2} (l_n^{+N_t} e^{i\theta_n} + e^{-\mu/T})$$

Low-T limit(large Nt, fixed a, Vs)

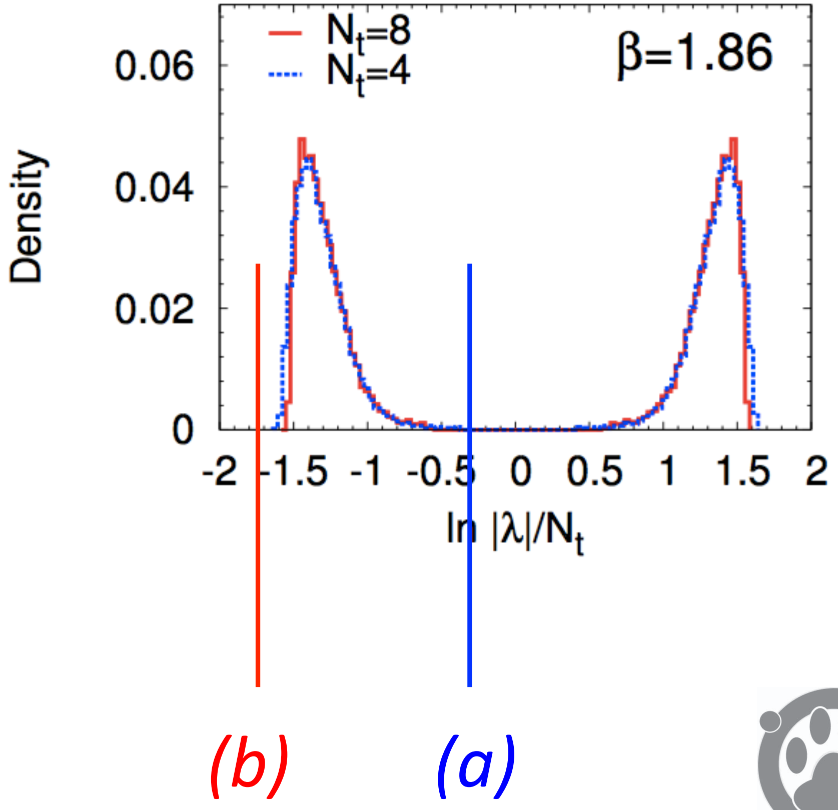
$$\det\Delta(\mu) = C_0\xi^{-N_r/2}\prod_{n=1}^{N_r/2}(\underline{l_n^{-N_t}}e^{i\theta_n} + \underline{e^{-\mu/T}})\prod_{n=1}^{N_r/2}(l_n^{+N_t}e^{i\theta_n} + \cancel{e^{-\mu/T}})$$

a. small μ $\xi > |\lambda_{S,\max}|$

$$\det\Delta(\mu) = C_0\prod_{n=1}^{Nr/2}\lambda_n^L \in R$$

b. large μ $\xi < |\lambda_{\min}|$

$$\det\Delta(\mu) = C_0\xi^{-Nr/2}\det Q \in R$$



Low-T limit

a. small μ

$$\det\Delta(\mu) = C_0 \prod_{i=1}^{N_r/2} \lambda_n^L$$

$$\langle n \rangle = 0$$

b. large μ

$$\det\Delta(\mu) = C_0 \xi^{-N_r/2} \det Q$$

$$= e^{2N_c N_s^3 \mu/T} \prod_{i=1}^{N_t} \det(B_i r_+ - 2\kappa r_-)$$

$$\langle n \rangle = 2N_c N_f$$

- ✓ *Quarks move on a fixed time slice, there is no propagation in t-direction(t-link vanishes)*
- ✓ *Z3 invariant*
- ✓ *ev's are not needed.*

Interpretation

- *Fugacity coefficients is related to free energy*

$$\det\Delta(\mu) = C_0 \sum c_n \xi^n \qquad\qquad Z(\mu) = \sum Z_n \xi^n$$

$$Z_n = e^{-E_n/T} \propto \langle c_n \rangle$$

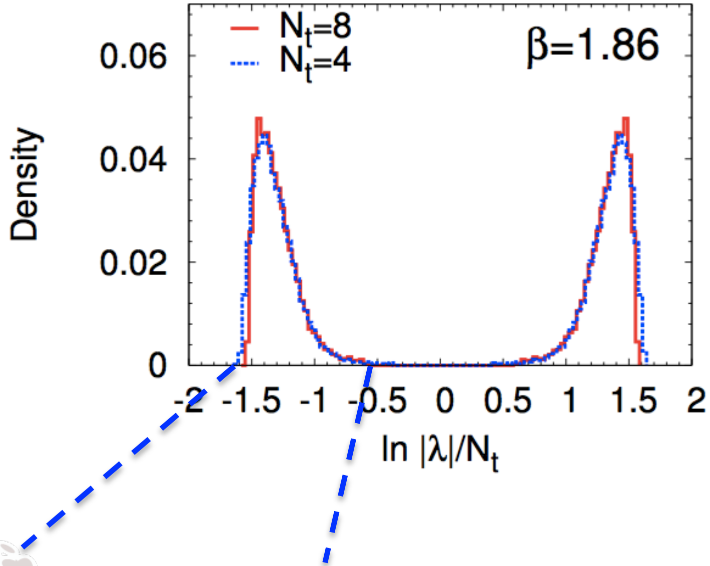
- *An ev near the gap is related to pion mass. (Gibbs('86))*
- *Some of hadron masses are obtained from the evs in thermodynamical approach (Fodor, Szabo Toth ('07))*

Interpretation

- *EV of Q is related to a (microscopic) energy of quark*

$$Q = \cdots U_4(t_1) \cdots U_4(t_2) \cdots U_4(t_{N_t})$$

$$\lambda_n = e^{-\epsilon_n/T + i\theta} \qquad \ln|\lambda_n|/N_t = -\epsilon_n a$$



max. energy
~ cutoff (?)

min. energy ~ mpi /2

Summary

- *We have considered the low-T limit of det D*
  - *by extrapolation based on Nt-scaling property of the reduced matrix.*
  - *with fixed spatial volume and lattice spacing*
- *In low T limit, there are two sign-free cases*
  - *small μ : $<n>=0$ (μ < mpi/2)*
  - *large μ : $<n>=2$Nf Nc (μ > cutoff)*

Summary

- *For confirmation & establishment*
  - ✓*Nt-scaling law for larger Nt*
  - ✓*gap for smaller quark mass*
  - ✓*thermodynamical limit.*
  - ✓*eigenvalues for finer lattice*
- *Application*
  - ✓*numerical simulation at low-T & large* μ *limit*

Low-T limit

a. small μ

$$\det\Delta(\mu) = C_0\xi^{-N_r/2}\prod_{n=1}^{N_r/2}(\cancel{l_n^{-N_t}e^{i\theta_n}} + e^{-\mu/T})\prod_{n=1}^{N_r/2}(l_n^{+N_t}e^{i\theta_n} + \cancel{e^{-\mu/T}})$$

$$\det\Delta(\mu) = C_0\prod_{n=1}^{Nr/2}\lambda_n^L \in R$$

b. large μ

$$\det\Delta(\mu) = C_0\xi^{-N_r/2}\prod_{n=1}^{N_r/2}(l_n^{-N_t}e^{i\theta_n} + \cancel{e^{-\mu/T}})\prod_{n=1}^{N_r/2}(l_n^{+N_t}e^{i\theta_n} + \cancel{e^{-\mu/T}})$$

$$\det\Delta(\mu) = C_0\xi^{-Nr/2}\det Q \in R$$

Nt-dep. of det D

$$\det \Delta(\mu) = C_0 \xi^{-N_r/2} \det(Q+\xi)$$

$$|Q - \lambda I| = 0$$

$$= C_0 \xi^{-N_r/2} \prod_{n=1}^{N_r} (\lambda_n + \xi)$$

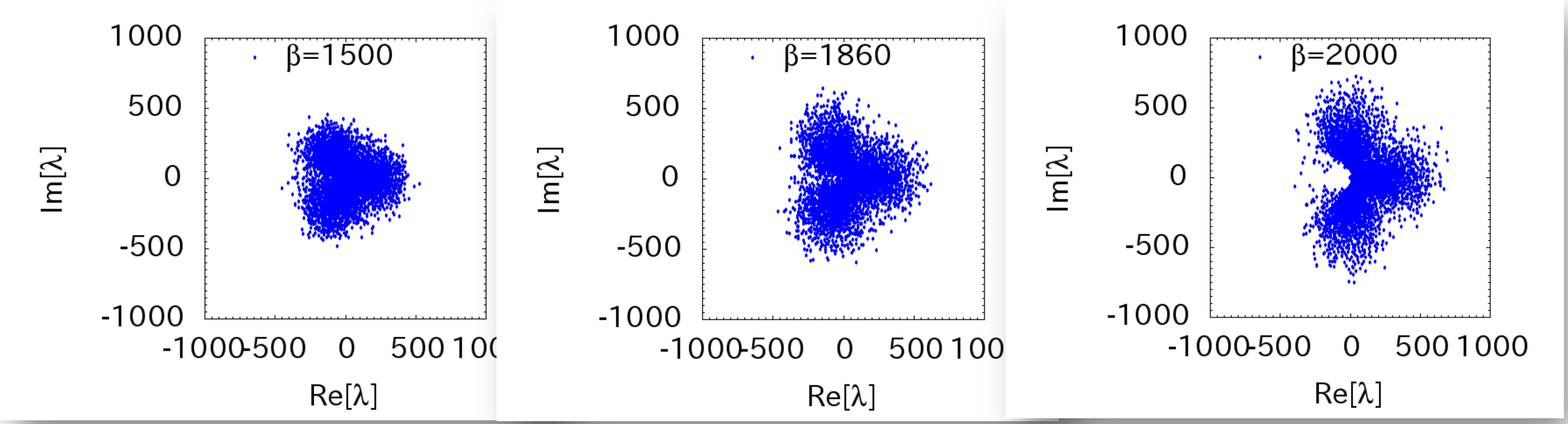
$$\lambda \leftrightarrow 1/\lambda^*$$

$$= C_0 \xi^{-N_r/2} \prod_{n=1}^{N_r/2} (1/\lambda_n^* + \xi) \prod_{n=1}^{N_r/2} (\lambda_n + \xi)$$

$$\lambda_n = l_n e^{i\theta_n}$$

$$= C_0 \xi^{-N_r/2} \prod_{n=1}^{N_r/2} (l_n^{-N_t} e^{i\theta_n} + e^{-\mu/T}) \prod_{n=1}^{N_r/2} (l_n^{+N_t} e^{i\theta_n} + e^{-\mu/T})$$

Eigenvalue distribution



T<Tc

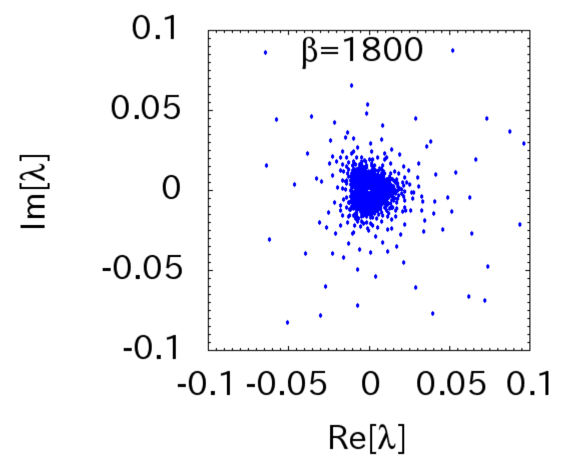
T=Tc

T>Tc

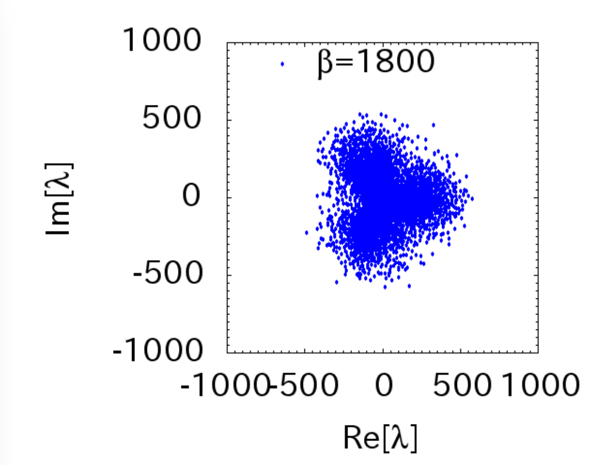
- *At low T : Z3-like distribution*
- *At high T : breaking of Z3 and broadening*

Symmetry of Eigenvalues

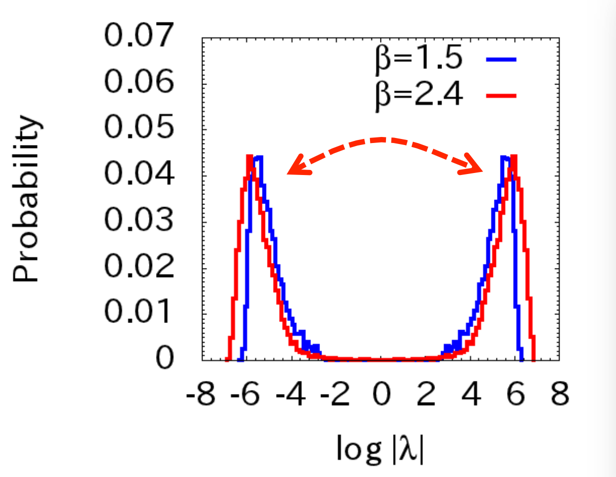
near the origin



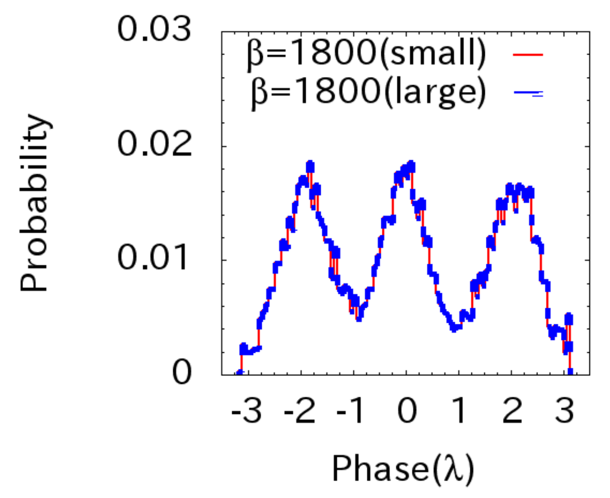
All ev's



Magnitude



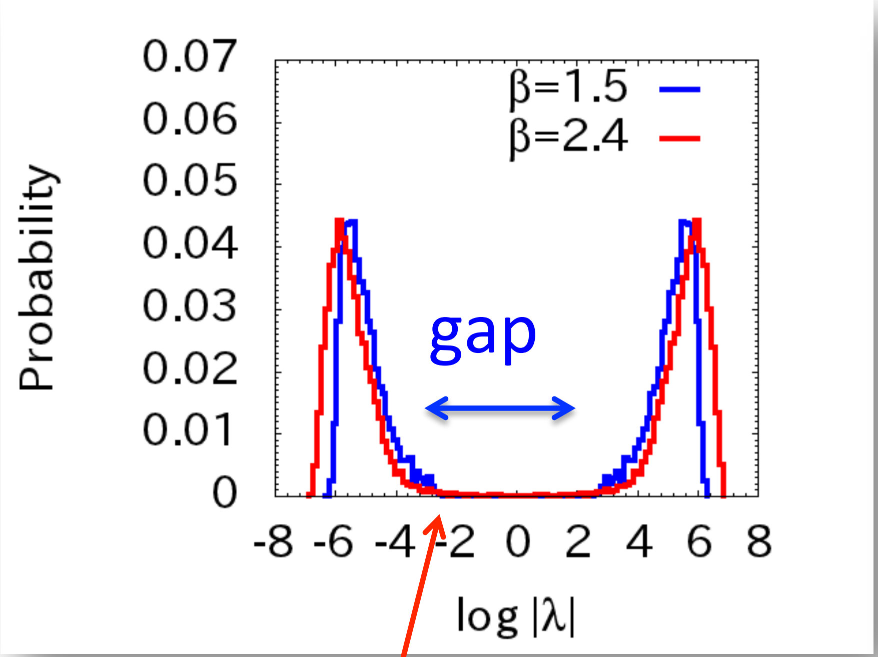
Phase

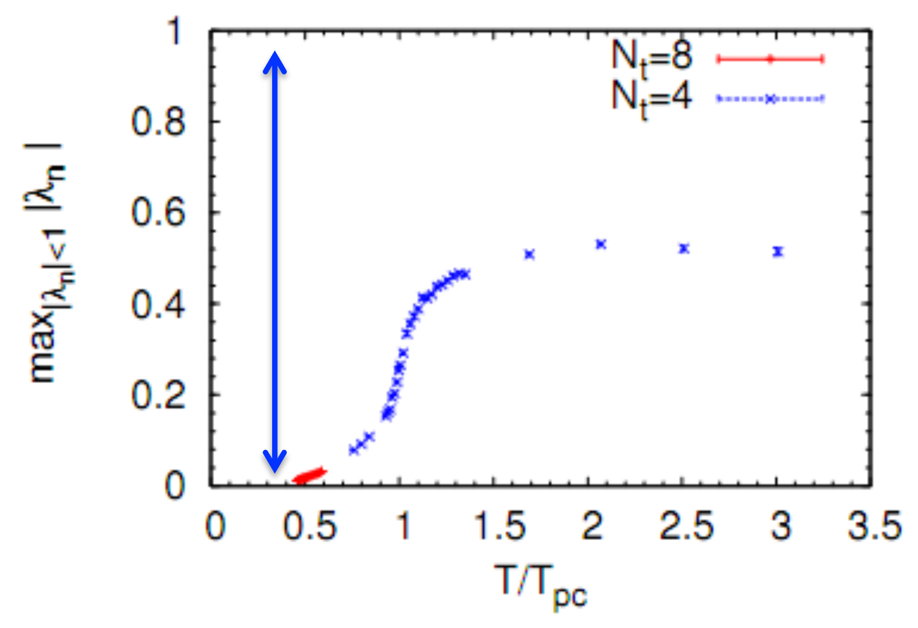


$$\lambda \leftrightarrow \frac{1}{\lambda^*}$$

Gap of ev's

$$\left\langle \max_{|\lambda_n|<1} |\lambda_n| \right\rangle$$





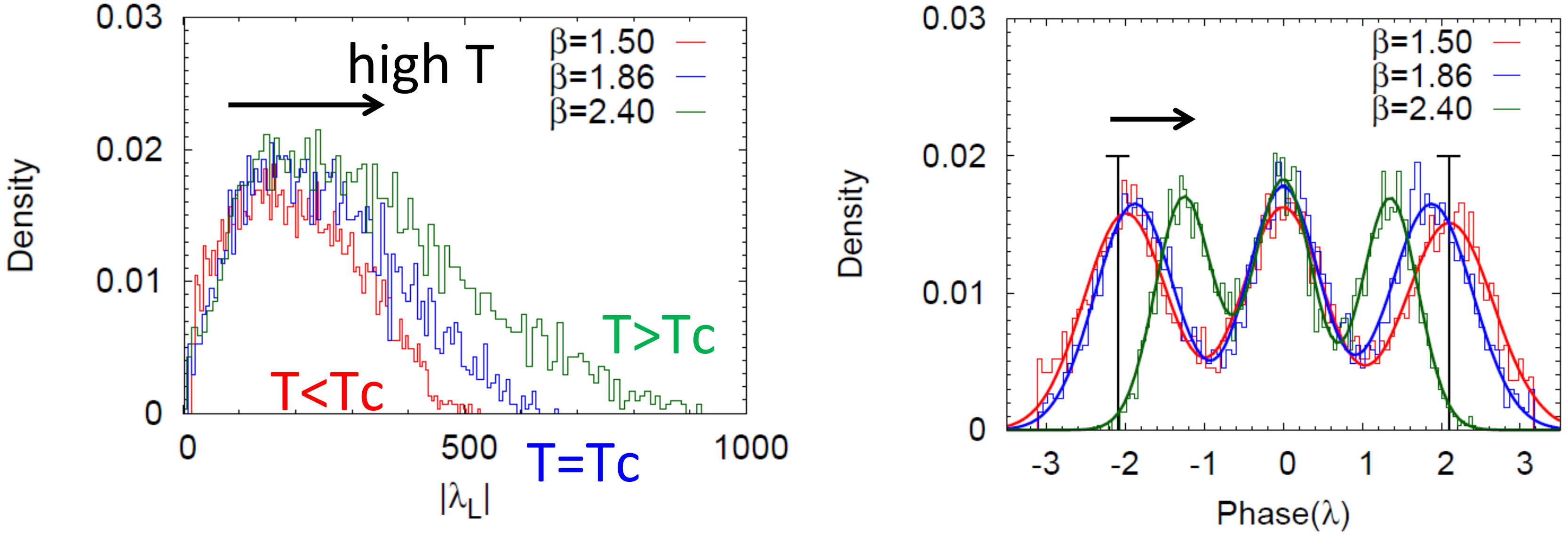
$$\max_{|\lambda_n|<1} |\lambda_n|$$

Gap is related to pion mass

Gibbs('86), Fodor, Szabo, Toth ('06).

β(or *T, a*)- *dependence*

Histogram : Absolute (left) and phase (right)



- *At low T : Z3-like distribution*
- *At high T : breaking of Z3 and broadening*

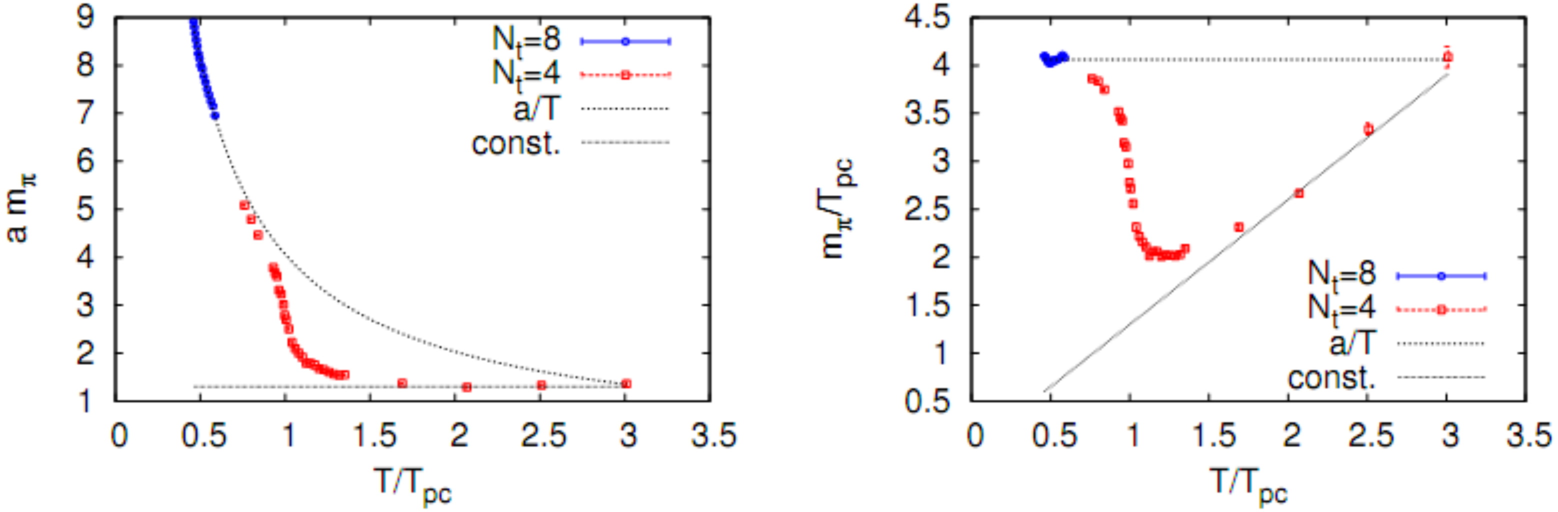
Gap is related to pion mass

$$am_\pi = -\frac{1}{Nt}\ln\max_{|\lambda_n|<1}|\lambda_n|^2$$

Gibbs('86). Eigenvalues and mpi

See also, Fodor, Szabo, Toth ('06). Eigenvalues and hadron spectrum

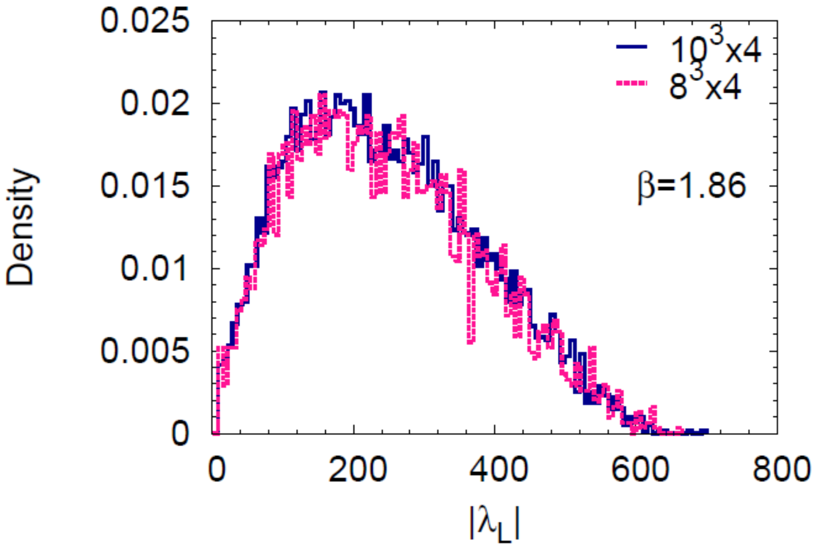
Left : mpi/T. Right : mpi/Tpc

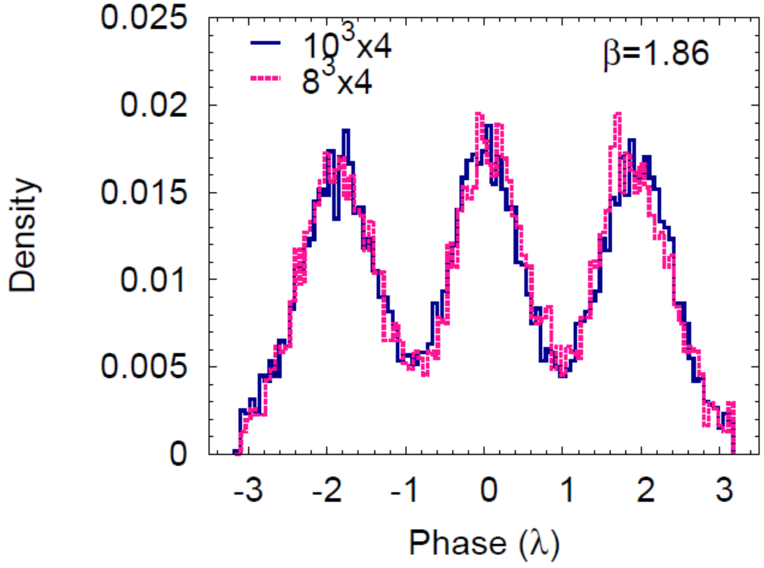


- *At low T, mpi/T is well fitted with a/T, a = 4 Tpc (mq heavy)*
- *At high T, mpi approaches to a constant*

Volume dependence

Histograms : |ev| (Left), arg(ev) (Right)
blue : 10^3 x4, red : 8^3 x 4





- *Increasing Ns(8->10) does not change eigenvalue distribution.*

quark mass dependence

mps / mv = 0.6 (red), 0.8 (blue)
Histograms : |ev| (Left), arg(ev) (Right)
confinement (top), deconfinement(bottom)