Sigma commutators
from an SU(3) chiral extrapolation

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Outline

1. Motivation
2. Method
3. Fits
4. Checks
5. Conclusion
History

- Idea of symmetries and broken symmetries important in physics since the 50s
- Understand hadron structure within QCD in terms of symmetries
- Want to understand / quantify small violations of fundamental symmetries
- Would like to measure matrix elements of symmetry breaking density
We consider

- **Sigma terms**
  - Matrix element of even number of charge commutators of the Hamiltonian
  - Modern definition: usually two commutators
  - e.g., $\sigma_N = \frac{1}{3} \langle N | [ [^5 Q_i, [^5 Q_i, H]], N \rangle$

- **Double commutator: picks out symmetry breaking part of $H$**
  - For QCD, that is $m \bar{\psi} \psi$
  - Operator identity: $[^5 Q_i, [^5 Q_j, H]] = \delta_{ij} \int d\vec{r} m \bar{\psi}(\vec{r}) \psi(\vec{r})$
  - So $\sigma_N = \bar{m} \langle N | \bar{u} u + \bar{d} d | N \rangle$
Definitions

For a baryon $B$, the sigma terms are scalar form-factors evaluated in the limit of vanishing momentum transfer. For each quark flavor $q$,

$$
\sigma_{Bq} := m_q \langle B | \bar{q} q | B \rangle \quad (1)
$$

$$
\bar{\sigma}_{Bq} := \sigma_{Bq} / M_B \quad (2)
$$

For the nucleon, define (traditional)

$$
\sigma_{\pi N} := m_l \langle N | \bar{u} u + \bar{d} d | N \rangle \quad (3)
$$

$$
\sigma_s := m_s \langle N | \bar{s} s | N \rangle \quad (4)
$$
Motivation - why study $\sigma$ terms?

To understand

- Nature of explicit chiral symmetry breaking in QCD
- Decomposition of the mass of the nucleon
- Nucleon structure

Also important for

- Interpretation of experimental searches for dark matter
  - e.g., neutralino
  - Interactions with hadronic matter determined by couplings to $\sigma_I$, $\sigma_s$
  - See recent papers\textsuperscript{1,2}

\textbf{Particularly topical: Dark Matter}

\textsuperscript{1}S. J. Underwood et al., [arXiv:1203.1092 [hep-ph]]

Traditional evaluation

$\sigma_{\pi N}$

- Experimental: $\sigma_{\pi N}$ determined from $\pi N$ scattering data
  - not well known
- $\sigma_{\pi N} \sim 45-70$ MeV
- Controversial, limited precision (but better than $\sigma_s$)

$\sigma_s$

- Indirect: $\sigma_s$ evaluated using $\sigma_{\pi N}$ and
  $\sigma_0 = m_1 \langle N | \bar{u}u + \bar{d}d - 2\bar{s}s | N \rangle$
- $\sigma_s \sim 300$ MeV
  - up to $\frac{1}{3} M_N$ from non-valence quarks
  - incompatible with constituent quark models
- Determination has limited precision
- $\sigma_s$ has uncertainty $\sim 90$ MeV, even with perfect $\sigma_{\pi N}$

$\sigma_s$ difficult to pin down experimentally
Lattice QCD results

Need to reduce uncertainty on $\sigma_s$ - lattice QCD

Many calculations in recent years, give $\sigma_s$ from the lattice

- Global: 20-50 MeV
- This work: $21 \pm 6$ MeV (See later)

Common methods:
1. Direct calculation
2. Feynman-Hellman Theorem (e.g., this work)
Feynman-Hellman Relation

\[ \sigma_{Bq} = m_q \frac{\partial M_B}{\partial m_q} \]

Gell-Mann-Oakes-Renner:

\[ m_l \rightarrow m_\pi^2 / 2 \]
\[ m_s \rightarrow m_K^2 - m_\pi^2 / 2 \] (5)

(6)

Lattice QCD:

- Baryon masses evaluated at different \((m_\pi, m_K)\)
- Most simulations not yet at physical point
Our method

- **Have:** PACS-CS\(^1\) lattice data for octet baryon masses at different \((m_\pi, m_K)\) values

- **Fit:** Chiral perturbation theory baryon mass function to data
  - Extrapolate to physical point
  - Fit constrains slope at physical point
  - Correlated error analysis

- **Differentiate:** Use Feynman-Hellman relation to evaluate sigma terms

\(^1\)S. Aoki et al. [PACS-CS Collaboration], *Phys. Rev.* **D79**, 034503 (2009)
Chiral perturbation theory (χPT) with finite-range regularisation (FRR)

**Idea:** Write baryon masses as a function of quark mass

**But:** Traditional low-order polynomial fits not consistent with symmetries of QCD

- Chiral symmetry broken → pion as Goldstone boson with small mass
- Goldstone boson loops contribute to hadron properties
- Give terms \( \propto \) odd powers or logs of pion mass
- GMOR: \( m_{\pi}^2 \propto m_q \)
- Therefore need non-analytic functions of quark mass

**Solution:** Chiral perturbation theory (χPT)
Chiral perturbation theory ($\chi$PT)

An effective field theory for QCD

- Goldstone bosons (pions) become the fundamental degrees of freedom
- Built on the symmetries of QCD
- Naturally generates observed non-analyticity
- i.e., Preserves correct chiral behavior of QCD
Formulae

- Obtain effective Lagrangian at leading order in $\chi$PT
- Power series expansion in powers of derivatives and (chiral-symmetry breaking) quark mass matrix
- Chiral loops account for non-analytic behavior

Expansion of baryon mass:

$$M_B = \{\text{terms analytic in } m_q\} + \{\text{chiral loop corrections}\}$$

Figure: One-loop graphs at order $m_q^{(3/2)}$


Finite-range regularisation (FRR)

- Need to regularize infinities arising from loop diagram integrals
- QCD: Goldstone bosons emitted/absorbed by composite objects made from quarks, gluons
  - Form factors suppress these processes for momenta greater than $R^{-1}$
- FRR: Introduce finite ultra-violet cutoff into loop integrals
- Physical results independent of shape of regulator

Summary:
- Physically motivated way to regularize infinities
- Shown to be effective over a large range of quark masses
Our method - Reminder

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\(^1\)S. Aoki et al. [PACS-CS Collaboration], *Phys. Rev.* **D79**, 034503 (2009)
Fit to PACS-CS lattice data
Comparison with physical values

<table>
<thead>
<tr>
<th>$B$</th>
<th>Mass (GeV)</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}$</td>
<td>0.959(24)(9)</td>
<td>0.939</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>1.129(15)(6)</td>
<td>1.116</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>1.188(11)(6)</td>
<td>1.193</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>1.325(6)(2)</td>
<td>1.318</td>
</tr>
</tbody>
</table>
Lattice data locations in $m_l - m_s$ plane

PACS-CS lattice data

Physical point
Lattice data locations in $m_l - m_s$ plane

PACS-CS lattice data

Physical point

derivative along purple arrow $\leftrightarrow \sigma_s$
Lattice data locations in $m_l - m_s$ plane

- PACS-CS lattice data
- Physical point
- UKQCD-QCDSF $24^3$ lattice
- UKQCD-QCDSF $32^3$ lattice
Lattice data locations in $m_l - m_s$ plane

$\begin{align*}
2m_K^2 - m_\pi^2 \text{ (GeV}^2) & \\
\end{align*}$

$\begin{align*}
m_\pi^2 \text{ (GeV}^2) & \\
\end{align*}$

- PACS-CS lattice data
- Physical point
- UKQCD-QCDSF $24^3$ lattice
- UKQCD-QCDSF $32^3$ lattice
This is NOT a fit to the data shown…

\[ X_\pi^2 = \frac{1}{3} (2m_K^2 + m_\pi^2) \quad X_N = \frac{1}{3} (M_N + M_\Sigma + M_\Xi) \]

W. Bietenholz et al. [QCDFS-UKQCD Collaboration], [arXiv:1102.5300[hep-lat]]
Results

<table>
<thead>
<tr>
<th>$B$</th>
<th>$\bar{\sigma}_{Bl}$</th>
<th>$\bar{\sigma}_{Bs}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$</td>
<td>0.047(6)(5)</td>
<td>0.022(6)(0)</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>0.026(3)(2)</td>
<td>0.141(8)(1)</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>0.020(2)(2)</td>
<td>0.172(8)(1)</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>0.0089(7)(4)</td>
<td>0.239(8)(1)</td>
</tr>
</tbody>
</table>

For the nucleon, dimensionful results

$$\sigma_{\pi N} = 45 \pm 6 \text{ MeV}$$

$$\sigma_s = 21 \pm 6 \text{ MeV}$$
Comparison with direct results from lattice QCD

- Our method = easy to evaluate $\sigma_{Bq}$ at any $(m_\pi, m_K)$
- QCDSF Collaboration\(^1\): precise direct calculations of $\sigma_{\pi N}$ and $\sigma_s$ from lattice QCD
  - at $(m_\pi, m_K) = (281, 547)$ MeV
  - $\sigma_{\pi N}^{\text{QCDSF}} = 106(11)(3)$ MeV
  - $\sigma_s^{\text{QCDSF}} = 12^{+23}_{-16}$ MeV

- We find
  - at $(m_\pi, m_K) = (281, 547)$ MeV
  - $\sigma_{\pi N} = 131(11)(5)$ MeV
  - $\sigma_s = 16(5)(1)$ MeV

- Another consistent check!

\(^1\)G. S. Bali et al. [QCDSF Collaboration], Phys. Rev. **D85**, 054502 (2012)
Success: Precise determination of $\sigma_s$ and $\sigma_{\pi N}$ from lattice QCD

$$\sigma_s = 21 \pm 6 \text{ MeV}$$

$$\sigma_{\pi N} = 45 \pm 6 \text{ MeV}$$
Backup - Check not leaving PCR

\[ \overline{\sigma}_l \]

\[ \overline{\sigma}_s \]

Number of Mass Points Fit

- \( \Xi \)
- \( \Sigma \)
- \( \Lambda \)

- \( \Xi \)
- \( \Sigma \)
- \( \Lambda \)

- \( \Xi \)
- \( \Sigma \)
- \( \Lambda \)

\( 5 \) (All) 4 3

\( 5 \) (All) 4 3
Backup - Sigma terms as input to dark matter cross-sections

Note: Write sigma commutator as $M_N f_{Tq} := \langle N | m_q \bar{q} q | N \rangle$

- e.g., Neutralino
  - Particle candidate for dark matter
  - Weakly interacting fermion
  - Mass $\sim 100$ GeV or more
  - Density: a few per Liter

- Neutralino-Hadron cross-section (spin-indep)
  - $\sigma_{SI} = 4 \left( \frac{N_N^2}{\pi} \right) f^2$
  - $f = \sum_{q=u,d,s} \frac{\alpha_3 q f_{Tq}}{m_q} + \sum_{Q=c,b,t} \frac{\alpha_3 Q f_{TQ}}{m_Q}$

For details see: S. J. Underwood et al., [arXiv:1203.1092 [hep-ph]]