Critical couplings and string tensions from two-lattice matching of RG decimations

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Introduction

- The RG-based method of lattice matching relates physical quantities on different lattices.
- This provides a method for computation of a quantity at different lattice spacings (couplings).
- To apply it one needs to implement RG block transformations on the approach to the Wilsonian Renormalized Trajectory (RT). This can be done in various ways.
    - MCRG
    - RG Recursion Relations (explicitly computable to various degrees).

Outline

1. RG Recursion scheme
2. The lattice matching method
3. Application to computation of critical couplings and string tensions in $SU(2)$ and $SU(3)$ pure gauge theories
4. Conclusions

Action - Character Expansion

Plaquette action $A_p(U_p, n)$ on lattice of spacing $b^n a$:

$$\exp\left(-A_p(U_p, n)\right) = \sum_j d_j F_j(n) \chi_j(U_p) \qquad (1)$$

The sum over all inequivalent irreducible representations labeled by j, characters χ_j of dimension d_j.
The action itself completely specified by the set of $F_j(n)$ coefficients and vice versa; of general form:

$$A_p(U_p, n) = \sum_j \frac{1}{d_j} \beta_j(n) \frac{1}{2 l_j} [\chi_j(U_p) + \chi_j(U_p^{-1})] \qquad (2)$$

with $l_j = 1$ for self-conjugate and $l_j = 2$ for non-self-conjugate representations. (For $SU(2)$, in particular, $l_j = 1$ for all j.)

Effective coupling

It is useful to define an effective coupling $g^{(n)}$ characterizing a given action of the form (2):

$$\beta^{(n)} = \frac{2N}{g^{(n)\,2}} \equiv 2N \frac{d^2 A_p(e^{i\theta \hat{m}\cdot t}, n)}{d\theta^2}\bigg|_{\theta=0} \tag{3}$$

where $\{t\}$ are the $SU(N)$ generators and $|\hat{m}| = 1$ (independent of the direction $\hat{m}$).

Note In the perturbative regime this reduces to the usual definition of gauge coupling.

We adopt (3) to track the RG recursion flows; it provides an effiicient parametrization of the renormalized trajectory below.

Decimation recursion

Lattice block step $b^n a \longrightarrow b^{n+1} a$:

Implemented as a prescription for character expansion coefficients $F_j(n+1)$ in terms of the $F_j(n)$:

$$F_j(n+1) = \left[\int dU \left[\sum_k d_k \, F_k(n) \, \chi_k(U) \right]^{\zeta^{(d-2)}} \frac{1}{d_j} \chi_j^*(U) \right]^{r^2} . \qquad (4)$$

Specify renormalization parameters ζ, r:

$$\zeta = b \left[1 - c \, g^{(n)\,2} \right] \qquad (5)$$

$$r = b \left[1 - c \, g^{(n)\,2} \right] \qquad (6)$$

with c an adjustable decimation parameter.

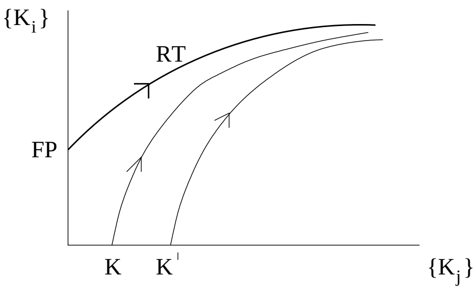
RG flows and Lattice Matching

General lattice system given by action Action $A(K)$ with set of couplings $K = \{K_i\}$.
RG blockings by a scale factor b generate a flow in action space:

$$K \to K^{(1)} \to K^{(2)} \to \cdots \to K^{(n)} \to \cdots$$

where

$$K^{(n)} = \{K_i^{(n)}\} = \text{couplings after } n \text{ blocking steps}$$



If flows from K and K' reach same point on RT after n and n' steps, then lattice correlation lengths ξ, ξ' and spacings a, a' related as

$$\xi' = b^{-(n-n')}\xi \qquad a' = b^{(n-n')}a\,. \tag{7}$$

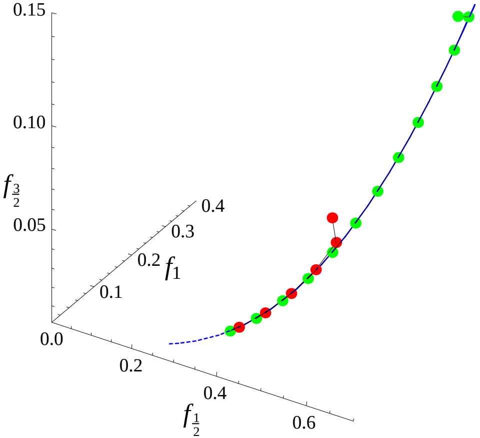
To identify such pairs of couplings we need ascertain that after n and n' RG steps, respectively, the same point is reached on the RT.
This can be done in two ways:

1. Show that corresponding actions coincide: $A(K^{(n)}) = A(K'^{(n')})$. This requires that one obtain the blocked action at each step.
2. Show that the expectations of every operator, measured after performing the corresponding number of blocking steps from the initial two actions, agree.

Either way, identifying such pairs (K, n), and (K', n') is referred to as two-lattice matching.

RG recursion flow features

- Even after a single block step resulting blocked action is plaquette action with generally infinite set of representations.
  Note: MCRG constructions of blocked actions indicate that the most relevant action terms for long distance dynamics are precisely actions of this type.
- Accurate stepping function (beta function) in the large β scaling region. (Exact in the strict $\beta \longrightarrow \infty$ limit)
- Flows rapidly reach unique Renormalized Trajectory.



Flow from the $SU(2)$ fund. Wilson action with $\beta = 4$ (green dots) and $\beta = 2.5$ (red dots). First three non-trivial expansion coefficients shown.

Critical couplings $\beta_c(N_\tau)$

$T = 1/aN_\tau$: lattice N_τ, a and lattice N'_τ, a' related by:

$$a' = \frac{N_\tau}{N'_\tau} a \,.$$

If after blocking n and n' times, respectively, the two flows reach the same point on the RT:

$$a' = b^{(n-n')} a \qquad \Longrightarrow \qquad n - n' = \log_b \left(\frac{N_\tau}{N'_\tau} \right) .$$

So,, at $T = T_c$:

$$\beta^{(n)}(\beta_c(N_\tau)) = \beta^{(n')}(\beta_c(N'_\tau)) \,. \tag{8}$$

Procedure:

- Assume $\beta_c(N_\tau)$ known for one N_τ (by MC, ...)
- Take

$$\begin{aligned} n &= \log_b N_\tau + m \\ n' &= \log_b N'_\tau + m \end{aligned}$$

Integer $m = 0, 1, \ldots$ chosen so n, n' large enough to be on the RT.

- Solve

$$\beta^{(n)}(\beta_c(N_\tau)) = \beta^{(n')}(\beta_c(N'_\tau))$$

for $\beta_c(N'_\tau)$, i.e., adjust starting point of the flow N'_τ lattice is adjusted to satisfy this equality.

String tensions ($T = 0$)

Two RG flows from Wilson action at β_0 and β_1 ending up at same RT point after n_0 and n_1 steps, respectively. Then:

$$\beta^{(n_0)}(\beta_0) = \beta^{(n_1)}(\beta_1) \tag{9}$$

and

$$a_1\sqrt{\sigma} = b^{(n_0-n_1)}a_0\sqrt{\sigma} \tag{10}$$

- Suppose $a_0\sqrt{\sigma}$ known.
- Choose n_0 large enough to be on the RT.
- Determine n_1 so that (9) is satisfied.
- $a_1\sqrt{\sigma}$ then obtained directly from (10).

Numerical results

Character expansion series

- $SU(2)$: 50 characters retained $\Longrightarrow$ omitted character size - remainders at $\beta = 5$ of order 10^{-45}.
- $SU(3)$: retained characters $j \equiv (p, q)$ with $p \leq 19$, $q \leq 19 \Longrightarrow$ remainders at $\beta = 10$ of less than 10^{-12}.

Scale factor $\quad b = 2$

Decimation parameter

- $SU(2)$: $\quad c = 0.10$
- $SU(3)$: $\quad c = 0.24$

$SU(2)$ critical couplings

N_τ	β_c	β_c	β_c(MC)
3	2.1875	2.1957	2.1768(30)
4	2.2909	<u>2.2991</u>	2.2991(02)
5	2.3600	2.3683	2.3726(45)
6	2.4175	2.4258	2.4265(30)
8	2.5097	2.5180	2.5104(02)
12	<u>2.6355</u>	2.6440	2.6355(10)
16	2.7275	2.7361	2.7310(20)
32	2.9487	2.9574	

$SU(3)$ critical couplings

N_τ	β_c	β_c	β_c(MC)
4	5.6501	5.6329	5.6925(002)
6	<u>5.8941</u>	5.8773	5.8941(005)
8	6.0773	6.0595	6.0010(250),6.0625(18)
10	6.2018	6.1837	6.1600(70)
12	6.3084	6.2900	6.2680(120),6.3385(55)
14	6.4015	<u>6.3830</u>	6.3830(100)
16	6.4845	6.4658	6.4500(500)
32	6.9024	6.8829	

$SU(2)$ string tensions

β	$a\sqrt{\sigma}$	$a\sqrt{\sigma}$	$a\sqrt{\sigma}$(MC)
2.2	0.5019	0.5161	0.4690(100)
2.3	0.3654	0.3756	0.3690(30)
2.4	0.2619	0.2696	0.2660(20)
2.5	0.1903	0.1957	0.1905(08)
2.5115	<u>0.1836</u>	0.1888	0.1836(13)
2.6	0.1373	0.1415	0.1360(40)
2.7	0.1002	0.1031	0.1015(10)
2.74	0.0884	<u>0.0911</u>	0.0911(08)
2.85	0.0622	0.0641	0.0630(30)

$SU(3)$ string tensions

β	$a\sqrt{\sigma}$	$a\sqrt{\sigma}$	$a\sqrt{\sigma}$(MC)
5.54	0.5580	0.5878	0.5727(52)
5.6	0.5070	0.5295	0.5295(09), 0.5064(28)
5.7	0.4205	0.4264	0.4099(12), 0.3879(39)
5.8	0.3486	0.3508	0.3302(15)
5.9	0.2919	0.2931	0.2702(19)
6.0	0.2465	0.2433	0.2269(62), 0.2209(23)
6.2	0.1698	0.1671	0.1619(19), 0.1604(11)
6.4	0.1214	0.1180	0.1214(12), 0.1218(28)
6.5	0.1010	0.0983	0.1068(09)
6.8	0.0616	0.0599	0.0738(20)

Conclusions

- Lattice matching via these RG recursions provides an inexpensive method for obtaining reliable critical couplings and string tensions in $SU(2)$ and $SU(3)$ pure gauge theories.
- RG recursion blocked actions are plaquette actions with large (infinite) number of group representations.
- Would be interesting to consider:
  - Other quantities, e.g. spatial string tensions at finite T.
  - More elaborate block transformations involving also non-plaquete terms and further decimation parameters - likely needed for computation of observables over different scales.
  - Inclusion of fermions (much harder).