Perturbative subtraction of lattice artefacts in the computation of renormalization constants

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Outline

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Motivation I

- Connection between "lattice world" and "real world": renormalization constants Z
- Must know them as accurate as possible
- Perturbative approach: complicated, slow convergence, mixing problems, ...
- Nonperturbative approach: widely used scheme is RI-MOM scheme
  - Simple implementation
  - Gauge fixing required
- Simulations at finite lattice constant $a \to$ problem of lattices artefacts

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RGI scheme

We define the so-called renormalization group invariant (RGI) operator as (see e.g., *Göckeler et al., PR D82 (2010) 114511)*

$$\mathcal{O}^{\mathrm{RGI}} = \Delta Z^{\mathcal{S}}(M)\, \mathcal{O}^{\mathcal{S}}(M) = Z^{\mathrm{RGI}}(a)\, \mathcal{O}_{\mathrm{bare}}$$

with

$$\Delta Z^{\mathcal{S}}(M) = \left(2\beta_0 \frac{g^{\mathcal{S}}(M)^2}{16\,\pi^2}\right)^{-(\gamma_0/2\beta_0)} \exp\left\{\int_0^{g^{\mathcal{S}}(M)} dg' \left(\frac{\gamma^{\mathcal{S}}(g')}{\beta^{\mathcal{S}}(g')} + \frac{\gamma_0}{\beta_0 g'}\right)\right\}$$

and

$$Z^{\mathrm{RGI}}(a) = \Delta Z^{\mathcal{S}}(M)\, Z^{\mathcal{S}}_{\mathrm{bare}}(M, a)$$

$g^{\mathcal{S}}$, $\gamma^{\mathcal{S}}$ and $\beta^{\mathcal{S}}$ are the coupling constant, the anomalous dimensions and the β-function in scheme $\mathcal{S}$

RGI scheme

Knowing Z^{RGI} and $\Delta Z^{\mathcal{S}}(M)$ one can compute $\mathcal{O}$ in any scheme $\mathcal{S}$ and at any scale M

On the lattice a widely used scheme is RI'-MOM:

$$Z_q^{-1}\, Z_{bare}^{\mathrm{RI'-MOM}}\, \frac{1}{12}\, \mathrm{tr}\left(\Gamma(p)\, \Gamma_{\mathrm{Born}}(p)^{-1}\right) = 1$$

with

$$Z_q(p) = \frac{\mathrm{tr}\left(-i\sum_\lambda \gamma_\lambda \sin(ap_\lambda) aS^{-1}(p)\right)}{12\sum_\lambda \sin^2(ap_\lambda)}$$

Γ - amputated Green function of $\mathcal{O}$

S - quark propagator

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- RI'-MOM is not covariant for most operators → not suitable for computing anomalous dimensions
- Two-step procedure: RG'-MOM → $\widetilde{MOM}$gg → RGI
- $Z^{\mathrm{RGI}}(a) = \Delta Z^{\widetilde{MOMgg}}(M=\mu_p)\, Z^{\widetilde{MOMgg}}_{\mathrm{RI'-MOM}}(M=\mu_p)\, Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(\mu_p, a)$
- $\Delta Z^{\widetilde{MOMgg}}$ and $Z^{\widetilde{MOMgg}}_{\mathrm{RI'-MOM}}(M=\mu_p)$ computed in continuum PT
- $Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(\mu, a)$ is the nonperturbatively measured Z-factor
- In MC simulations $(a\mu)$ is not small → lattice artefacts!
- Artefacts under control → determination of Z^{RGI} with better accuray!

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Operators under consideration

Operator	notation	Repr.	Members
$\bar{u}\, d$	$\mathcal{O}^{\mathrm{S}}$	$\tau_1^{(1)}$	$\mathcal{O}^{\mathrm{S}}$
$\bar{u}\,\gamma_5\, d$	$\mathcal{O}^{\mathrm{P}}$	$\tau_4^{(1)}$	$\mathcal{O}^{\mathrm{P}}$
$\bar{u}\,\gamma_\mu\, d$	$\mathcal{O}^{\mathrm{V}}_\mu$	$\tau_1^{(4)}$	$\mathcal{O}^{\mathrm{V}}_1, \mathcal{O}^{\mathrm{V}}_2, \mathcal{O}^{\mathrm{V}}_3, \mathcal{O}^{\mathrm{V}}_4$
$\bar{u}\,\gamma_5\gamma_\mu\, d$	$\mathcal{O}^{\mathrm{A}}_\mu$	$\tau_4^{(4)}$	$\mathcal{O}^{\mathrm{A}}_1, \mathcal{O}^{\mathrm{A}}_2, \mathcal{O}^{\mathrm{A}}_3, \mathcal{O}^{\mathrm{A}}_4$
$\bar{u}\,\sigma_{\mu\nu}\, d$	$\mathcal{O}^{\mathrm{T}}_{\mu\nu}$	$\tau_1^{(6)}$	$\mathcal{O}^{\mathrm{T}}_{12}, \mathcal{O}^{\mathrm{T}}_{13}, \mathcal{O}^{\mathrm{T}}_{14}, \mathcal{O}^{\mathrm{T}}_{23}, \mathcal{O}^{\mathrm{T}}_{24}, \mathcal{O}^{\mathrm{T}}_{34}$
$\bar{u}\,\gamma_\mu\, \overleftrightarrow{D}_\nu\, d$	$\mathcal{O}_{\mu\nu} \to \mathcal{O}_{v_{2,a}}$	$\tau_3^{(6)}$	$\mathcal{O}_{\{12\}}, \mathcal{O}_{\{13\}}, \mathcal{O}_{\{14\}}$, $\mathcal{O}_{\{23\}}, \mathcal{O}_{\{24\}}, \mathcal{O}_{\{34\}}$
$\bar{u}\,\gamma_\mu\, \overleftrightarrow{D}_\nu\, d$	$\mathcal{O}_{\mu\nu} \to \mathcal{O}_{v_{2,b}}$	$\tau_1^{(3)}$	$1/2(\mathcal{O}_{11} + \mathcal{O}_{22} - \mathcal{O}_{33} - \mathcal{O}_{44})$, $1/\sqrt{2}(\mathcal{O}_{33} - \mathcal{O}_{44})$, $1/\sqrt{2}(\mathcal{O}_{11} - \mathcal{O}_{22})$

Table: Operators and their representations as discussed.. $\{..\}$ means toal symmetrization.

Operators under consideration

- If dimension of multiplet > 1 → trace depends on direction of momentum → O(4) covariance is violated
- For each member a different Z-factor
- Conversion to covariant scheme → common Z-factor for all members
- Here: average over all members

$$Z_q^{-1}\, Z\, \frac{1}{N} \sum_{j}^{N} \frac{1}{12}\, \mathrm{tr}\left(\Gamma_j(p)\, \Gamma_{j,\mathrm{Born}}(p)^{-1}\right) = 1$$

where j runs over all members of the multiplett → common Z-factor

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Settings, parameters

Following settings are used for the discussed examples

- Clover improved Wilson fermions + plaquette gauge action
- Landau gauge
- $24^3 \times 48$ lattice, $\beta = 5.4$
- $\frac{r_0}{a} = 8.285$, $r_0 \Lambda_{\overline{\mathrm{MS}}} = 0.73, r_0 = 0.5$
- $P = 0.562499 \rightarrow g^2 = 1.11111, g_B^2 = \frac{g^2}{P} = 1.97531$

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One-loop subtraction (all)

General one-loop expression for the Z-factor

$$Z(\mu, a)_{\rm pert} = 1 + \frac{g^2\, C_F}{16\, \pi^2}\, F(p, a) + O(g^4)$$

In "conventional" LPT we have $\tilde{F}(p, a) \simeq \gamma\, \ln(a^2\, p^2) + \Delta$

Difference $D(p, a) = F(p, a) - \tilde{F}(p, a) \rightarrow$ **lattice artefacts in one-loop LPT**

Define the **subtracted** Z-factor in RI'-MOM scheme

$$Z(\mu, a)^{\rm RI'-MOM}_{\rm bare}(p, a)_{\rm MC,sub} = Z(\mu, a)^{\rm RI'-MOM}_{\rm bare}(p, a)_{\rm MC} - \frac{g_B^2\, C_F}{16\, \pi^2}\, D(p, a)$$

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One-loop subtraction (all)

- Procedure ensures complete $O(a^n)$ subtraction in one-loop
- $D(p, a)$ is **computed numerically** $\rightarrow$ large computational effort, for operators with more than one covariant derivative impractical
- Use of boosted coupling g_B is justified a posteriori by the result
- Z-factor in RGI-scheme:

$$Z^{\mathrm{RGI}} = \Delta Z^{\widetilde{MOMgg}} Z^{\widetilde{MOMgg}}_{\mathrm{RI'-MOM}} Z^{\mathrm{RI'-MOM}}_{\mathrm{bare,MC,sub}}$$

- Ideally, Z^{RGI} does not depend on scale p, but we may have significant deviations due to
  - Remaining $O(a)$ artefacts
  - Truncation of perturbation theory
- Remaining scale dependence is fitted - constrained by lattice symmetry and perturbative "ingredients"

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One-loop subtraction (all) - results

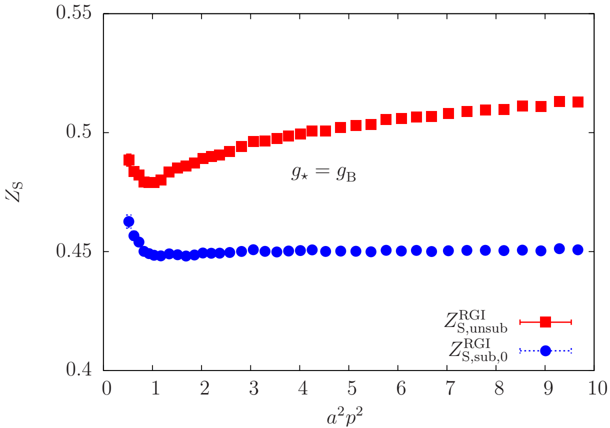

Figure: Unsubtracted and subtracted renormalization constants Z^{RGI} for the operator $\mathcal{O}^S$.

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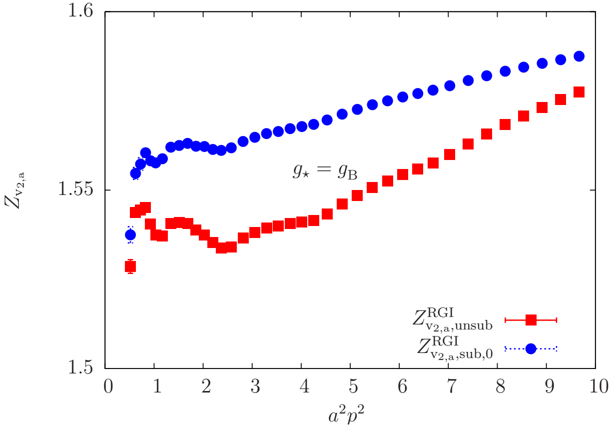

Figure: Unsubtracted and subtracted renormalization constants Z^{RGI} for the operator $\mathcal{O}_{v_{2,a}}$.

One-loop subtraction $O(a^2p^2)$

- Look for procedure which can be applied to more general cases - eventually with "less" correction effect
- Cyprus group pioneered diagrammatic $O(g^2a^2p^2)$ approach (see e.g., *M. Constantinou, V. Lubicz, H. Panagopoulos and F. Stylianou, JHEP* **0910** *(2009) 064)*
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One-loop subtraction $O(a^2p^2)$

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- Cyprus group pioneered diagrammatic $O(g^2a^2p^2)$ approach (see e.g., *M. Constantinou, V. Lubicz, H. Panagopoulos and F. Stylianou, JHEP* **0910** *(2009) 064)*
- Results for local and one-link bilinears and different actions and general mass terms; higher derivative operators are possible (but also not very easy)

$\mathcal{O}^{\rm S}$

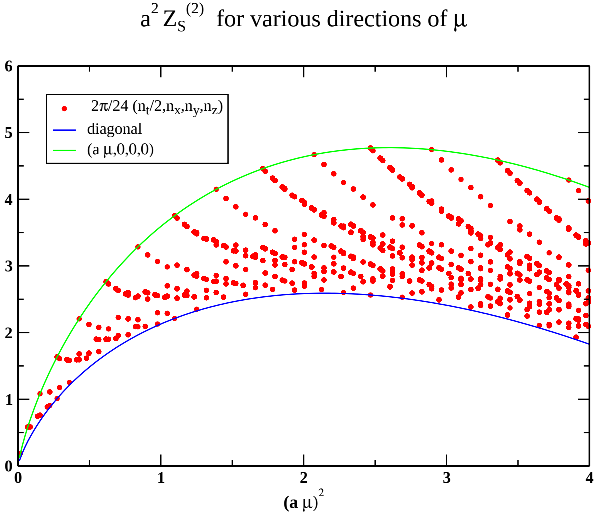
Example: scalar operator $\mathcal{O}^{\rm S}$

$$
\begin{aligned}
Z^{\rm S}(a,p) &= 1 + \frac{g^2 C_F}{16\,\pi^2}\Bigg(-23.3099453215 + 3\,\log(a^2 p^2) \\
&\quad + (a^2 p^2)\Big(1.6408851782248 - \frac{239}{240}\,\log(a^2 p^2) \\
&\quad + \frac{p4}{(p^2)^2}\Big(1.9510436778 - \frac{101}{120}\,\log(a^2 p^2)\Big)\Big)\Bigg) \\
&\equiv 1 + g^2\, Z_1^{\rm S} + g^2 a^2 p^2\, \Delta Z_2^{\rm S} \equiv 1 + \frac{g^2 C_F}{16\,\pi^2}\, Z_{\rm S}^{(1)} + \frac{g^2 C_F}{16\,\pi^2}\, a^2 Z_{\rm S}^{(2)}
\end{aligned}
$$

with $p4 = \sum_{\lambda=1}^{4} p_\lambda^4$

$\mathcal{O}^S - O(g^2a^2p^2)$ artefacts

$O(g^2a^2p^2)$ lattice artefacts for a general momentum set ($24^3 \times 48$)



Subtraction procedures

Following procedures are possible

- $$Z_{\rm bare}^{\rm RI'-MOM}(p,a)_{\rm MC,sub,1} = Z_{\rm bare}^{\rm RI'-MOM}(p,a)_{\rm MC} - g_\star^2\,\delta Z_{a^2}$$
- $$Z_{\rm bare}^{\rm RI'-MOM}(p,a)_{\rm MC,sub,2} = Z_{\rm bare}^{\rm RI'-MOM}(p,a)_{\rm MC} \times \left(1 - g_\star^2\,\delta Z_{a^2}\right)$$
- $$Z_{\rm bare}^{\rm RI'-MOM}(p,a)_{\rm MC,sub,3} = Z_{\rm bare}^{\rm RI'-MOM}(p,a)_{\rm MC} \,/\, \left(1 + g_\star^2\,\delta Z_{a^2}\right)$$

$g_\star$ can be chosen to be either the bare lattice coupling g or the boosted coupling $g_{\rm B}$.

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Subtraction procedures

Following procedures are possible

- $$Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC,sub,1}} = Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC}} - g_\star^2\,\delta Z_{a^2}$$

- $$Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC,sub,2}} = Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC}} \times \left(1 - g_\star^2\,\delta Z_{a^2}\right)$$

- $$Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC,sub,3}} = Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC}} \,/\, \left(1 + g_\star^2\,\delta Z_{a^2}\right)$$

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Effect of subtraction

For the scalar operator we get

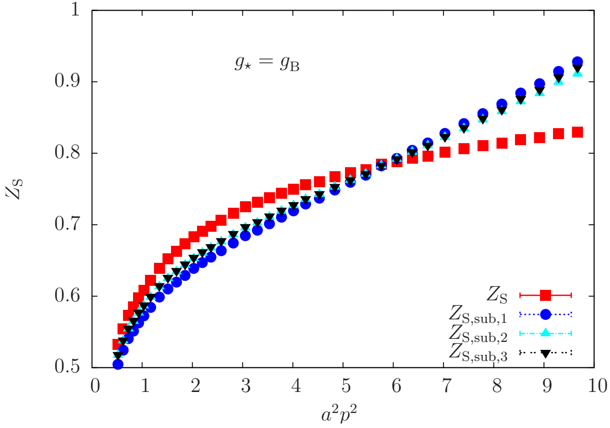

Figure: Z_S and $Z_\mathrm{S,sub,i}$ as function of a^2p^2.

Fit of lattice artefacts

Parametrizing the remaining lattice artefacts

$$\left(Z^{\mathrm{RGI}}(a) = \Delta\, Z^{\widetilde{MOMgg}}(p)\, Z^{\widetilde{MOMgg}}_{\mathrm{RI'-MOM}}(p)\, Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC,sub}} \right)$$

$$\begin{aligned} Z^{\widetilde{MOMgg}}_{\mathrm{RI'-MOM}}(p)\, Z^{\mathrm{RI'-MOM}}_{\mathrm{bare}}(p,a)_{\mathrm{MC,sub}} \;=\;& Z^{\mathrm{RGI}}(a)\,/\Delta Z^{\widetilde{MOMgg}}(p) + \beta\, a^2\, p2 \\ & + \gamma\, a^2\, p4/p2 + \epsilon\, a^2\, p6/p2^2 \\ & + \kappa\, a^4\, p2^2 + \lambda\, a^4\, p4 \\ & + \mu\, a^6\, p2^3 + \nu\, a^6\, p4\, p2 + \omega\, a^6\, p6 \end{aligned}$$

$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$ fit the lattice artefacts
$pn = \sum_i p_i^n$

Fit procedure

1. We chose of a set of momentum intervals $a^2p^2_{i,min} \leq a^2p^2 \leq a^2p^2_{i,max}$. In order to avoid the region of Landau poles we demand for all i the lower limit $(a^2p^2_{i,min}) \geq 0.5$.
2. Among all fits we extract those with $\chi^2 \leq \chi^2_{min}$. It turned out that $\chi^2_{min} = 2$ is a good choice.
3. We determine $Z^{\mathrm{RGI}}(a)$ by investigation of the resulting histogram.
4. The calculated parameters $(\beta, \gamma, \epsilon)$ are used to investigate the remaining $O(a^2p^2)$ dependence.

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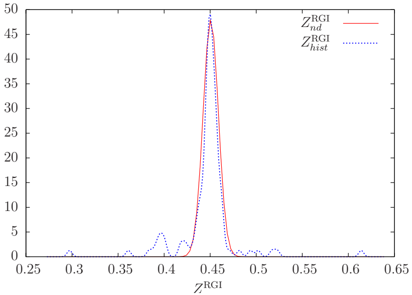
Determination of Z^{RGI}

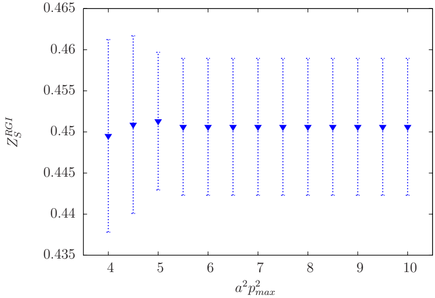
Smoothed histogram → normal distribution → $\langle Z_S^{\mathrm{RGI}} \rangle$ for the choice $a^2p^2_{max} = 5.5$ and $\chi^2 \leq 2$



$\longrightarrow \langle Z_S^{\mathrm{RGI}} \rangle = 0.45062(834)$

Determination of Z^{RGI}

Repeat this procedure for various $a^2p^2_{i,max}$ but keep the condition $\chi^2 \leq 2$



For $a^2p^2_{max} \geq 5.5$ no new data enter the fit with $\chi^2 \leq 2 \rightarrow$ stable result

Determination of Z^{RGI} - different variants

Common setting: $\chi^2 \leq 2,\ a^2p^2_{max} = 5.5, \{p\}_{all}$

Determination of Z^{RGI} - different variants

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$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.45062(834)

Determination of $Z^{\rm RGI}$ - different variants

Common setting: $\chi^2 \leq 2$, $a^2p^2_{max} = 5.5, \{p\}_{all}$

$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.45062(834)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.44746(93)

Determination of Z^{RGI} - different variants

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$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.44746(93)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g_B^2 * \delta Z_{a^2})$	0.46117(818)

Determination of Z^{RGI} - different variants

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$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g_B^2 * \delta Z_{a^2})$	0.46000(92)

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$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.45670(862)

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$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.45670(862)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.44745(95)

Determination of $Z^{\rm RGI}$ - different variants

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$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46252(857)

Determination of Z^{RGI} - different variants

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$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.45062(834)
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$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46252(857)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46603(37)

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$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g^2 * \delta Z_{a^2})$	0.46235(855)

Determination of Z^{RGI} - different variants

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$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.45062(834)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g_B^2 * \delta Z_{a^2}$	0.44746(93)
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$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g_B^2 * \delta Z_{a^2})$	0.46066(813)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g_B^2 * \delta Z_{a^2})$	0.46000(92)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.45670(862)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.44745(95)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46252(857)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46603(37)
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$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g^2 * \delta Z_{a^2})$	0.46620(32)

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$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g_B^2 * \delta Z_{a^2})$	0.45943(86)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g_B^2 * \delta Z_{a^2})$	0.46066(813)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g_B^2 * \delta Z_{a^2})$	0.46000(92)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.45670(862)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} - g^2 * \delta Z_{a^2}$	0.44745(95)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46252(857)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC} * (1 - g^2 * \delta Z_{a^2})$	0.46603(37)
$(\beta,\gamma,\epsilon,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g^2 * \delta Z_{a^2})$	0.46235(855)
$(0,0,0,\kappa,\lambda,\nu,\mu,\omega)$, $Z_{MC}/(1 + g^2 * \delta Z_{a^2})$	0.46620(32)
subtract-all (*Göckeler et al.*)	0.45155(80)(568)(15)

Determination of Z^{RGI} - different variants

- Current data set contains momenta near the diagonal $\rightarrow$ procedure should be tested for more non-diagonal momenta
- Selection of more non-diagonal momenta out of the current data - no significant change (number of data points decreases, however)
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Remaining $O(a^2p^2)$ dependence

After subtraction of $O(g^2a^2p^2)$ terms remain

1. $O(g^2(a^2p^2)^n), n > 1$ contributions
2. $O(g^{2n}\, a^2p^2), n > 1$ contributions
3. $O(g^{2n}\,(a^2p^2)^m), n, m > 1$ contributions

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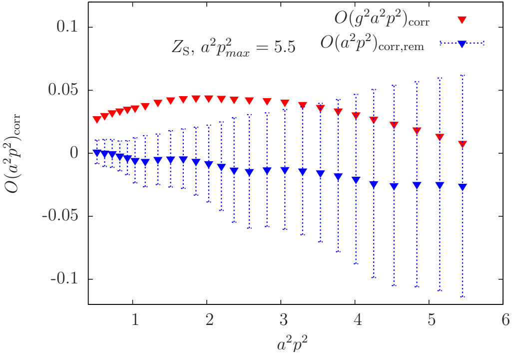
Remaining $O(a^2p^2)$ dependence

Compare $O(g^2a^2p^2)_{corr}$ and the remaining $O(a^2p^2)_{corr,rem}$ after subtraction (described by the parameters $(\beta, \gamma, \epsilon)$)



$\rightarrow$ for scalar operator the $O(g^2a^2p^2)_{corr}$ does a good job already

Fit with $\beta = \gamma = \epsilon = 0 \rightarrow \langle Z_{\rm S}^{\rm RGI} \rangle = 0.44746(93)$

Some more results

Same procedure for one-link operators ($g_\star = g_B$, $Z_{MC} - g_B^2 \delta Z_{a^2}$)

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$\langle Z_S^{\mathrm{RGI}} \rangle$	
$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$	0.45062(834)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$	0.44746(93)

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$\langle Z_{v_{2,a}}^{\mathrm{RGI}} \rangle$	
$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$	1.5572(65)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$	1.5590(16)

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Göckeler et al.	1.5526(54)(−159)(6)
$\langle Z_{v_{2,b}}^{\mathrm{RGI}} \rangle$	
$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$	1.5537(99)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$	1.5625(12)

Some more results

Same procedure for one-link operators ($g_\star = g_B$, $Z_{MC} - g_B^2 \delta Z_{a^2}$)

$\langle Z_S^{\mathrm{RGI}} \rangle$	
$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$	0.45062(834)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$	0.44746(93)
Göckeler et al.	0.45155(80)(568)(15)
$\langle Z_{v_{2,a}}^{\mathrm{RGI}} \rangle$	
$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$	1.5572(65)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$	1.5590(16)
Göckeler et al.	1.5526(54)(−159)(6)
$\langle Z_{v_{2,b}}^{\mathrm{RGI}} \rangle$	
$(\beta, \gamma, \epsilon, \kappa, \lambda, \nu, \mu, \omega)$	1.5537(99)
$(0, 0, 0, \kappa, \lambda, \nu, \mu, \omega)$	1.5625(12)
Göckeler et al.	1.5555(28)(−155)(−6)

Summary

- Described subtraction procedures for getting rid of lattice artefacts
- Numerical procedure for subtraction of all lattice artefacts in one-loop very efficient
- Not suited for more complicated operators
- Subtraction of one-loop $O(a^2 p^2)$ terms can be done for a more general class of operators
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Acknowledgements

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