New algorithms and new results for Strong Coupling LQCD

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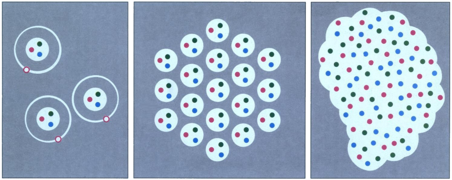
Lattice 2012, Cairns

26.06.2012



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

1. Motivation for Strong Coupling LQCD in Continuous Time
   - Continuous Time Limit and $a/a_t = f(\gamma)$
   - Continuous Time Partition Function $Z(\beta)$
2. Quantum Monte Carlo Methods
   - Spin Representation
   - Stochastic Series Expansion
3. Application: 2 flavor SC-LQCD
   - Generalization of Spin Representation
   - Preliminary Results

Why Strong Coupling Lattice QCD?

Look at Lattice QCD in a regime where the finite baryon density **sign problem** can be made mild:

$$\beta = \frac{2N_c}{g^2} \to 0$$

- allows to integrate out the gauge fields completely, as **link integration factorizes**
  $\Rightarrow$ no fermion determinant
- drawback: strong coupling limit is converse to asymptotic freedom, lattice is maximally coarse

Strong coupling LQCD shares important features with QCD:

- exhibits **confinement**, i.e. only color singlet degrees of freedom survive:
  - **mesons** (represented by monomers and dimers)
  - **baryons** (represented by oriented self-avoiding loops)
- and **spontaneous chiral symmetry breaking/restoration:** (restored at T_c)
  $\Rightarrow$ SC-LQCD is a great laboratory to study the full T–μ **phase diagram**

SC-LQCD is a useful toymodel for **nuclear matter**

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SC-LQCD is a **1-parameter deformation of QCD**

SC-LQCD at finite temperature

How to vary the temperature?

- $aT = 1/N_\tau$ is discrete with N_τ even
- $aT_c \simeq 1.5$, i.e. ${N_\tau}^c < 2$ $\Rightarrow$ we cannot address the phase transition!

Solution: introduce an **anisotropy** γ in the Dirac couplings:

$$\mathcal{Z}(m_q, \mu, \gamma, N_\tau) = \sum_{\{k,n,l\}} \prod_{b=(x,\mu)} \frac{(3-k_b)!}{3!k_b!} \gamma^{2k_b\delta_{\mu 0}} \prod_x \frac{3!}{n_x!} (2am_q)^{n_x} \prod_l w(\ell, \mu)$$

Should we expect $a/a_\tau = \gamma$, as suggested at weak coupling?

- **No**: meanfield predicts $a/a_\tau = \gamma^2$, since $\gamma_c^2 = N_\tau \frac{(d-1)(N_c+1)(N_c+2)}{6(N_c+3)}$

  $\Rightarrow$ sensible, N_τ-independent definition of the temperature: $aT \simeq \frac{\gamma^2}{N_\tau}$

- Moreover, SC-LQCD partition function is a function of γ^2

However: **precise correspondence between a/a_τ and γ^2 not known**

SC-LQCD at finite Temperature and Continuous Time:

Strategy for **unambiguous** answer: the **continuous Euclidean time limit** (CT-limit):

$$N_\tau \to \infty, \qquad \gamma \to \infty, \qquad \gamma^2/N_\tau \equiv aT \quad \text{fixed}$$

- same as in analytic studies: $a_\tau = 0$, $aT = \beta^{-1} \in \mathbb{R}$

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Several **advantages** of continuous Euclidean time approach:

- ambiguities arising from the functional dependence of observables on the anisotropy parameter will be circumvented, **only one parameter** setting the temperature
- no need to perform the continuum extrapolation $N_\tau \to \infty$
- allows to estimate critical temperatures more precisely, with a faster algorithm (about 10 times faster than $N_t = 16$ at T_c)
- baryons become static in the CT-limit, the **sign problem is completely absent!**

Continuous Time Partition Function

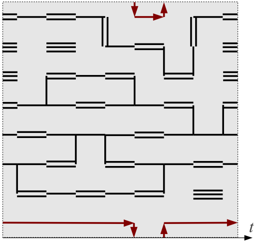
Partition function in the limit of large γ^2, N_τ with $T = \gamma^2/N_\tau$:

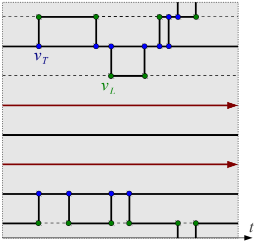
$$\mathcal{Z}(\gamma, N_\tau) = \sum_{\{k, n_B(x)\}} e^{3 n_B \mu / T} \prod_{x \in V_M} \left(\frac{v_L}{\gamma}\right)^{n_L(x)} \left(\frac{v_T}{\gamma}\right)^{n_T(x)}, \quad n_B(x) \in \{-1, 0, 1\}$$

- $V_M \subseteq {N_\sigma}^3$ is mesonic subvolume and $n_B = \sum_x n_B(x)$ is baryon number
- multiple spatial dimer become resolved into **single spatial dimers** as $a_t \to 0$

typical (2-dimensional) configurations in discrete and continuous time at the same temperature:

- weight of configuration given by **number of spatial dimers and vertices** (v_L, v_T) **regardless of time coordinates**
- **baryons** become **static** in continuous time!





Diagrammatic Partition Function

Rewrite partition function in inverse temperature $\beta = 1/aT$:

- sum over all spatial dimer time coordinantes $\sim N_\tau/2 \Rightarrow$ **expansion in $\beta = N_\tau/\gamma^2$**
- number of spatial dimers: $\kappa = \frac{1}{2}\sum_{x\in V_M}(n_L(x) + n_T(x))$

$$\mathcal{Z}(\beta) = \sum_{\kappa\in 2\mathbb{N}} \frac{(\beta/2)^\kappa}{\kappa!} \sum_{\mathcal{C}\in\Gamma_\kappa} v_T^{n_T(\mathcal{C})} e^{\beta 3\mu B(\mathcal{C})}, \quad n_T = \sum_x n_T(x)$$

- Γ_κ is the set of equivalence classes of configurations with κ spatial dimers, time coordinates of spatial dimers irrelevant

Importance sampling of diagrams described in terms of a perturbative series:

- each term Γ_κ is represented by a **world line configuration**
- perturbative series may not converge, but for any finite volume and temperature, only a finite number of orders contribute
- important QMC techniques: **continuous time worm** (Beard & Wiese), loop cluster algorithm (Evertz et al.), **stochastic series expansion** (Sandvik)

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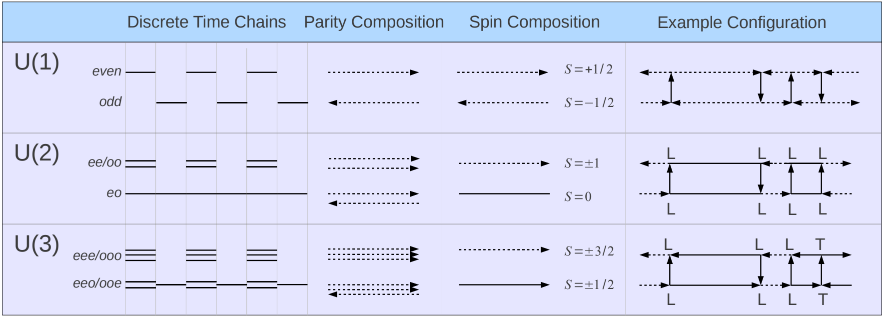
results for SC-LQCD obtained with CT-Worm algorithm reported last year (phase diagram in the μ–T plane) [hep-lat/1111.1434]

Mapping of 1-flavor SU(N_c) to a spin system

Continuous time methods can be applied to any gauge group SU(N_c):

- baryons become static for $N_c \geq 3$
- mesonic discrete time chains **classified by parity:**



$\Rightarrow$ mesonic CT line types **classified by "spin"**: $S = -N_c/2 \ldots N_c/2$ (remnant of staggered even/odd ordering), $\Delta S = \pm 1$ (absorption/emission)

- generalizes to arbitrary $U(N_c)$

Stochastic Series Expansion

Idea: rewrite partition function, based on decomposition in diagonal and non-diagonal elements $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, truncation L:

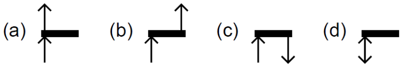
$$\mathcal{Z}(\beta) = \mathrm{Tr}\left\{e^{-\beta\mathcal{H}}\right\} = \sum_{\chi}\sum_{S_L} \frac{\beta^\kappa (L-\kappa)!}{L!} \left\langle \chi \left| \prod_{i=1}^{L} \mathcal{H}_{a_i,b_i} \right| \chi \right\rangle, \qquad \begin{aligned} \mathcal{H}_{1,b} &= \varepsilon \mathbb{1},\ \varepsilon \geq 0 \\ \mathcal{H}_{2,b} &= \tfrac{1}{2} S_x^+ S_y^- \end{aligned}$$

with S_L a **time-ordered sequence** of operator-indices: $S_L = [a_1, b_1], [a_2, b_2], \ldots [a_L, b_L]$

- $a_i = 0$: identity, $a_i = 1, 2$: diagonal/non-diagonal matrix element
- $b_i = \langle x, y \rangle \in Vd$ denotes a bond

Two kinds of **updates:**

1. **changing order in β,** $\kappa \mapsto \kappa \pm 1$:
   $P([1,b]_p \mapsto [0,0]_p) = \frac{L-\kappa+1}{Vd\beta\langle\chi|\mathcal{H}_{1,b}|\chi\rangle}$, $\quad P([0,0]_p \mapsto [1,b]_p) = \frac{Vd\beta\langle\chi|\mathcal{H}_{1,b}|\chi\rangle}{L-\kappa}$
2. **operator loop:** visit bonds b_i successively from an input leg, determine output leg with heatbath probability $\langle \chi_x \chi_y | \mathcal{H}_{a_i b_i} | \chi'_x \chi'_y \rangle$



L is set larger than $\kappa_{\max}$ $\Rightarrow$ **SSE is approximation free** (like CT-Worm)

SSE applied to SC-LQCD

Strong Coupling $U(1)$ is identical to **XY Model** in zero field!

- new observable: spin susceptibility $\chi_S = \beta \left\langle (\sum_i S_i^z)^2 \right\rangle / N$

Extension to U(3) for SC-LQCD straightforward:

$$\mathcal{H} = \frac{1}{2} \sum_{\langle x,y \rangle} J_x^+ J_y^-$$

with $J^+ = \begin{pmatrix} 0 & & & \\ v_L & 0 & & \\ & v_T & 0 & \\ & & v_L & 0 \end{pmatrix}$

and $J^- = (J^+)^T$ for absorption/emission

- state vector characterizing time slice: $|S^z\rangle(t) \in \left\{ \bigotimes_{\vec{x}\in V} S^z_{\vec{x}} | S^z_{\vec{x}} \in \{-N_\mathrm{c}/2, \ldots N_\mathrm{c}/2\} \right\}$
- **oriented spatial dimers** act at time t_i on $|S_x\rangle$ by raising/lowering spin at absorption/emission site
- lowest/highest weight: $J^+|N_\mathrm{c}/2\rangle = 0, \quad J^-|-N_\mathrm{c}/2\rangle = 0$
- S^z counts net number of (odd-even) time like meson sites at each site
- $\frac{N_\mathrm{c}}{2}[J^+, J^-] = J^z = \mathrm{diag}(-N_\mathrm{c}/2, \ldots, N_\mathrm{c}/2)$ fulfilled, $J^z|S^z\rangle = S^z|^z\rangle$

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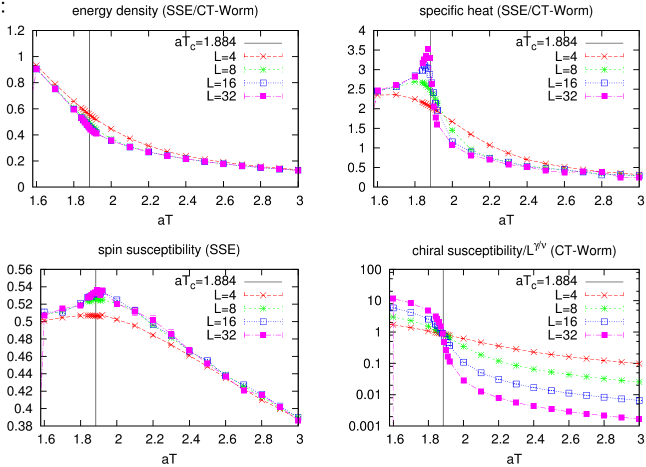
"Spin" is a conserved quantity, generalizes for arbitrary N_c!

Comparison of Continuous Time Worm and SSE

Observables in both algorithms: energy, specific heat (obtained from # spatial dimers)

- CT-Worm faster if truncation L is unlimited in SSE
- SSE faster if L is fixed and not too large

$N_c = 3$:



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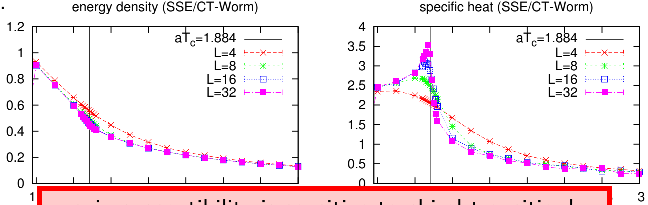
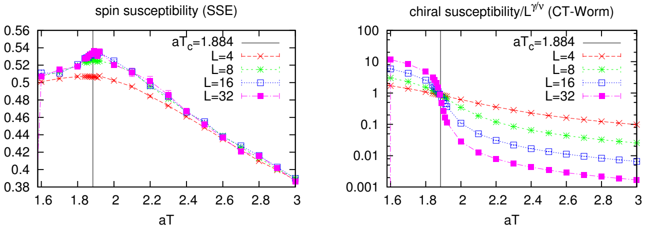


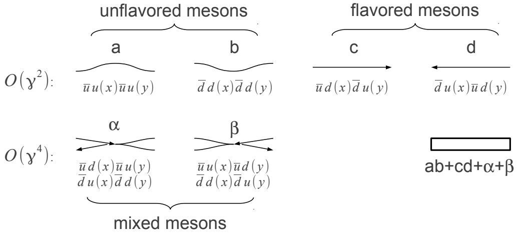
spin susceptibility is sensitive to chiral transition!



Application: Generalization of SC-LQCD to 2 chiral flavors!

Aim: obtain phase diagram for 2-flavor SC-LQCD, where **pion exchange** may play a crucial role for nuclear transition, but:

- at present, no 2-flavor formulation for staggered SC-LQCD suitable for MC
- already the mesonic sector has a severe **sign problem** (worse than for finite μ HMC)
- 2 new types of mixed dimers give negative sign in mesonic loops already for U(2):



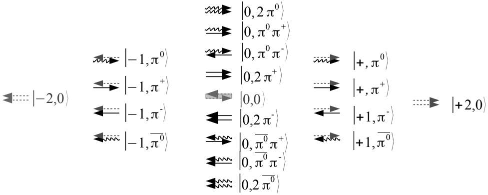
Observation in continuous time formulation:

- **static lines** for 2 different flavors can be composed such that **cancellations appear**
- first step in this direction: use SSE (simpler than CT-Worm)

Continuous Time Transition Rules

Flavored static lines:

- new classification in terms of **quantum numbers** $|S^z, Q_{\pi^0}, Q_{\pi^+}\rangle$
- in total: 19 types of lines, 18 have weight $1/16$ per $2a_t$, "vacuum state" $|0,0,0\rangle$ has weight $1/8$



Quantum Numbers:

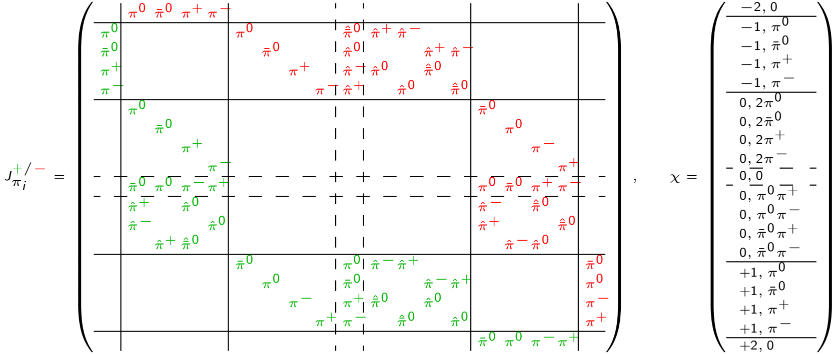
- "spin" S^z counts number of emission/absorption events (remnant of even/odd decomposition of lattice for staggered fermions) $S^z = -\frac{1}{2}N_c N_f, \ldots, +\frac{1}{2}N_c N_f$
- "charges" $Q_i = -N_c, \ldots, +N_c$ denote the flavor content
- **spin/charge conservation:** transitions at spatial dimers, raising charges at one site, lowering at a neighboring site:

$$|\Delta S^z| = 1, \quad |\Delta Q_{\pi^0}| + |\Delta Q_{\pi^+}| = 1$$

Hamiltonian for $N_f = 2$, $N_c = 2$

$$\mathcal{H} = \frac{1}{2} \sum_{\langle x,y \rangle} \left(J^{+}_{\pi^0(x)} J^{-}_{\pi^0(y)} + J^{+}_{\bar{\pi}^0(x)} J^{-}_{\bar{\pi}^0(y)} + J^{+}_{\pi^+(x)} J^{-}_{\pi^+(y)} + J^{+}_{\pi^+(x)} J^{-}_{\pi^-(y)} \right)$$

Absorption ($J^{+}_{\pi_i}$, lower left triangle) and Emission ($J^{-}_{\pi_i}$, upper triangle), state vector



The vertex weights are $v_{\pi_i} = 1$ for vertices not mixing the two charges Q_i, and $v_{\hat{\pi}_i} = \frac{1}{\sqrt{2}}$ for vertices mixing the charges

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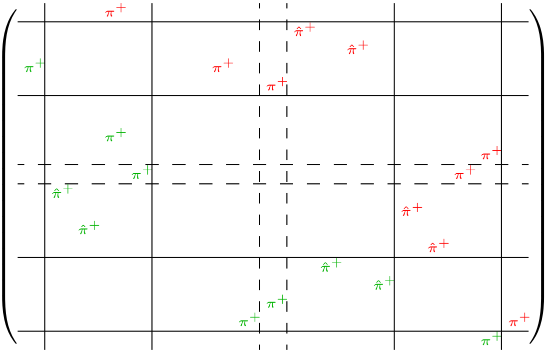
$J^{+/-}_{\pi^+} =$



, $\chi = \begin{pmatrix} -2,\,0 \\ -1,\,\pi^0 \\ -1,\,\bar{\pi}^0 \\ -1,\,\pi^+ \\ -1,\,\pi^- \\ 0,\,2\pi^0 \\ 0,\,2\bar{\pi}^0 \\ 0,\,2\pi^+ \\ 0,\,2\pi^- \\ 0,\,0 \\ 0,\,\pi^0\pi^+ \\ 0,\,\pi^0\pi^- \\ 0,\,\bar{\pi}^0\pi^+ \\ 0,\,\bar{\pi}^0\pi^- \\ +1,\,\pi^0 \\ +1,\,\bar{\pi}^0 \\ +1,\,\pi^+ \\ +1,\,\pi^- \\ +2,\,0 \end{pmatrix}$

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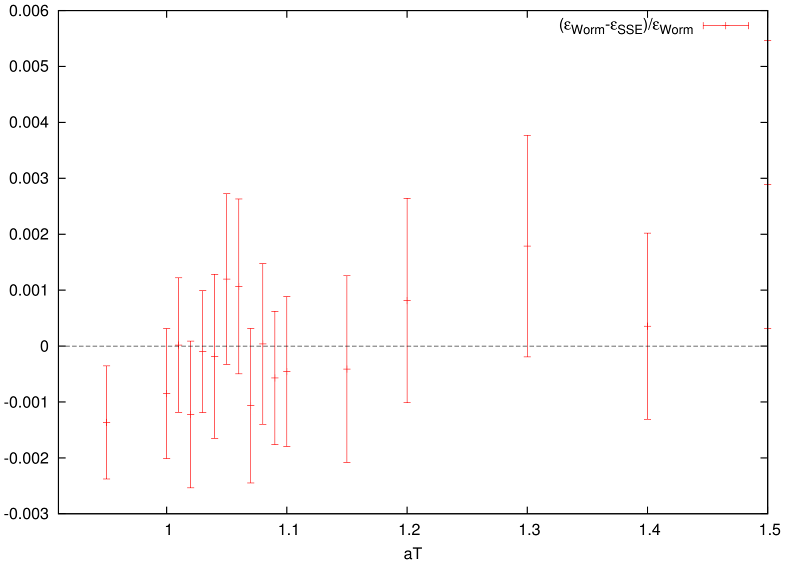
Conclusions

Achievements:

- CT partition function: new formulation as a **spin system!**
- "spin" formulation and Hamiltonian follow from conservation laws for even/odd chains of time-like dimers and flavors - this **generalizes to arbitrary N_c, N_f**
- new observable: spin susceptibility, sensitive to chiral transition
- quantum Monte Carlo applicable: e.g. continuous time worm or **stochastic series expansion** (most convenient)
- now also applied to U(2) with two flavors (incorporates pion exchange)
- extension to SU(3) with finite baryon chemical potential straightforward (Hamiltonian worked out, but no simulations yet)

Comparison of SSE and CT-Worm



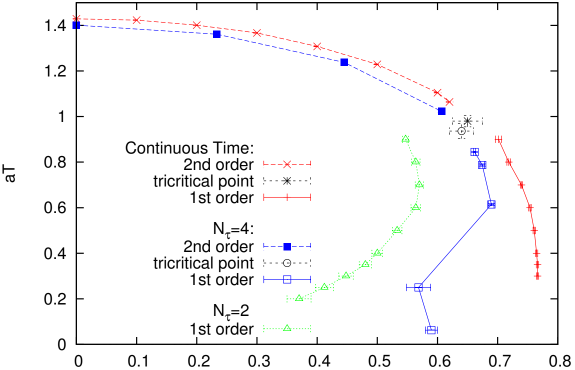
SC-QCD Phase Diagram

Studied via CT-Worm algorithm: arXiv:1111.1434 [hep-lat]
Comparison of phase diagram with $N_\tau = 4$ data (M. Fromm, 2010):

- CT-data compared to $N_\tau = 4$ data for identification

$$a\mu = \gamma^2 a_\tau \mu$$

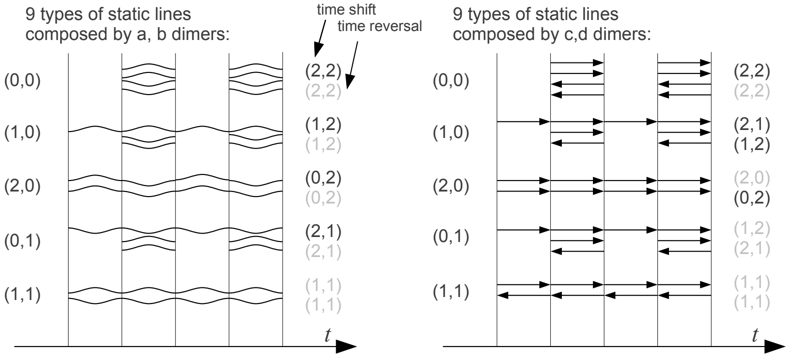
- behavior at low μ agrees well, location of TCP agrees within errors
- no re-entrance is seen at small temperatures



Static Line Rules

Combine temporal dimers of alternating orders in γ^2 (here for $N_c = 2$):

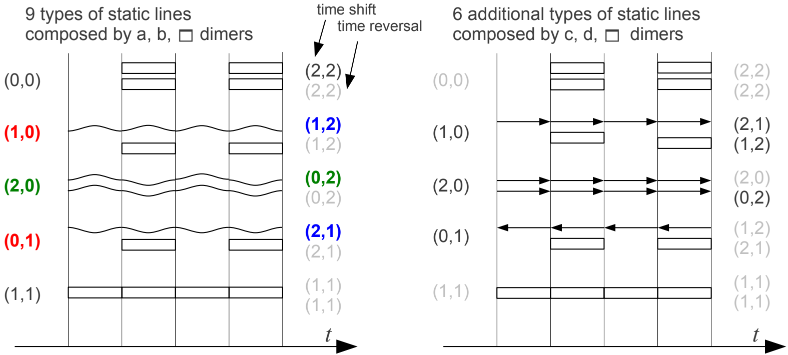
- first: consider (a, b) and (c, d) dimers sepaterely
- then: resum them to obtain flux representation



Static Line Rules

Combine temporal dimers of alternating orders in γ^2 (here for $N_c = 2$):

- resummation of $a - b - a$ and $b - a - b$ chains
- resummation of $ab + cd + \alpha + \beta$ into $\square$ dimers



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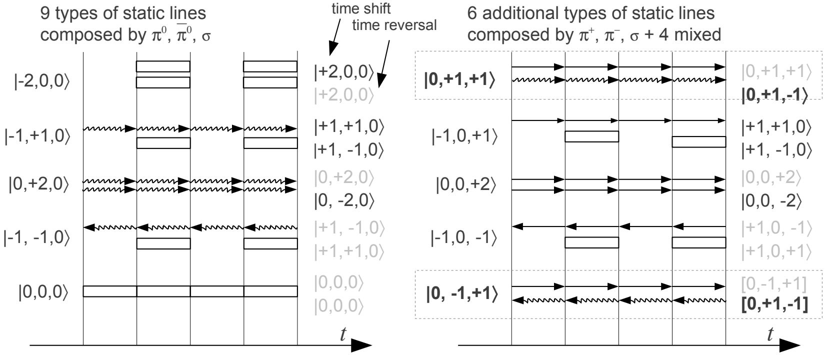
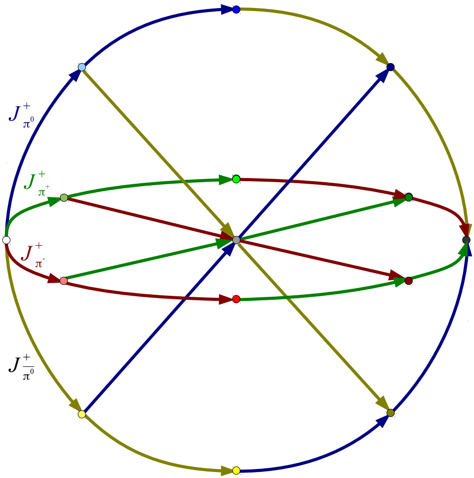


The Transition Rules Encoded in $J^{\pm}$



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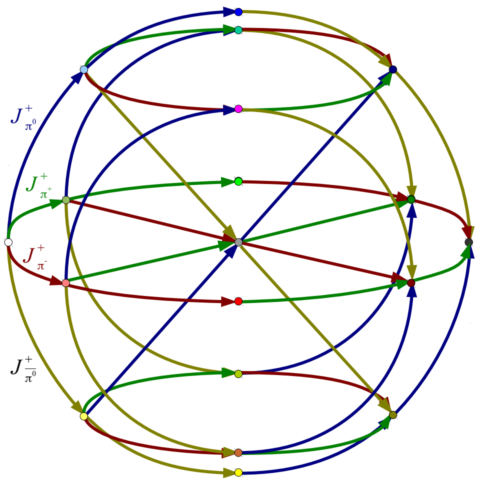

Why Study Strong Coupling QCD on the Lattice?

Two possible scenarios for the relation between
SC-LQCD (back) and the (L)QCD phase diagram for four flavors (front):