Three loops renormalization constants in Numerical Stochastic Perturbation Theory

Masayasu Hasegawa,
Michele Brambilla, and Francesco Di Renzo

University of Parma and INFN

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Introduction

My main purpose is to present three loops renormalization constants for $N_f = 2$ Symanzik and $N_f = 4$ Iwasaki gauge action (Wilson fermions).

Topics I will talk about;

- Motivations.
- What is Numerical Stochastic Perturbation Theory and how can it enable high orders computations?
- How to take the continuum limit (lattice spacing $\to 0$) and remove finite size effects from our data?
- Finally, I report on three loops renormalization constants of quark bilinears for $N_f = 2$ Symanzik and $N_f = 4$ Iwasaki gauge actions.

1. Purpose of this study

We want to bridge perturbative and non-perturbative computations. There are different systematic involved, of which one should be aware:

- In the perturbative case, one issue is truncation errors.
- In the non-perturbative case, one most often has to deal with chiral extrapolations
- Both in perturbative and non-perturbative case, one should take the continuum limit, and (often) tame finite size effects.

Our goal is to the take a good control on all these issues.

2. Numerical Stochastic Perturbation Theory

In the Stochastic Quantization framework

$$\frac{\partial}{\partial t}\phi_\eta(x,t) = -\frac{\delta S[\phi]}{\delta\phi_\eta(x,t)} + \eta(x,t).$$

$$\lim_{t\to\infty}\langle\phi(x_1,t)\ldots\phi(x_n,t)\rangle_\eta = \langle\phi(x_1)\ldots\phi(x_n)\rangle.$$

we expand the solution to Langevin equation

$$\phi_\eta(x,t) = \phi_\eta^{(0)}(x,t) + \sum_{n>0} g^n \phi_\eta^{(n)}(x,t)$$

and compute observables order by order

$$O\left[\sum_n g^n \phi_\eta^{(n)}(x,t)\right] = \sum_n g^n O^{(n)}(x,t).$$

These computations are something like a perturbative Monte Carlo!

2.1 RI'-MOM scheme

We compute quark bilinears bracketted in fixed momentum states and amputate them to Γ functions

$$\int dx \, \langle p | \, \overline{\psi}(x) \Gamma \psi(x) \, | p \rangle \;=\; G_\Gamma(p) \qquad\qquad G_\Gamma(p) \;\rightarrow\; \Gamma_\Gamma(p)$$

We project on tree-level structures

$$O_\Gamma(p) = Tr \left(\hat{P}_{O_\Gamma} \Gamma_\Gamma(p) \right) .$$

We define the field renormalization

$$Z_q(\mu, g) = -i \frac{1}{12} \frac{Tr(\not{p} S^{-1}(p))}{p^2}$$

and finally define renormalization constants

$$Z_{O_\Gamma}(\mu, g) Z_q^{-1}(\mu, g) O_\Gamma(p)|_{p^2=\mu^2} = 1$$

Much is known in this scheme! 3 loops (J. Gracey)

2.2 We can take anomalous dimensions for free (an example)

We can write a renormalization constant as an expansion in the renormalized coupling, with a lattice cutoff (a) in place

$$Z = 1 + \sum_{n>0} d_n(L)\alpha(\mu)^n \qquad d_n(L) = \sum_{i=0}^{n} d_n^{(i)} L^i \qquad L = log(\mu a)$$

We different the log of Z with respect to $\log(\mu a)$ and obtain a result which we equate to the expansion of the anomalous dimension

$$\gamma = \sum_{n>0} \gamma_n \alpha(L)^n$$

Since the anomalous dimension is finite, all the logs should cancel, which determines the coefficients in front of the logs in the original formula. The last step is to express the result as an expansion in the bare coupling α_0.

For example, in Landau gauge the field renormalization reads ($\gamma_q^{(1)} = 0$ and K_1 comes from the matching $\alpha(\mu) = \alpha_0 + (K_1 - 2b_0 L)\alpha_0^2 + \ldots$)

$$\begin{aligned} Z_q(\hat{\mu}) &= 1 + Z_q^{(1)}\alpha_0 + \left[Z_q^{(2)} - 2\gamma_q^{(2)} L\right]\alpha_0^2 + \\ &+ \left[Z_q^{(3)} - \left(4\gamma_q^{(2)} K_1 + 2\gamma_q^{(3)} + 2\gamma_q^{(2)} Z_q^{(1)}\right) L + 4\beta_0\gamma_q^{(2)} L^2\right]\alpha_0^3 \end{aligned}$$

2.3 But we need 2 loop matchings of the couplings

In the general case, also K_2 shows up, which is the non-log part of the 2 loop matching of the couplings

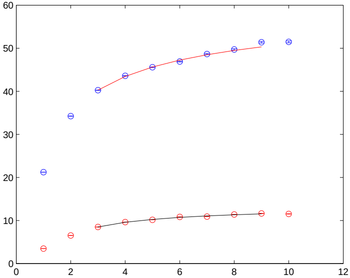
$$\alpha(\mu) = \alpha_0 + (K_1 - 2b_0 L)\alpha_0^2 + (4b_0^2 L^2 - (2b_1 + 4b_0 K_1)L + K_2)\alpha_0^3 + \ldots$$

Since this were not known, we computed it going through an intemediate matching with α_V

$$V(R) = 2\delta m - C_F \frac{\alpha_V(R)}{R}$$

The potential was in turn computed out of Wilson loops.

Figure shows the fitting procedure both for Symanzik (blue) - and for Iwasaki (red)



3. No chiral extrapolation

No chiral extrapolation is needed, since we stay at zero mass.

In order to stay at zero mass, since in the Wilson quark self-energy there is a counter term (critical mass)

(In our notation $\hat{p} = pa$)

$$
\begin{aligned}
a\Gamma_2(\hat{p}, \hat{m}_{cr}, \beta^{-1}) &= aS(\hat{p}, \hat{m}_{cr}, \beta^{-1})^{-1} \\
&= i\hat{\not{p}} + \hat{m}_W(\hat{p}) - \hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1})
\end{aligned}
$$

$$
\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_c(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_V(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_o(\hat{p}, \hat{m}_{cr}, \beta^{-1})
$$

$$
\hat{\Sigma}(0, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_c(0, \hat{m}_{cr}, \beta^{-1}) = \hat{m}_{cr}
$$

we have to plug it in order by order.

Critical mass

Critical mass up to 3 loops results (expansions are in β^{-1})

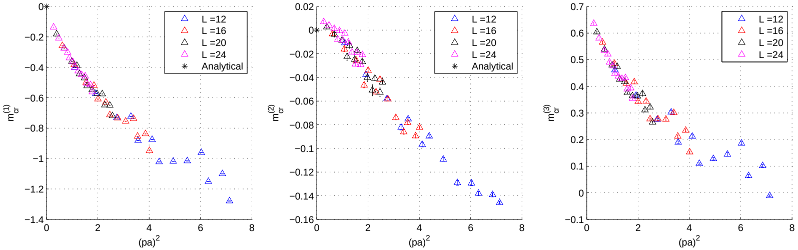
$N_f = 2$ Symanzik

$m_{cr}^{(1)}$	1.3209 (analytic)
$m_{cr}^{(2)}$	0.1911 (analytic)
$m_{cr}^{(3)}$	3.94(4)

$N_f = 4$ Iwasaki

$m_{cr}^{(1)}$	2.0489 (analytic)
$m_{cr}^{(2)}$	1.9836 (analytic)
$m_{cr}^{(3)}$	0.77(2)

Figure refers to critical mass for $N_f = 4$ Iwasaki action



4. Hypercubic Taylor expansion and Continuum limit

As an example, let's go back to the quark self-energy and look for the field renormalization.

(In our notation $\hat{p} = pa$)

$$\hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1}) = \hat{\Sigma}_c(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_V(\hat{p}, \hat{m}_{cr}, \beta^{-1}) + \hat{\Sigma}_o(\hat{p}, \hat{m}_{cr}, \beta^{-1})$$

Let's H4-Taylor expand it (after subtracting the logs!)

$$\hat{\Sigma}_V = i \sum_\mu \gamma_\mu \hat{p}_\mu \left(\hat{\Sigma}_V^{(0)} + \hat{p}_\mu^2 \hat{\Sigma}_V^{(1)} + \hat{p}_\mu^4 \hat{\Sigma}_V^{(2)} + \dots \right)$$

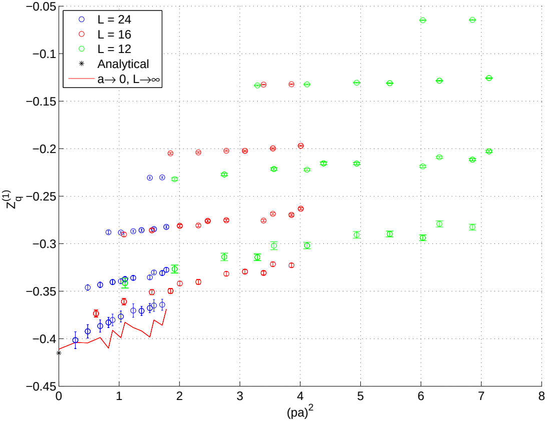
$\Sigma^{(n)}$ are also H4-Taylor expanded

$$\hat{\Sigma}_V^{(n)} = \alpha_1^{(n)} 1 + \alpha_2^{(n)} \sum_\nu \hat{p}_\nu^2 + \alpha_3^{(n)} \sum_\nu \hat{p}_\nu^4 + \alpha_4^{(n)} \sum_{\nu \neq \rho} \hat{p}_\nu^2 \hat{p}_\rho^2 + \mathcal{O}(a^6)$$

The only term surviving the $a \to 0$ limit is $\alpha_1^{(0)}$.

4.1 Continuum limit at work

We display how we reconstruct first loop analytical result for field renormalization constant (actually, with finite size effects correction on top of the continuum limit: see next slide)



5. Finite size effects

We need to tame finite size effects to get $L \to \infty$ results.

On dimensional grounds we expect (take once again $\Sigma^{(n)}$) *pL effects*

$$\begin{aligned}\hat{\Sigma}_V^{(n)}(\hat{p}, pL) &= \hat{\Sigma}_V^{(n)}(\hat{p}, \infty) + \left(\hat{\Sigma}_V^{(n)}(\hat{p}, pL) - \hat{\Sigma}_V^{(n)}(\hat{p}, \infty)\right) \\ &= \hat{\Sigma}_V^{(n)}(\hat{p}, \infty) + \Delta\hat{\Sigma}_V^{(n)}(\hat{p}, pL)\end{aligned}$$

so that a better expansion to fit is

$$\begin{aligned}\hat{\Sigma}_V^{(n)}(\hat{p}, pL) &= \alpha_1^{(n)} 1 + \alpha_2^{(n)} \sum_\nu \hat{p}_\nu^2 + \alpha_3^{(n)} \sum_\nu \hat{p}_\nu^4 + \\ &\quad + \alpha_4^{(n)} \left(\sum_\nu \hat{p}_\nu^2\right)^2 + \Delta\hat{\Sigma}_V^{(n)}(\hat{p}, pL) + \ldots\end{aligned}$$

In first approximation

$$\Delta\hat{\Sigma}_V^{(n)}(\hat{p}, pL) \sim \Delta\hat{\Sigma}_V^{(n)}(pL)$$

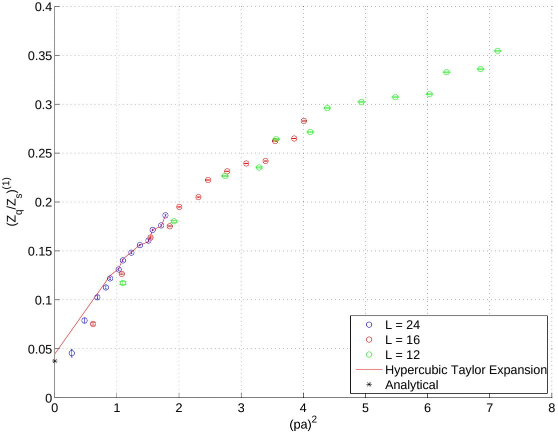
But

$$p_\mu L = \frac{2\pi n_\mu}{L} L = 2\pi n_\mu!$$

i.e. same correction on different lattice sizes for the same $\{n_1, n_2, n_3, n_4\}$.

6. Evidences for finite size effects

By taking PL effects into account (simultaneous fit to many lattice sizes) we reconstruct correct one loop result for the scalar current



7.1 Currents renormalization coefficients: Symanzik

- **One, two and three loops of renormalization constants for $N_f = 2$ Symanzik gauge action** (expansions are in β^{-1})

Analytic $Z_S^{(1)}$	-0.6893
$Z_S^{(1)}$	-0.683(5)
$Z_S^{(2)}$	-0.789(13)
$Z_S^{(3)}$	-1.9(1)
Analytic $Z_P^{(1)}$	-1.1010
$Z_P^{(1)}$	-1.098(2)
$Z_P^{(2)}$	-1.18(2)
$Z_P^{(3)}$	-3.1(2)

Analytic $Z_V^{(1)}$	-0.8411
$Z_V^{(1)}$	-0.834(6)
$Z_V^{(2)}$	-0.896(12)
$Z_V^{(3)}$	-1.88(3)
Analytic $Z_A^{(1)}$	-0.6352
$Z_A^{(1)}$	-0.644(26)
$Z_A^{(2)}$	-0.618(9)
$Z_A^{(3)}$	-1.21(3)

7.2 Currents renormalization coefficients: Iwasaki

- **One, two, and three loops of renormalization constants for** $N_f = 4$ **Iwasaki gauge action** (expansions are in β^{-1})

Analytic $Z_S^{(1)}$	-0.4488
$Z_S^{(1)}$	-0.435(9)
$Z_S^{(2)}$	-0.16(2)
$Z_S^{(3)}$	-0.82(5)
Analytic $Z_P^{(1)}$	-0.7433
$Z_P^{(1)}$	-0.735(3)
$Z_P^{(2)}$	-0.202(8)
$Z_P^{(3)}$	-1.01(9)

Analytic $Z_V^{(1)}$	-0.5623
$Z_V^{(1)}$	-0.553(6)
$Z_V^{(2)}$	-0.073(9)
$Z_V^{(3)}$	-0.37(4)
Analytic $Z_A^{(1)}$	-0.4150
$Z_A^{(1)}$	-0.410(4)
$Z_A^{(2)}$	-0.055(9)
$Z_A^{(3)}$	-0.079(16)

At each order, leading irrelevant corrections to Iwasaki renormalization constants appear to be small, which makes the fit harder. Perturbative corrections themselves are quite small.

Summary and Conclusions

Summary

- I briefly introduced Numerical Stochastic Perturbation Theory for RI'-MOM scheme
- I discussed the techniques by which we can control systematics errors, and in particular
  - continuum limit
  - infinite volume limit
- I gave results for quark bilinear three loops renormalization constants for $N_f = 2$ Symanzik, and $N_f = 4$ Iwasaki gauge actions.

Conclusions

Three loop computations for renormalization constants are feasible, with good control on systematics.

Perturbative corrections to leading order appear to be small for Iwasaki action.