TWO-FLAVOR QCD AT FINITE QUARK OR ISOSPIN DENSITY

Leonardo Cosmai
INFN Bari

work done in collaboration with
P. Cea, M. D'Elia, A. Papa, F. Sanfilippo

LATTICE 2012

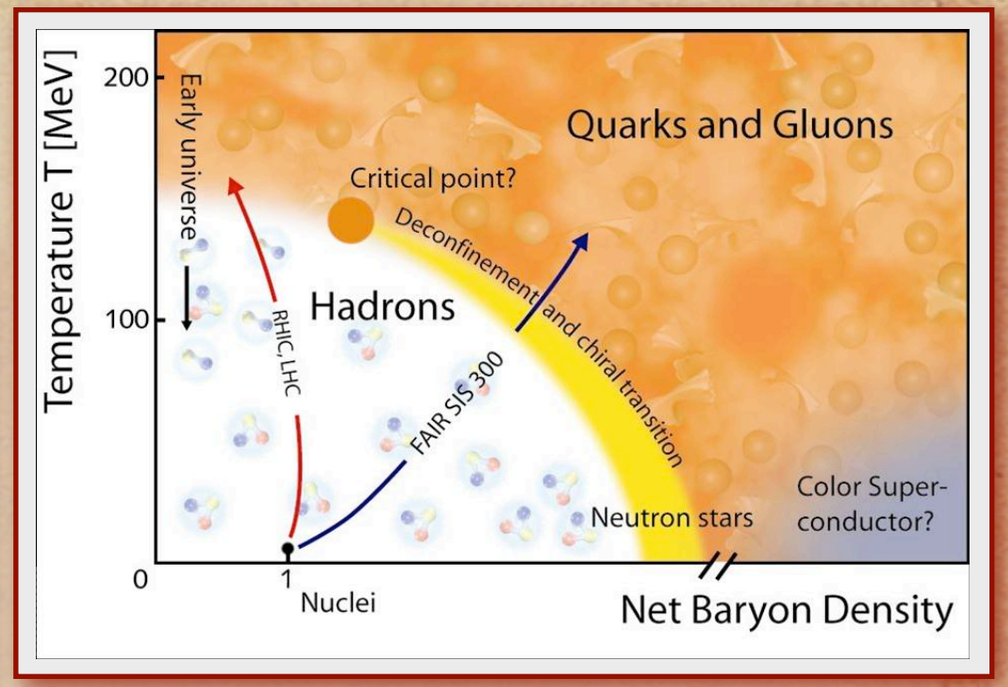
June 24-29, Cairns, Australia

Outline

- Introduction
- QCD with two degenerate staggered fermions:
  - finite isospin density
  - finite quark density
- Conclusions

Introduction

- The determination of the QCD phase diagram in the temperature–quark density plane is becoming increasingly important, due to its impact in cosmology and in the physics of compact stars and of heavy-ion collisions.



- The first-principle nonperturbative approach of discretizing QCD on a space-time lattice and performing numerical Monte Carlo simulations is plagued, at nonzero quark chemical potential, by the well-known sign problem: the fermionic determinant is complex and the Monte Carlo sampling becomes unfeasible.

- Analytic continuation amongst the possible alternatives to solve (approximately) the sign problem:
  - imaginary chemical potential $\mu = i\mu_{\rm IM}$ ➜ positive measure
  - interpolation of the results at imaginary chemical potential
  - analytic continuation: $\mu_{\rm IM} \to -i\mu$
  - limitations due to ambiguity in the interpolation and nonanalyticities and periodicity, we expect reliable estimation for $\mathrm{Re}(\mu)/T \lesssim 1$

Alford-Kapustin-Wilczek (1999)
Lombardo (2000)
deForcrand-Philipsen (2003)
D'Elia-Lombardo (2003)

Roberge-Weiss (1986)

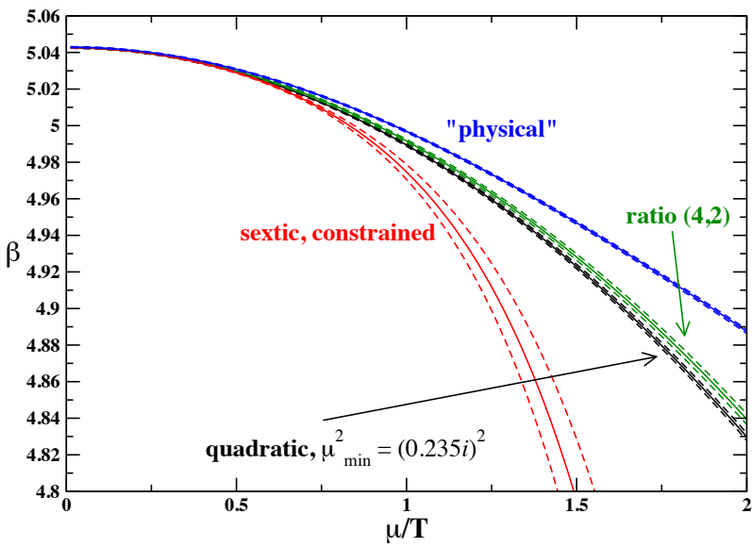
In previous works (Cea-Cosmai-D'Elia-Papa (2007,2008), Cea-Cosmai-D'Elia-Manneschi-Papa (2009)) we have studied the analytic continuation of the pseudocritical line in the case of:

- SU(2) with $n_f=8$ staggered fermions and finite quark density
- SU(3) with $n_f=8$ staggered fermions and finite isospin density

direct simulations at real chemical potentials available

it was found that the nonlinear terms in the dependence of β_c on μ^2 in general cannot be neglected and that the extrapolation to real μ may be wrong otherwise.

- SU(3) with $n_f=4$ staggered fermions and finite quark density



- deviations in the pseudocritical line from the linear behavior in μ^2 for larger absolute values of μ^2 were clearly seen
- several possible extrapolations to real μ up to $\mu/T \simeq 0.6$

Cea-Cosmai-D'Elia-Papa (2010)

In the present study we consider two-flavor QCD in presence of a quark or an isospin chemical potential in the standard staggered discretization for fermion fields

$$Z_{q/\mathrm{iso}}(T,\mu) \equiv \int \mathcal{D}U e^{-S_G} (\det M[\mu])^{\frac{1}{4}} (\det M[\pm\mu])^{\frac{1}{4}}$$

Our purposes:

- analytic continuation of the critical line, $T_c(\mu)$ from imaginary to real μ in the case of a finite μ_{iso} , where simulations are available both for imaginary and real μ_{iso} . Apply analytic continuation to the case of a finite μ_q, also on the basis of what learned in the case of a finite μ_{iso} ;

- comparison between the two theories at finite μ_q or μ_{iso} , quantifying systematic differences for quantities like the curvature of the pseudocritical line at zero chemical potential;

- nature of the transition as a function of the isospin chemical potential

Numerical simulations

- $16^3 \times 4$ lattice (apart from some special cases where we varied spatial size to investigate the critical behavior);

- bare quark mass am=0.05 corresponding to $m_\pi \sim 400$ MeV;

- Rational Hybrid Monte Carlo (RHMC) algorithm, properly modified for the inclusion of quark/isospin chemical potential;

- typical statistics 10k trajectories of 1 Molecular Dynamics unit for each run, 100k trajectories for 4-5 β values around the pseudocritical point, for each μ^2, in order to correctly sample the critical behavior at the transition;

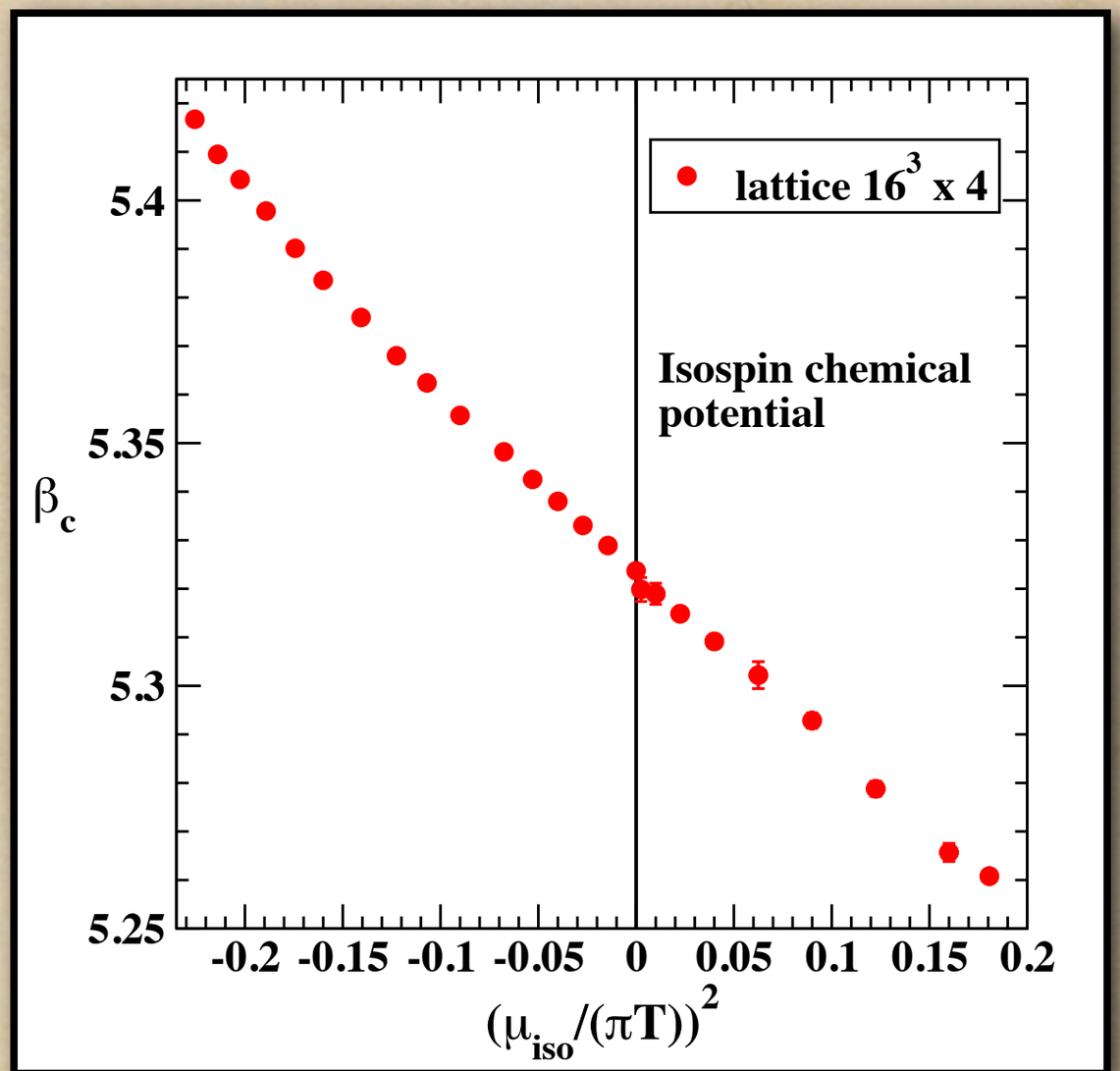
- the pseudocritical $\beta(\mu^2)$ has been determined as the value for which the susceptibility of the (real part of the) Polyakov loop exhibits a peak. We have verified that the determinations are consistent for other observables.

Nonzero isospin chemical potential

$$\beta_c \quad \text{vs.} \quad \mu^2$$

TABLE I. Summary of the values of $\beta_c(\mu^2)$ for finite isospin SU(3) with $n_f = 2$ on the $16^3 \times 4$ lattice with fermionic mass $am = 0.05$.

$\mu/(\pi T)$	β_c
$0.475i$	5.41670(31)
$0.4625i$	5.40948(40)
$0.450i$	5.40429(51)
$0.435i$	5.39780(59)
$0.4175i$	5.39012(49)
$0.400i$	5.38353(44)
$0.375i$	5.37588(61)
$0.350i$	5.36799(62)
$0.327i$	5.36239(64)
$0.300i$	5.35570(50)
$0.260i$	5.34820(47)
$0.230i$	5.3425(10)
$0.200i$	5.33800(52)
$0.165i$	5.33304(85)
$0.120i$	5.3289(12)
0.	5.32371(86)
0.050	5.3199(24)
0.100	5.3189(22)
0.150	5.31486(82)
0.200	5.3091(13)
0.250	5.3022(28)
0.300	5.2928(15)
0.350	5.2788(15)
0.400	5.2657(18)
0.425	5.26079(94)



Interpolating $\beta_c(\mu^2)$ ratio of polynomials

$$\beta_c(\mu^2) = \frac{a_0 + a_1(\mu/(\pi T))^2 + a_2(\mu/(\pi T))^4 + a_3(\mu/(\pi T))^6}{1 + a_4(\mu/(\pi T))^2}$$

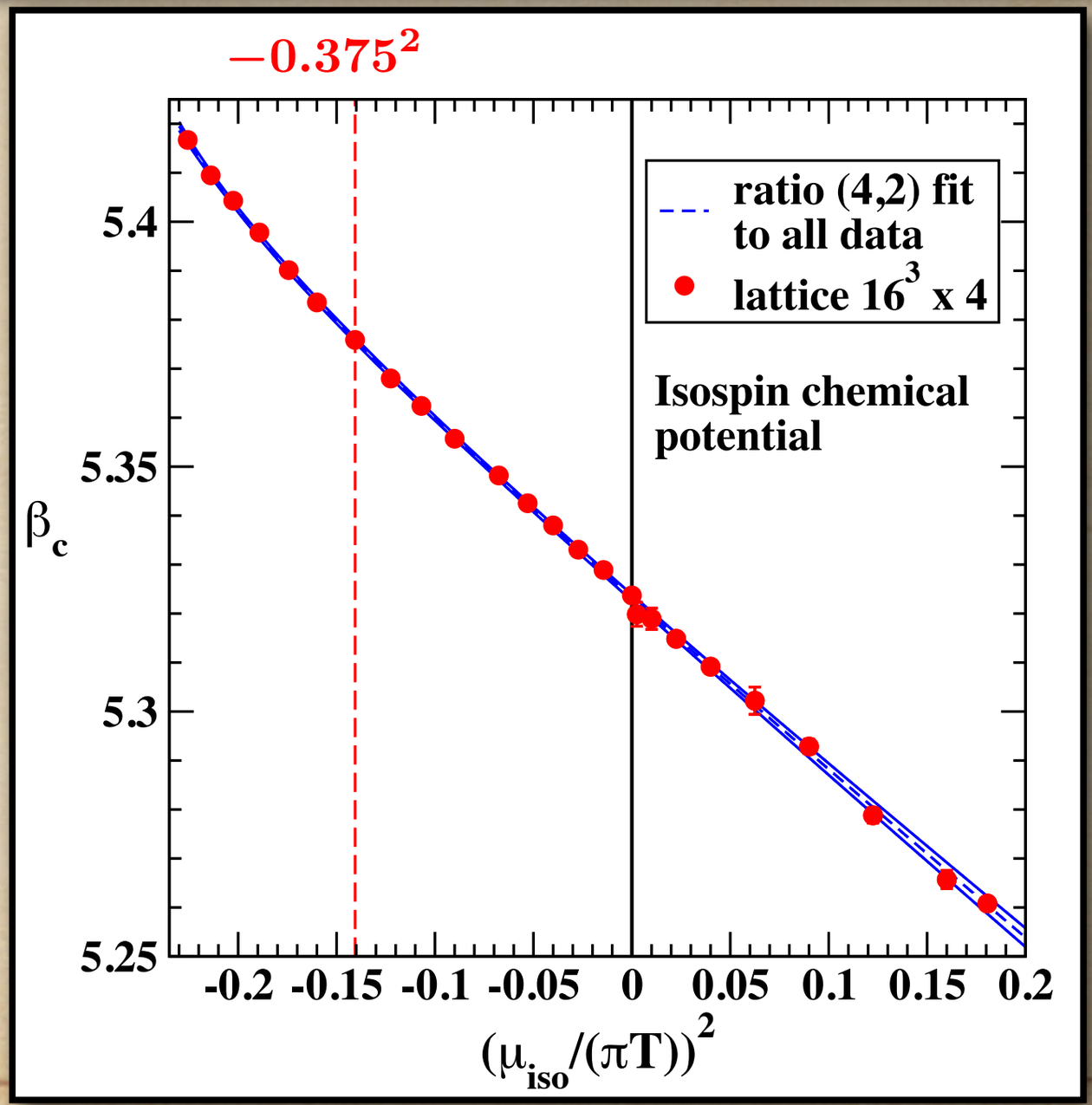
➡ Fit to all data requires at least a ratio (4,2)

➡ Fit to all data with

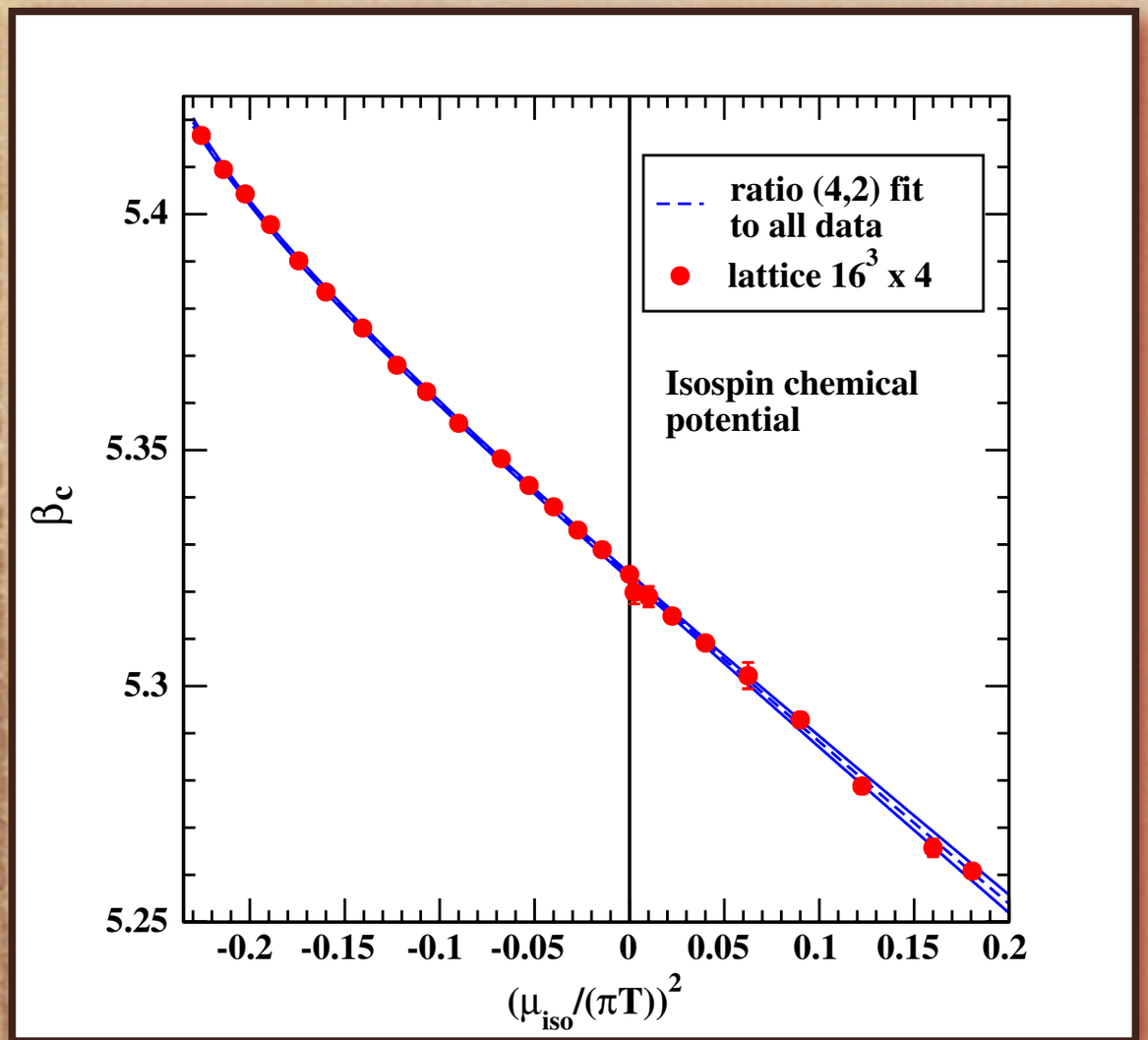
$$(\mu/\pi T)^2 \geq -0.375^2$$

linear (in μ^2) fit OK

i.e. non linear corrections more important for imaginary values of μ than for real ones (contrary to our previous studies)



Analytic continuation of the ratio (4,2) interpolation



Data at $\mu^2 \leq 0$ are precise enough to be sensitive to terms beyond the $O(\mu^2)$

Best interpolation using the ratio of polynomials $O(\mu^4)/O(\mu^2)$

$$\chi^2/\text{d.o.f.} = 0.49$$

(4 parameters)

TABLE II. Parameters of the fits to the pseudocritical couplings in finite isospin SU(3) with $n_f = 2$ on a $16^3 \times 4$ lattice with fermionic mass $am = 0.05$, according to the fit function $\beta_c(\mu^2) = (a_0 + a_1(\mu/(\pi T))^2 + a_2(\mu/(\pi T))^4 + a_3(\mu/(\pi T))^6)/(1 + a_4(\mu/(\pi T))^2)$. Blank columns stand for terms not included in the fit. The asterisk denotes a constrained parameter. Fits are performed in the interval $[(\mu/(\pi T))^2_{\text{min}}, (\mu/(\pi T))^2_{\text{max}}]$; the last two columns give the value of $(\mu/(\pi T))^2_{\text{min,max}}$.

a_0	a_1	a_2	a_3	a_4	χ^2/d.o.f.	$(\mu/(\pi T))^2_{\text{min}}$	$(\mu/(\pi T))^2_{\text{max}}$
5.32326(62)	16.755(10)	−1.072(26)		3.2143(19)	0.60	-0.475^2	0.425^2
5.32385(54)	−0.3597(60)				0.96	-0.375^2	0.425^2
5.31940(76)	−0.4192(47)				18.3	-0.475^2	0
5.3232(11)	−0.368(12)				0.59	-0.375^2	0
5.3255(14)	−0.286(25)	0.511(94)			1.85	-0.475^2	0
5.3235(21)	−0.374(68)	−0.36(63)	−2.4(1.7)		0.43	-0.475^2	0
5.3232*	−0.368*	−0.253(91)	−2.01(44)		0.62	-0.475^2	0
5.32403(94)	14.602(14)	−0.844(44)		2.8066(25)	0.49	-0.475^2	0

"Physical" fit

write the interpolating function in "physical" units:

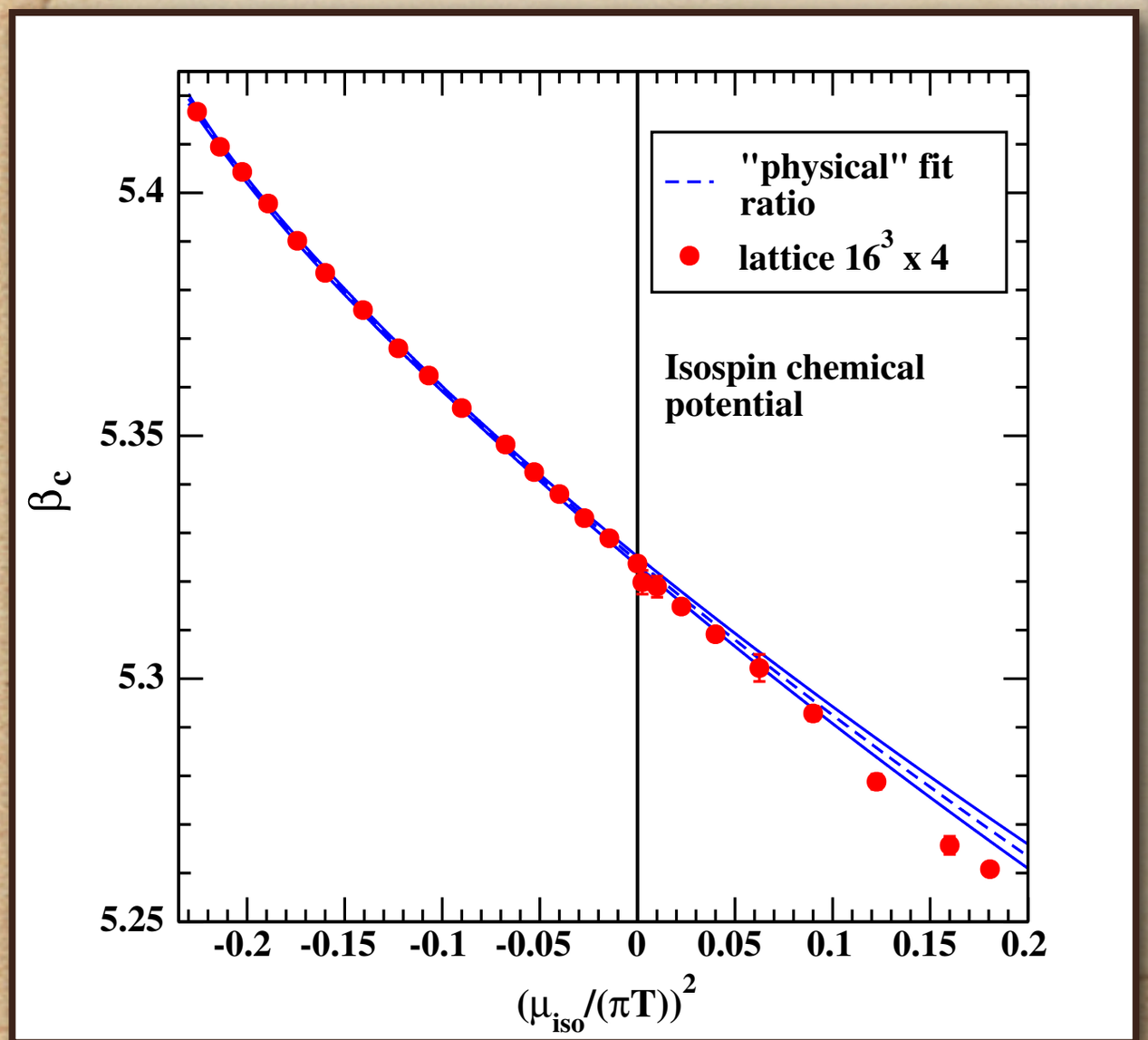
$$\left[\frac{T_c(0)}{T_c(\mu)}\right]^2 = \frac{1 + A\,x + B\,x^2}{1 + C\,x}$$

$$T = \frac{1}{(N_t a(\beta))}$$

$$x \equiv \mu/(\pi T)\,,\; T/T_c(0)$$

$$a^2(\beta_c(\mu^2))|_{2-\mathrm{loop}} = a^2(\beta_c(0))|_{2-\mathrm{loop}} \times \frac{1 + A\,x + B\,x^2}{1 + C\,x}$$

implicit relation between β_c and μ^2



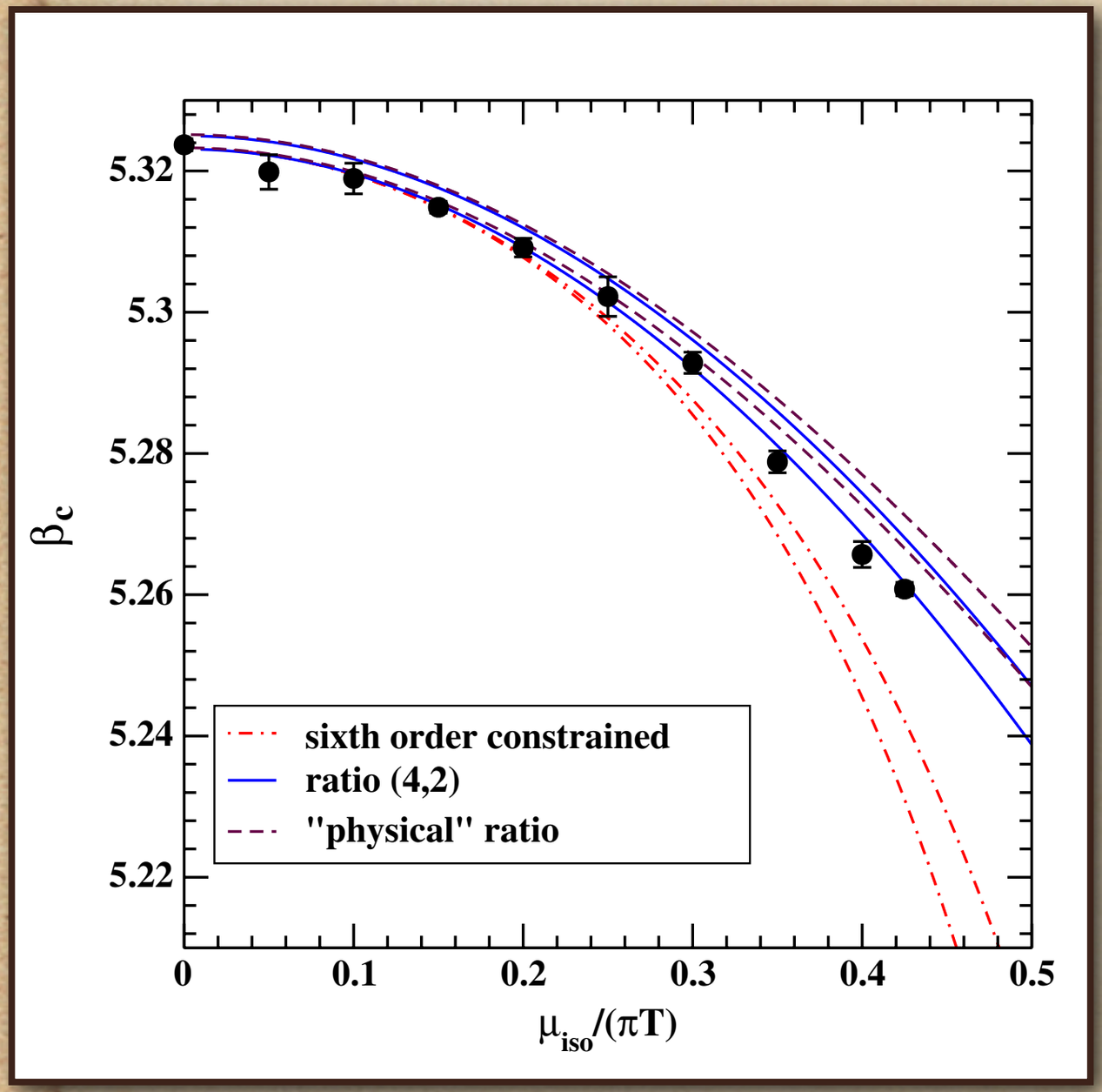
fit parameters

$$\beta_c(0) = 5.32422(94)\,,\; A = 4.077(23)$$

$$B = 2.659(77)\,,\; C = 3.221(26)$$

$$\chi^2/\mathrm{d.o.f.} = 0.53$$

Extrapolations to real isospin chemical potentials together with results from simulations at real values



Agreement between extrapolations as long as $\mu/(\pi T) \lesssim 0.2$

Sixth order constrained extrapolation deviates from the other two extrapolations

Different interpolations that well reproduce imaginary data, lead to distinct extrapolations (as we have seen for n_f=4 SU(3))

ratio (4,2) is quite close to the direct determinations of the pseudocritical couplings

Nonzero quark chemical potential

TABLE III. Summary of the values of $\beta_c(\mu^2)$ for finite density SU(3) with $n_f = 2$ on the $16^3 \times 4$ lattice with fermionic mass $am = 0.05$.

$\mathrm{Im}(\mu)/(\pi T)$	β_c
0.	5.32371(86)
0.100	5.3277(12)
0.180	5.33524(71)
0.200	5.33914(83)
0.245	5.34712(75)
0.260	5.35000(81)
0.270	5.35255(91)
0.280	5.35510(59)
0.290	5.35710(70)
0.300	5.35970(21)
0.310	5.36307(62)
0.320	5.36622(37)
0.327	5.36956(63)
1/3	5.37067(75)

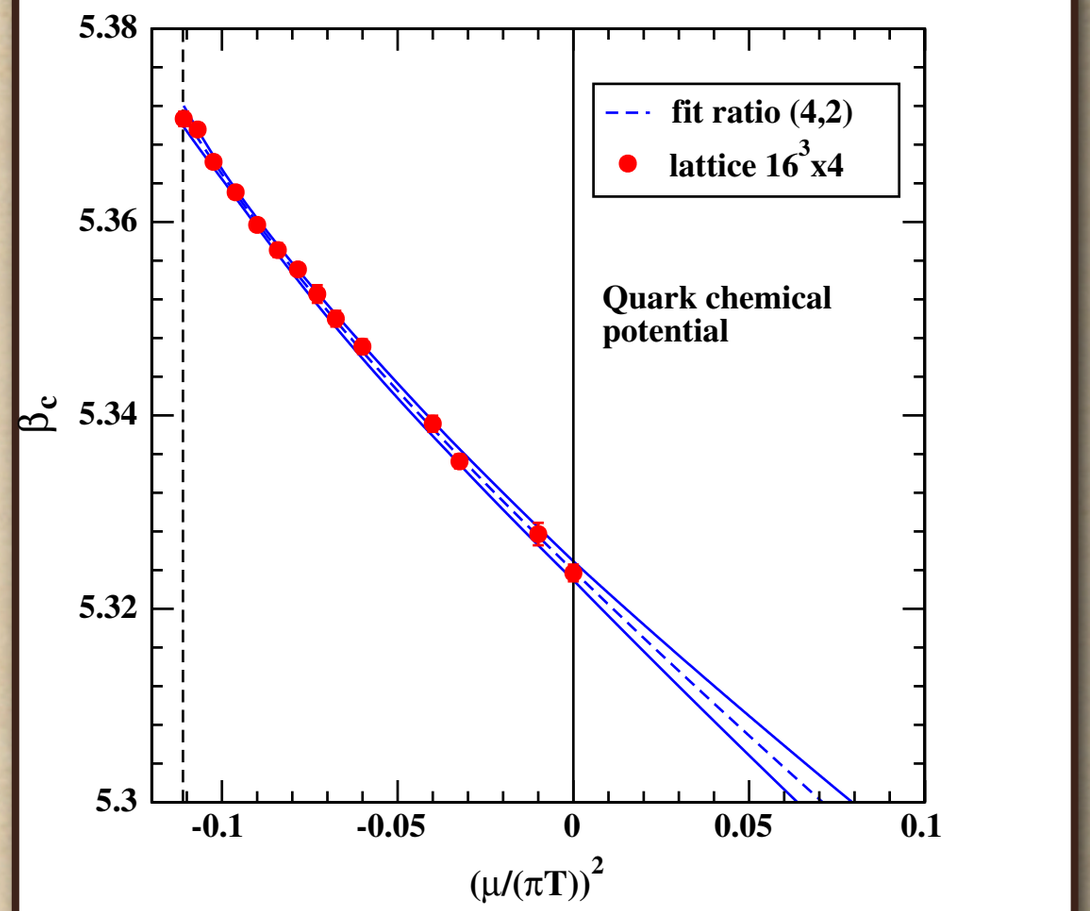


TABLE IV. Parameters of the fits to the pseudocritical couplings in finite density SU(3) with $n_f = 2$ on a $16^3 \times 4$ lattice with fermionic mass $am = 0.05$, according to the fit function $\beta_c(\mu^2) = (a_0 + a_1(\mu/(\pi T))^2 + a_2(\mu/(\pi T))^4 + a_3(\mu/(\pi T))^6)/(1 + a_4(\mu/(\pi T))^2)$. Blank columns stand for terms not included in the fit. The asterisk denotes a constrained parameter. Fits are performed in the interval $[(\mu/(\pi T))^2_{\mathrm{min}}, 0]$; the last column gives the value of $(\mu/(\pi T))^2_{\mathrm{min}}$.

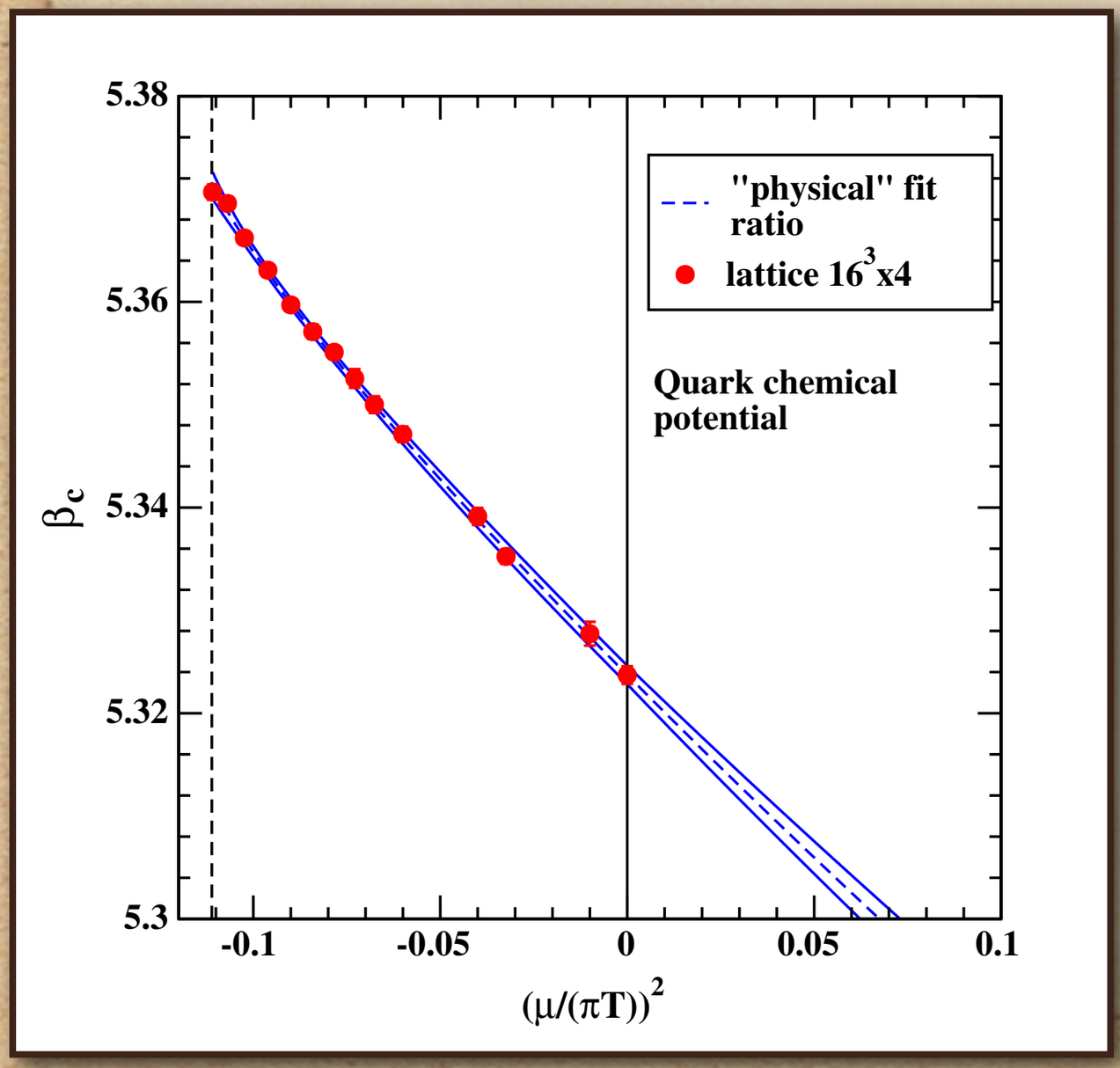
a_0	a_1	a_2	a_3	a_4	χ^2/d.o.f.	$(\mu/(\pi T))^2_{\mathrm{min}}$
5.32189(78)	−0.4262(90)				2.87	$-1/3^2$
5.32283(83)	−0.410(10)				0.63	-0.310^2
5.3242(13)	−0.314(44)	0.92(35)			0.85	$-1/3^2$
5.3226(12)	−0.446(86)	−1.7(1.7)	−14.4(9.7)		1.41	$-1/3^2$
5.32283*	−0.410*	−0.76(13)	−8.7(5.4)		0.65	$-1/3^2$
5.32394(98)	25.736(24)	−1.05(61)		4.9002(45)	0.60	$-1/3^2$

"Physical" fit at nonzero quark chemical potential

$$\left[\frac{T_c(0)}{T_c(\mu)}\right]^2 = \frac{1 + A\,x + B\,x^2}{1 + C\,x}$$

$$x \equiv \mu/(\pi T)$$

$$a^2(\beta_c(\mu^2))|_{2-\mathrm{loop}} = a^2(\beta_c(0))|_{2-\mathrm{loop}} \times \frac{1 + A\,x + B\,x^2}{1 + C\,x}$$



fit parameters

$$\beta_c(0) = 5.32373(90)\ ,\ A = 8.140(32)$$
$$B = 6.59(26)\ ,\ C = 7.201(35)$$
$$\chi^2/\mathrm{d.o.f.} = 0.51$$

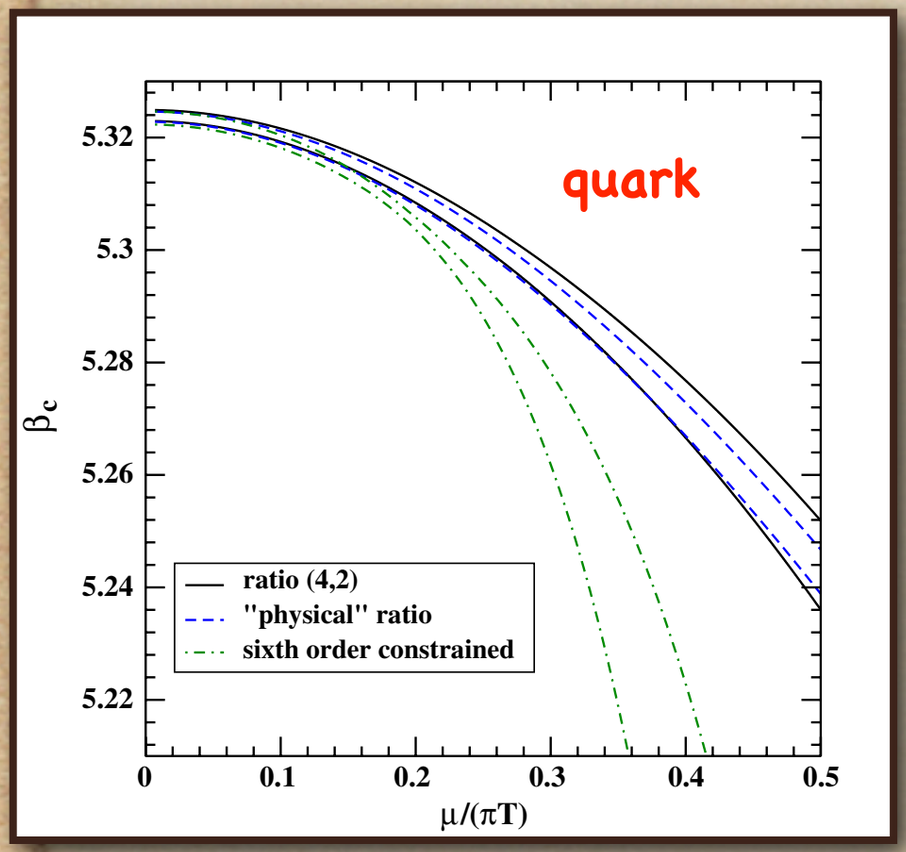
extrapolation to real chemical potentials down to T=0 (but there are systematic effects!)

$$\mu_c(T=0) = \sqrt{\frac{C}{B}}\,T_c(0) = 3.284(65)\,T_c(0)$$

$2.73(58)T_c(0)$

n_f=2 Wilson fermion (Nagata-Nakamura 2011)

Extrapolations to real quark chemical potentials

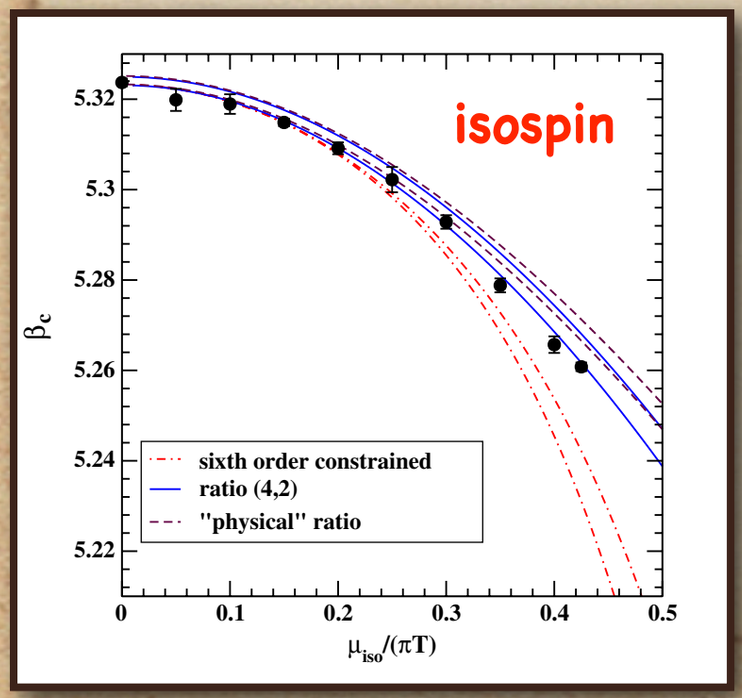


Comparison of different extrapolations:

$\mu/(\pi T) \lesssim 0.1$ OK

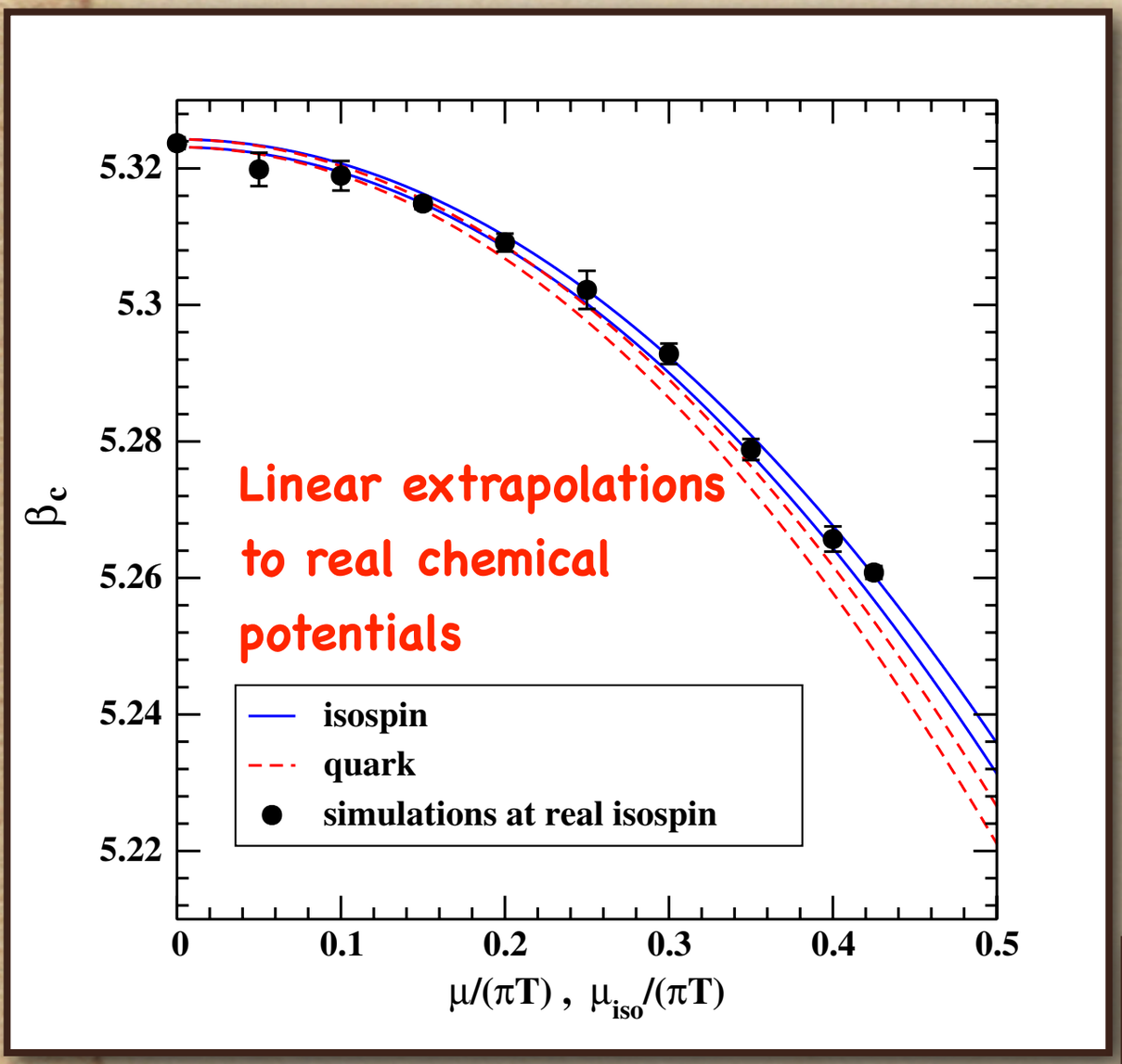
$\mu/(\pi T) > 0.1$

Sixth order constrained polynomial deviates from the other two curves



- Using analysis at finite isospin one could argue that "ratio (4,2)" is preferred.
- HOWEVER one cannot exclude possible systematic differences between QCD at finite quark and isospin chemical potentials.

THE CURVATURES OF THE CRITICAL LINES



Curvature of the pseudocritical line at μ=0

Common fit to all the pseudocritical couplings at imaginary (and real, when available) chemical potentials, including as many data points as compatible with a reasonable chisquare (*):

$$\beta_c(\mu_q, \mu_{\rm iso}) = \beta_c(0) + a_q \left(\frac{\mu_q}{\pi T}\right)^2 + a_{\rm iso} \left(\frac{\mu_{\rm iso}}{\pi T}\right)^2$$

$$a_q = -\,0.3997(87)\ , \quad a_{\rm iso} = -0.3606(67)$$
$$\beta_c(0) = 5.32370(57)\ , \quad \chi^2/{\rm d.o.f.} = 0.93\ .$$

$a_q \neq a_{\rm iso}$ by ~4 σ

(*) the mixed terms is absent, for two degenerate flavors the theory is even under reflection of μ_q and μ_{iso} separately (D'Elia-Sanfilippo 2009)

The curvatures in terms of dimensionless quantities

$$\frac{T_c(\mu_q,\mu_{\rm iso})}{T_c(0)} = 1 + R_q\left(\frac{\mu_q}{\pi T}\right)^2 + R_{\rm iso}\left(\frac{\mu_{\rm iso}}{\pi T}\right)^2$$

$$R_{q/\rm iso} = -\frac{1}{a}\frac{\partial\, a}{\partial\beta}\bigg|_{\beta_c(0)} \qquad a_{q/\rm iso} = \sqrt{\frac{N_c}{2\beta_c(0)^3}}\frac{1}{\beta_L(\beta_c(0),m_q)}a_{q/\rm iso}$$

a lattice spacing

$\beta_L = a\dfrac{\partial g_0}{\partial a}$ lattice beta-function (2-loop)

the expectation is: $R_q = R_{\rm iso}$ (at the leading order $1/N_c$) $\qquad R_{\rm q,iso} \sim N_f/N_c$

WE FIND:

$R_q = -0.515(11)$ $\qquad R_{\rm iso} = -0.465(9)$

$-0.500(34)$ deForcrand-Philipsen (2003) am=0.025 $\qquad -0.426(19)$ Kogut-Sinclair (2004) m=0.05 8^3x4 R algorithm

$-0.38(12)$ Nagata-Nakamura (2011) - Wilson fermions

$-0.792(10)$ 4 flavors (D'Elia-Lombardo 2004, Cea-Cosmai-D'Elia-Papa 2010) ➜ the curvature grows with n_f

(ratio: to avoid the systematic error related to the choice of the two-loop expression for β_L)

$$R_{q-\rm iso} = \frac{R_q - R_{\rm iso}}{R_q} = \frac{a_q - a_{\rm iso}}{a_q} = 0.098(26) \sim 10\% \sim O(1/N_c)$$

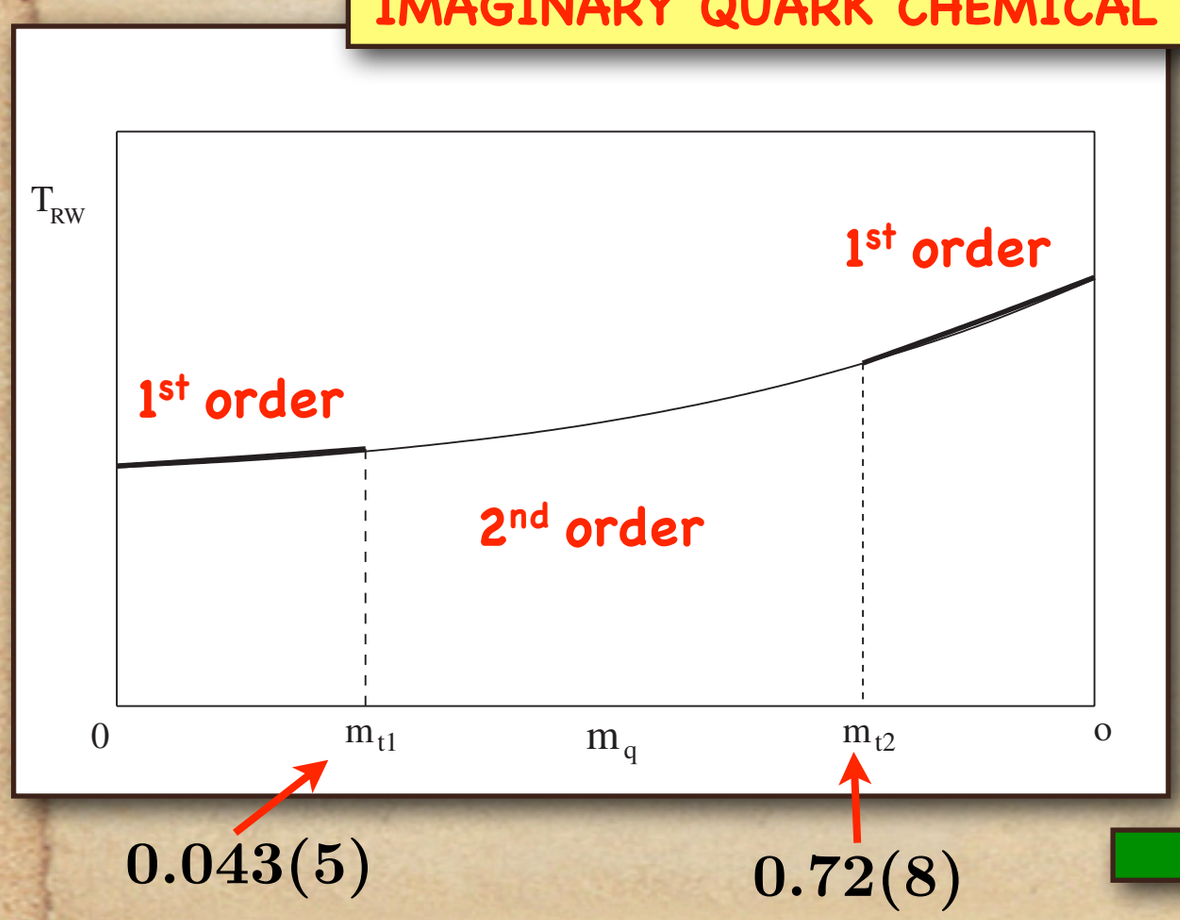
$$R_{\rm q} - R_{\rm iso} \sim O(1/N_c^2)$$

first lattice evidence of the $O(1/N_c^2)$ difference between the two thories at small chemical potentials ➳ D.Toublan (2005)

ORDER OF THE PHASE TRANSITION AT IMAGINARY CHEMICAL POTENTIALS

- The phase structure at finite T and imaginary chemical potential may be important of its own and teach us something about the nonperturbative properties of QCD also at zero and small chemical potential.
- The phase transition at the Roberge-Weiss endpoint could in principle have influence also far from the endpoint.

IMAGINARY QUARK CHEMICAL POTENTIAL



For N_f=2 the RW transition ($\mu_{IM}/\pi T$=1/3):

- first order for small and high quark masses
- second order for intermediate quark masses

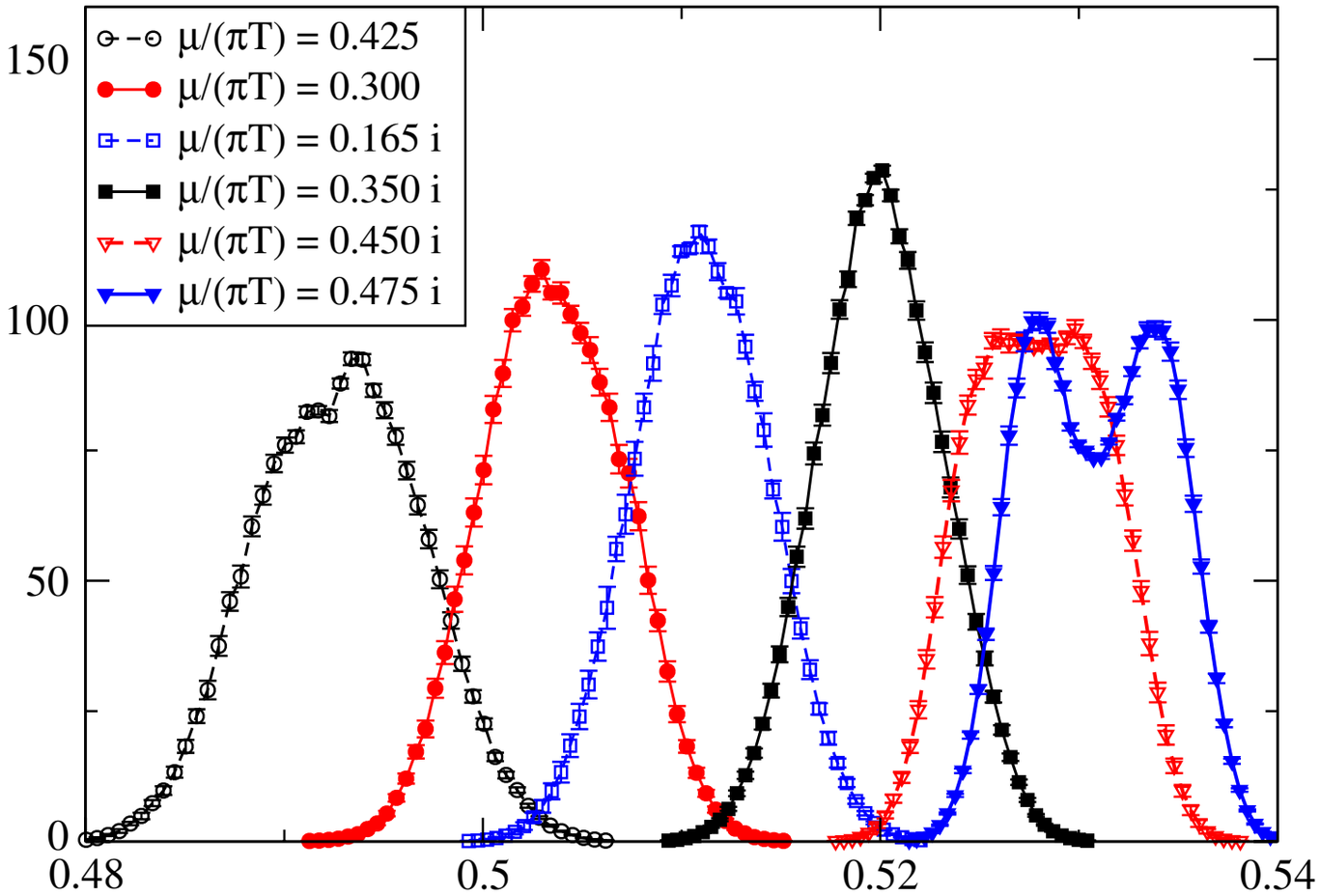
Bonati-Cossu-D'Elia-Sanfilippo (2011)

In the present study am=0.05 ➡ 2nd order RW for quark chemical potential

see what happens for the isospin chemical potential, where the available range of imaginary values is larger

ORDER OF THE PHASE TRANSITION

Imaginary isospin chemical potential may strengthen the transition as an imaginary quark potential does: a **first-order transition** could be manifest along the pseudocritical line (even for our quark mass value m=0.05).



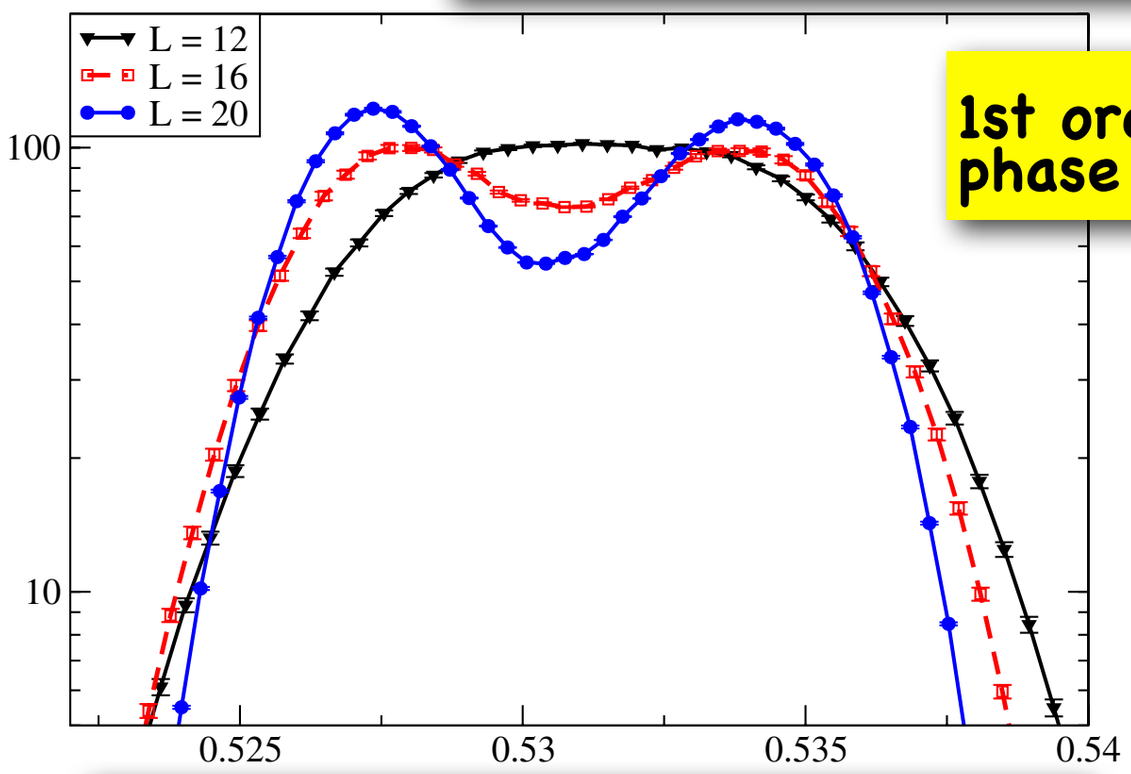
$$16^3 \times 4$$

ISOSPIN CHEMICAL POTENTIAL

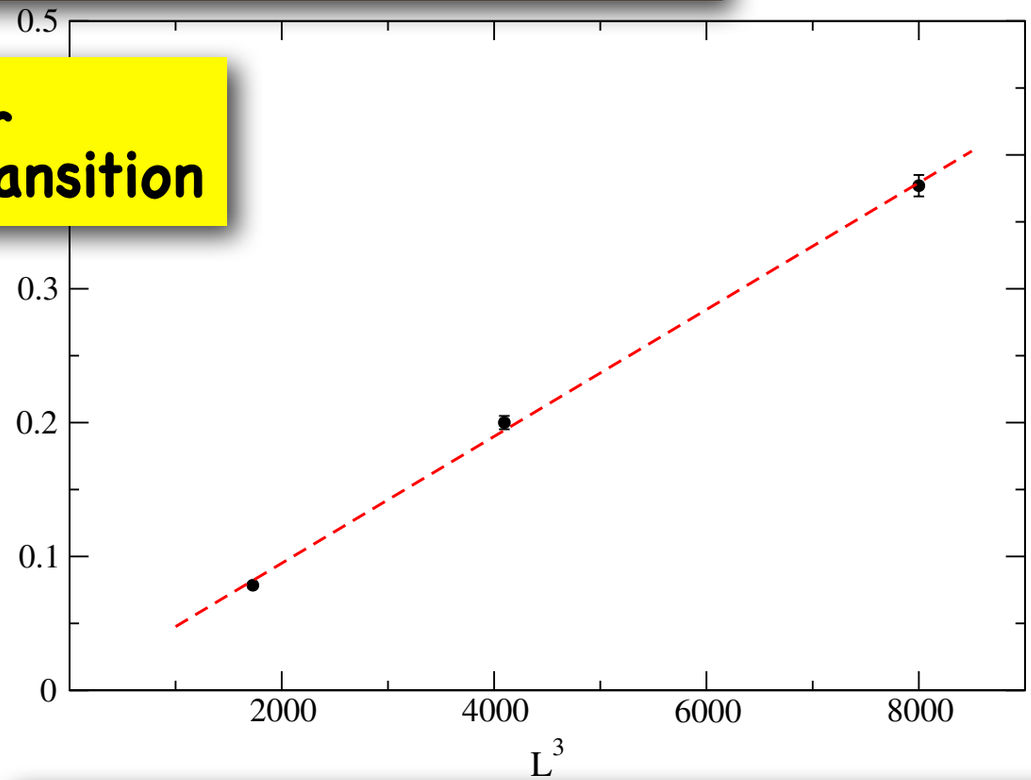
$$m = 0.05$$

Normalized plaquette distribution at the pseudocritical coupling for different values of the isospin chemical potential

$\mu_{\rm iso}/(\pi T) = 0.475i \quad am = 0.05$



1st order phase transition



Normalized plaquette distributions at the pseudocritical coupling for different spatial lattice sizes

Maxima of the plaquette susceptibility scale linearly with the spatial volume

We argue that (for n_f=2 staggered fermions of mass am=0.05) the transition is first order at $\mu_{iso}/\pi T$=0.475i and there is possibly a critical point along the line at some smaller value of $\mu_{iso}/\pi T$.

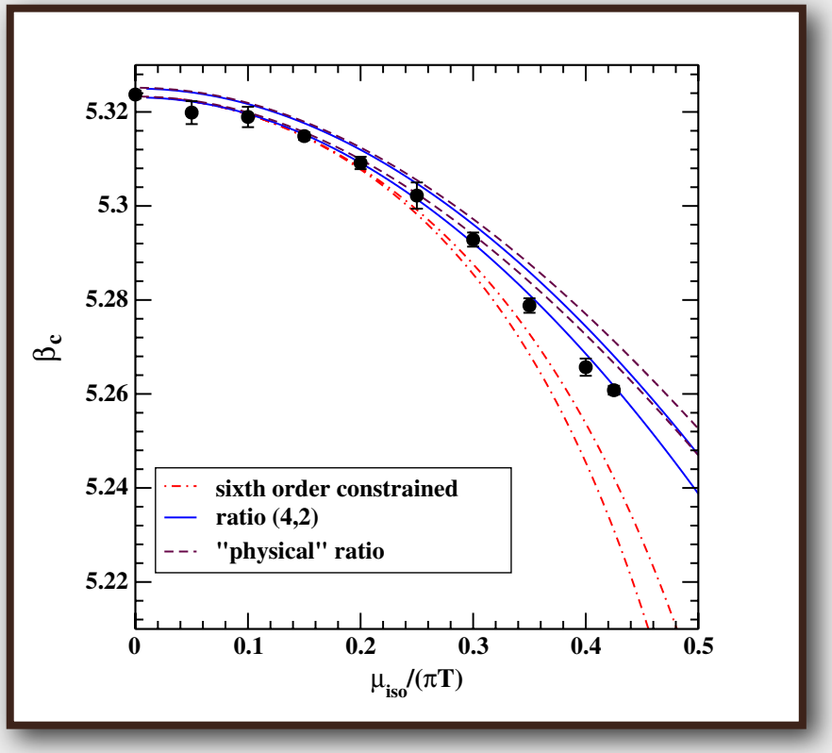
Such non-trivial behavior resembles what happens for quark chemical potentials and may have consequences on the general structure of the QCD phase diagram (deserves further investigations).

D'Elia-Sanfilippo (2009) deForcrand-Philipsen (2010) Bonati-Cossu-D'Elia-Sanfilippo (2011) Bonati-deForcrand-D'Elia-Philipsen-Sanfilippo (2011)

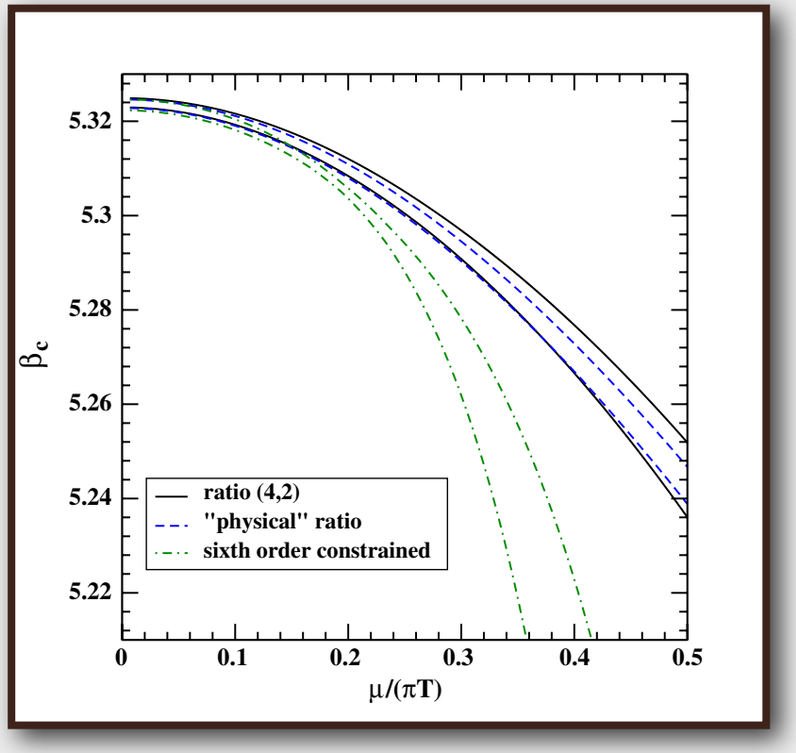
Summary & Conclusions (1/3)

- Localization of the pseudocritical line in the temperature-chemical potential plane, for the two cases of quark and isospin density by means of analytic continuation:
  - deviations from the linear behavior in μ^2 of the critical lines clearly seen for $\mu^2<0$ are nicely described by several analytic functions; however, the extrapolations to positive μ^2 overlap, within errors, only as long as

$$\mu_{\rm iso}/(\pi T) \simeq 0.2$$



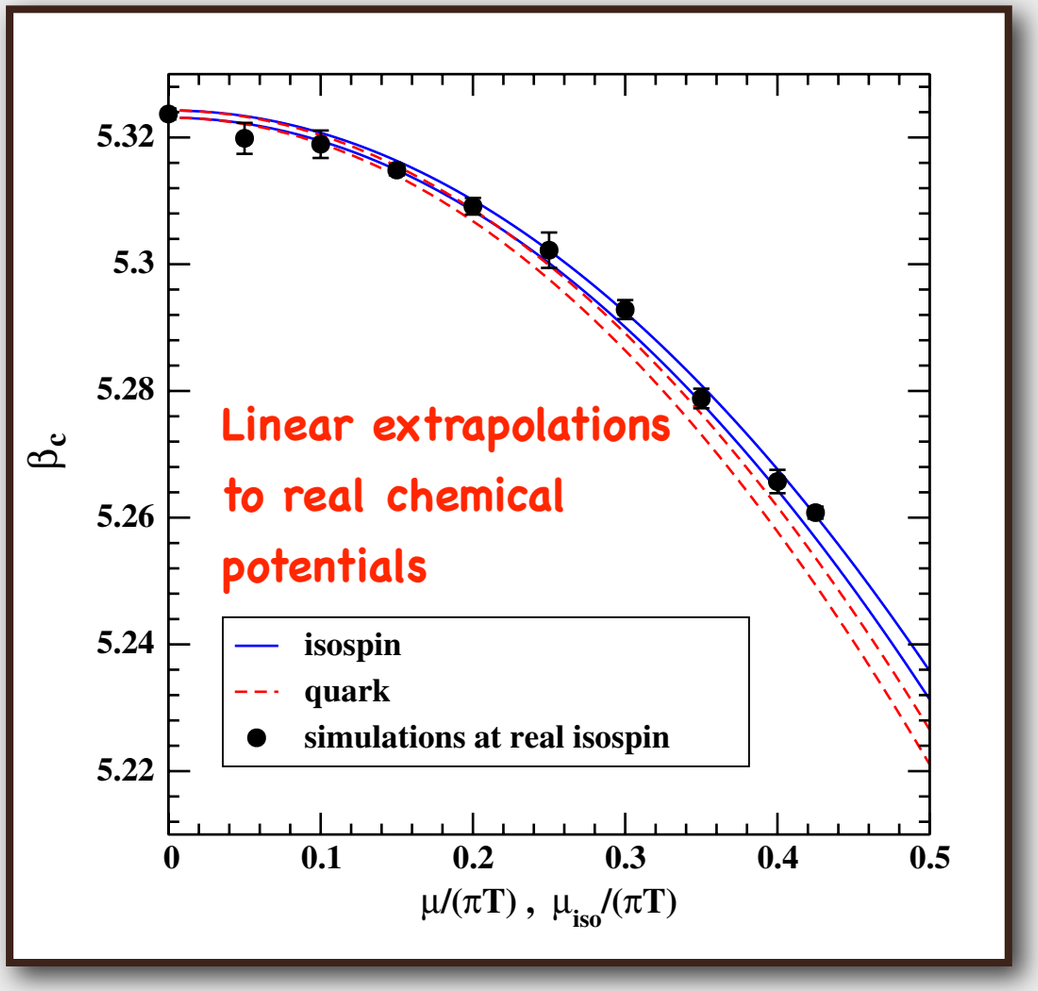
$$\mu_{\rm q}/(\pi T) \simeq 0.1$$



preference for extrapolations based on Padé approximants

Summary & Conclusions (2/3)

- We have performed a careful determination of the curvatures of the two critical lines at zero chemical potential



$$R_{q-\mathrm{iso}} = \frac{R_q - R_{\mathrm{iso}}}{R_q} = 0.098(26) \sim O(1/N_c)$$

First evidence for an

$$R_{\mathrm{q}} - R_{\mathrm{iso}} = O(1/N_c^2)$$

difference between the two theories at small chemical potentials

➳ D.Toublan (2005)

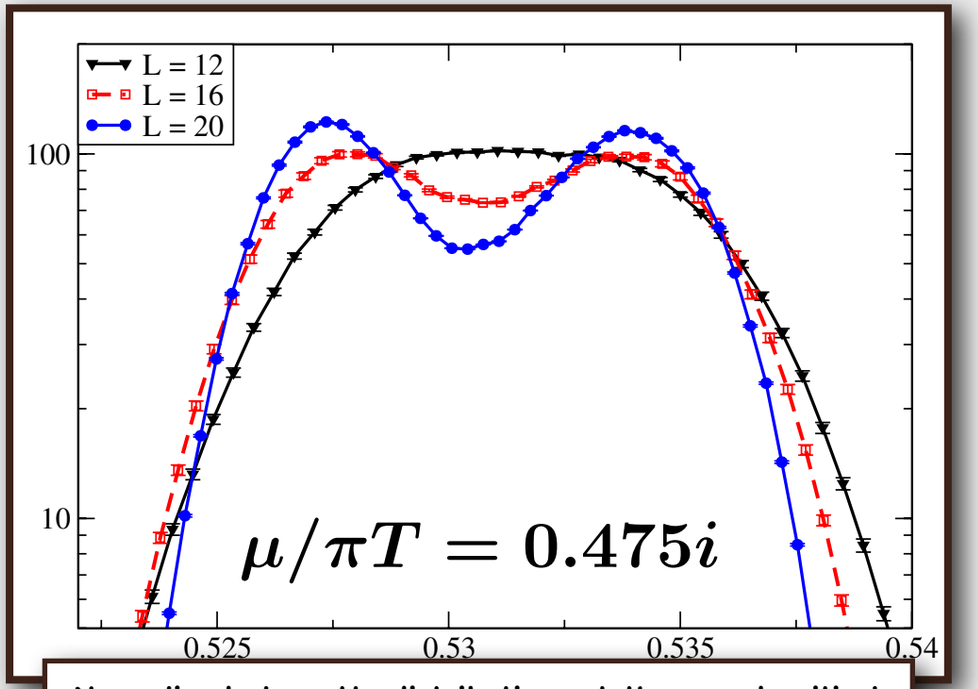
It would be interesting to explore how results change for different values of N_c

Summary & Conclusions (3/3)

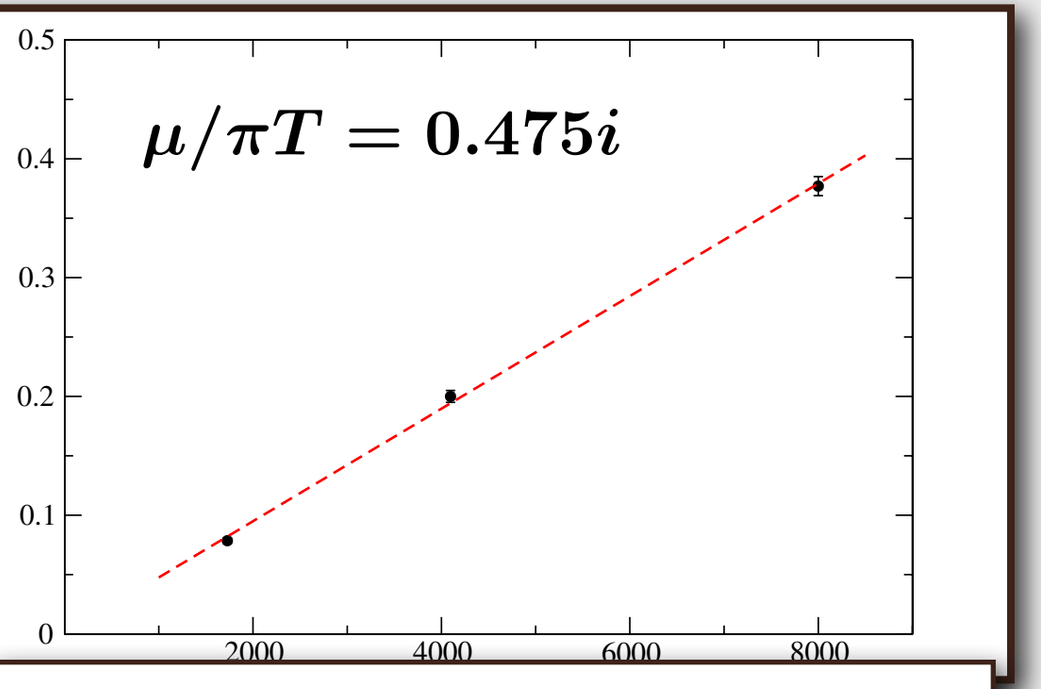
- Study of the order of the phase transition along the two critical lines:

- non zero quark density: no clear signatures of a first-order transition

- non zero isospin density: evidence of a first-order transition for large enough imaginary chemical potentials



Normalized plaquette distributions at the pseudocritical coupling for different spatial lattice sizes



Maxima of the plaquette susceptibility scale linearly with the spatial volume

"Australia, according to the Proposed divisions." from The Journal of the Royal Geographical Society, Volume 8, 1838 to accompany "Considerations on the Political Geography and Geographical Nomenclature of Australia by Captain Vetch, Royal Engineers, F.R.S."