Padé approximants and g-2 for the muon

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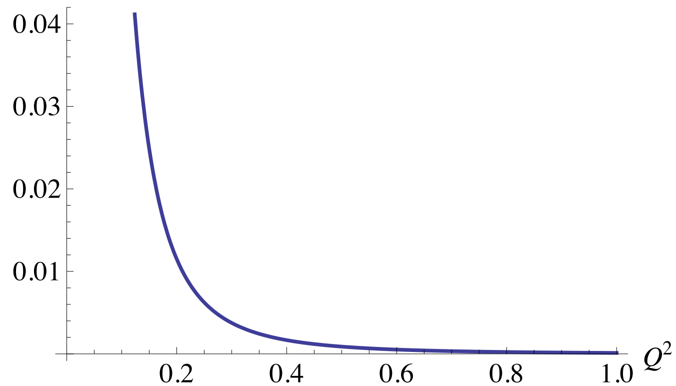
Contribution from lowest-order hadronic vacuum polarization (HLO)

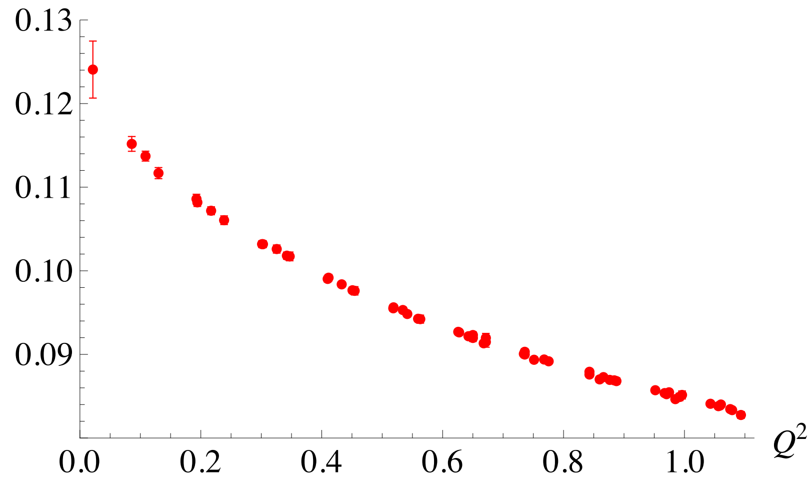
$$a_\mu^{\mathrm{HLO}} = 4\alpha^2 \int_0^\infty dQ^2\, f(Q^2)\left(\Pi(0) - \Pi(Q^2)\right)$$

(Lautrup, Peterman, de Rafael 1971, Blum 2002)

$f(Q^2) =$



Data for $\Pi(Q^2)$:



Need to fit data to compute integral; VMD introduces model dependence

Multi-point Padé approximants:

Write $\left(\Pi(0)-\Pi(Q^2)\right)/Q^2=\int_{4m_\pi^2}^{\infty}dt\,\frac{\rho(t)}{t(t+Q^2)}\equiv\Phi(Q^2)\ ,\quad \rho(t)\geq 0$

This integral is a Stieltjes function, analytic everywhere except cut $(-\infty,-4m_\pi^2]$.

Theorem: Given P points $(Q_i^2,\Phi(Q_i^2))$ a sequence of PAs can be constructed which converge to $\Phi(Q^2)$ on any closed, bounded region of the complex plane excluding the cut, in the limit $P\to\infty$. (Baker 1969, Barnsley 1973)

Construction: $$\Phi(Q^2)=\cfrac{\Psi_0}{1+\cfrac{(Q^2-Q_1^2)\Psi_1}{1+\cfrac{\ddots}{\cfrac{(Q^2-Q_{P-2}^2)\Psi_{P-2}}{1+(Q^2-Q_{P-1}^2)\Psi_{P-1}}}}}$$

with Ψ_i related to $\Phi(Q^2_{j\leq i+1})$ ($\Psi_0=\Phi(Q_1^2)$, etc.), yields a $[[(P-1)/2],[P/2]]$ PA.

Parametrization and strategy

Furthermore, can prove that (Baker, Barnsley)

$$\Pi(Q^2) = \Pi(0) - Q^2 \left(a_0 + \sum_{n=1}^{[P/2]} \frac{a_n}{b_n + Q^2} \right)$$

with $a_n > 0$ (positive residues) and $b_{[P/2]} > \cdots > b_1 > 4m_\pi^2$ (all poles on cut), $a_0 = 0$ for P even .

Fit this form for $P = 2, 3, 4, 5$; yields $[0,1],\ [1,1],\ [1,2],\ [2,2]$ PAs.

Compute $\quad a_\mu^{\mathrm{HLO},Q^2\leq 1} = 4\alpha^2 \int_0^{1\ \mathrm{GeV}^2} dQ^2\, f(Q^2)\ \left(\Pi(0) - \Pi(Q^2)\right)$

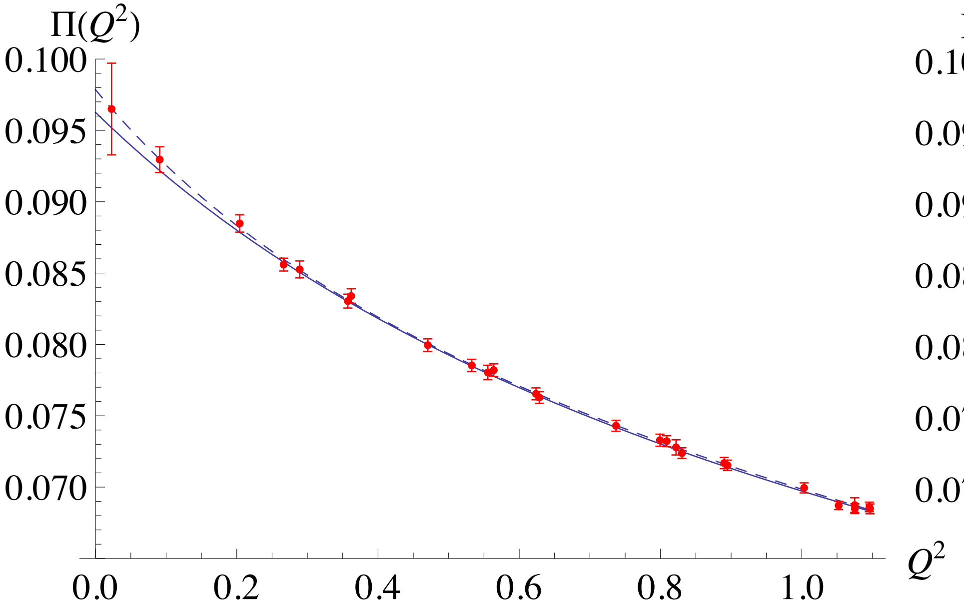
Note: VMD is same as $[0,1]$ PA with $b_1 = m_\rho^2$ fixed; NOT a (valid) PA!

Test on MILC lattices with $a = 0.09$ fm , $m_\pi = 480$ MeV

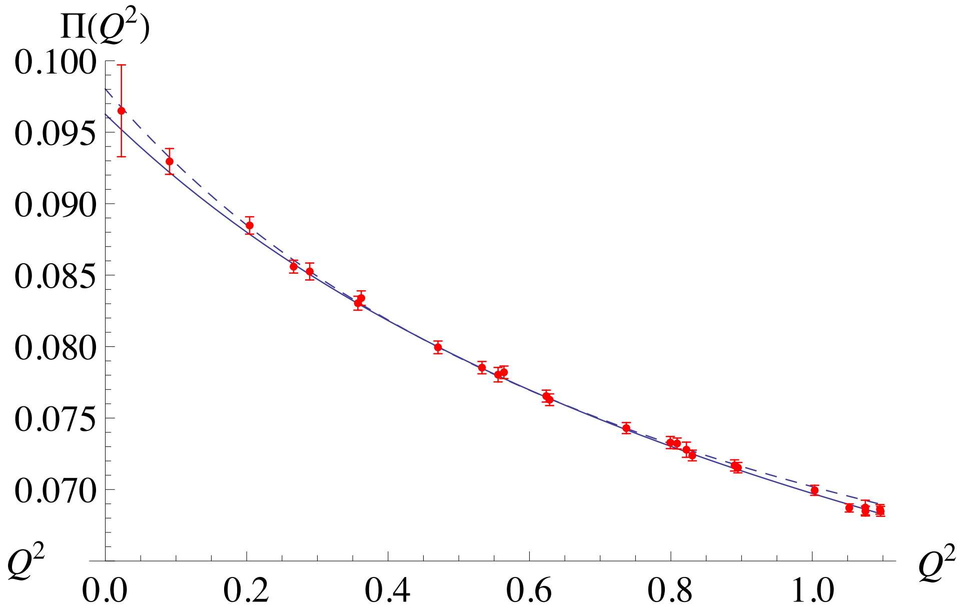
		correlated interval $0 < Q^2 \leq 0.6$ GeV^2		**uncorrelated** interval $0 < Q^2 \leq 1$ GeV^2	
PA	# parameters	χ^2/dof	$10^{10} a_\mu^{\mathrm{HLO},Q^2\leq 1}$	χ^2/dof	$10^{10} a_\mu^{\mathrm{HLO},Q^2\leq 1}$
VMD	2	5.86/3*	363(7)	4.37/18	413(8)
[0, 1]	3	11.4/8	338(6)	3.58/17	373(37)
[1, 1]	4	7.49/7	350(8)	3.36/16	424(116)
[1, 2]	5	7.49/6	350(8)	3.35/15	443(293)
[2, 2]	6	7.49/5	350(7)	3.35/14	445(432)

* interval $0 < Q^2 \leq 0.35$ GeV^2
uncorrelated VMD fit agrees with Aubin and Blum, 2007

- Correlated: VMD bad, clear improvement with addition of parameters
- Difficult to determine 2nd pole, but a_μ insensitive to higher poles
- Internal consistency, except uncorr. VMD (unknown systematic error!) and correlated PAs



[1,1] corr. (solid) and uncorr. (dashed)

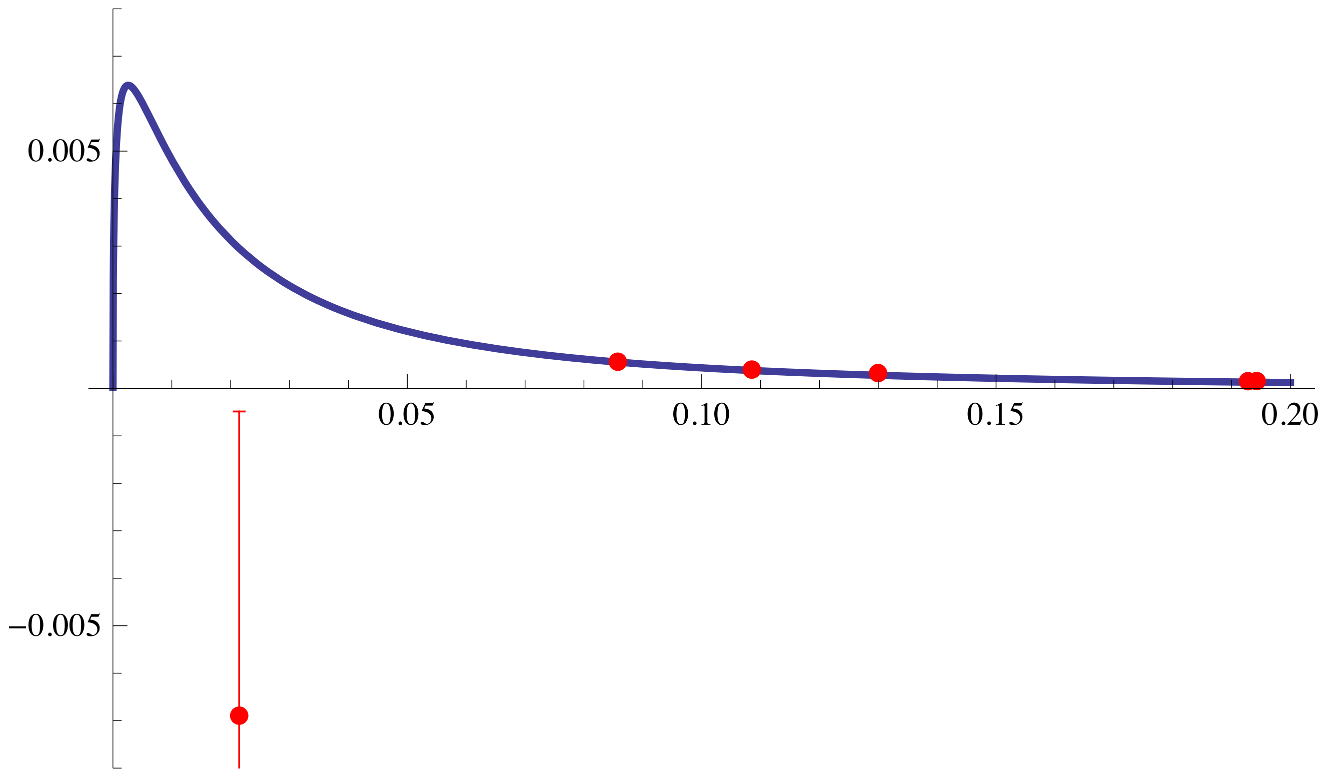


[1,1] corr. (solid) and VMD uncorr. (dashed)

- uncorrelated fits look better at small Q^2
- also considered MILC lattices with $a = 0.06\ \mathrm{fm}\ ,\ m_\pi = 220\ \mathrm{MeV}$ – similar

$a_\mu^{\mathrm{HLO},Q^2\leq 1} = 572(41) \times 10^{-10}$ [1,1] corr. , $a_\mu^{\mathrm{HLO},Q^2\leq 1} = 646(8) \times 10^{-10}$ VMD uncorr.

- not possible to decide which fit is best, based on current data



Integrand of $a_\mu^{\mathrm{HLO}}/(4\alpha^2)$ compared with data
(MILC, $a = 0.06\ \mathrm{fm}\ ,\ m_\pi = 220\ \mathrm{MeV}$)

⇒ need more data at low Q^2 with smaller errors! In progress…

Conclusions

- New method to parametrize hadronic vacuum polarization; avoid model dependence of vector meson dominance. Based on representation of vacuum polarization in terms of Stieltjes function.

- Tested on two examples of numerical data for vacuum polarization. Padé approximant fits can lead to larger statistical errors, but avoid unknown systematic error associated with VMD.

- Method looks promising, but data at lower momenta and smaller errors are indispensible (difference between $a_\mu(VMD)$ and $a_\mu([1,1])$ is about 17%).

- Note: long chiral extrapolation – also need data with small pion mass in order to control this source of error (well below 300 MeV).

Backup slide 1: comparison with polynomial fits

	Poly 3		Poly 4		PA [1,1]		PA [1,2]	
# points	χ^2/dof	$a_\mu^{(1)}$	χ^2/dof	$a_\mu^{(1)}$	χ^2/dof	$a_\mu^{(1)}$	χ^2/dof	$a_\mu^{(1)}$
16	9.6/12	543	9.5/11	483	9.7/12	564	9.7/11	565
18	11.4/14	526	10.5/13	596	11.2/14	541	11.5/13	561
20	13.1/16	536	13.1/15	535	13.9/16	572	13.9/15	572
22	16.5/18	541	15.9/17	513	18.5/18	566	18.5/17	566
24	16.6/20	537	16.4/19	521	19.4/20	583	19.4/19	583
26	30.7/22	505	23.6/21	580	26.8/22	557	26.7/21	560

- Poly 3, PA [1,1] and PA [1,2] correlated fits all good, not so Poly 4.
- Stability from PA [1,1] to PA [1,2], not from Poly 3 to Poly 4.
- Note: PA's converge everywhere except on cut;
  polynomials only within radius of convergence.

Backup slide 2: chiral extrapolation

Assume VMD, and approximate $\Pi(Q^2)_{subtr} = g_\rho^2 Q^2/m_\rho^2$ (continuum)

$\Pi(Q^2)_{subtr} = g_V^2 Q^2/m_V^2$ (lattice)

Define $I_H = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w\left((Q^2/m_\mu^2)(H_{phys}^2/H_{latt}^2)\right) \Pi(Q^2)_{subtr}$

(Feng et al. 2011)

Then $I_H = \left(\frac{H_{latt}}{H_{phys}}\right)^2 \frac{g_V^2}{g_\rho^2}\frac{m_\rho^2}{m_V^2}\, I_{model}$ hence choose $H_{latt} = m_V/g_V$

Whatever choice: model dependent! 1st PA pole not equal to m_V^2
Cannot avoid small pion masses (much smaller than 300 MeV)