Observables from the low-lying eigenmodes of the lattice Dirac operator

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Introduction

The quantum fluctuations of the QCD vacuum are the origin of many interesting nonperturbative physics (e.g., spontaneous chiral symmetry breaking, confinement).

Each gauge conf. of QCD possesses a well-defined topological charge Q_t. The fluctuation of Q_t can be measured by the topological susceptibility:

$$\chi_t = \int d^4x \{\rho_Q(x)\rho_Q(0)\} = \frac{\langle Q_t^2 \rangle}{\Omega} \tag{1}$$

where Q_t can be determined unambigously in lattice QCD through the Atiyah-Singer index theorem [Atiyah & Singer (1968)]:

$$Q_t \equiv \int d^4x\, \rho_Q(x) = n_+ - n_- = \mathrm{index}(\mathcal{D}) \tag{2}$$

In ChPT, χ_t for $N_f = 2$ at tree level [Leutwyler & Smilga (1992)] and NLO [Mao & Chiu, PRD (2009)] are:

$$\chi_t = \Sigma(m_u^{-1} + m_d^{-1})^{-1} \tag{3}$$

$$\frac{\chi_t}{m_q} = \frac{\Sigma}{2}\left[1 - 3\left(\frac{\Sigma m_q}{16\pi^2 F_\pi^4}\right)\ln\left(\frac{2\Sigma m_q}{F_\pi^2 \mu_{sub}^2}\right) + 32\left(\frac{\Sigma}{F_\pi^4}\right)(2L_6 + 2L_7 + L_8)m_q\right] \tag{4}$$

Introduction (cont.)

The Banks-Casher relation [Banks, Casher, NPB (1980)] shows that Σ is due to the non-vanishing density of near-zero modes of $\mathcal{D}$:

$$\Sigma = \pi \lim_{m_q \to 0} \lim_{\Omega \to \infty} \rho(0), \qquad \rho(\lambda) = \frac{1}{\Omega} \sum_k^{\infty} \langle \delta(\lambda - \lambda_k) \rangle \tag{5}$$

In principle, one can extract $\Sigma_{\rm eff}$ for each sea quark mass on a finite lattice [Damgaard, Fukaya, JHEP (2009)]:

- $\rho(\lambda) = \Sigma_{\rm eff}/\pi = -\pi^{-1} {\rm Re} \, \langle \bar{q} q \rangle |_{m_v = i\lambda}$
  $\rho(\lambda)$: eigenmode density of the Dirac operator on the finite lattice.
  $\langle \bar{q} q \rangle$: constructed from low-lying eigenmodes of the Dirac operator.
- $\rho(0) = \Omega^{-1} N_{\rm conf}^{-1} \left. (dN/d\lambda) \right|_{\lambda=0}$
  $N(\lambda) = \Omega \int_0^{\lambda} d\lambda' \rho(\lambda')$: accumulated mode number below λ.

In NLO ChPT [Gasser, Leutwyler, Annals Phys. (1984)],

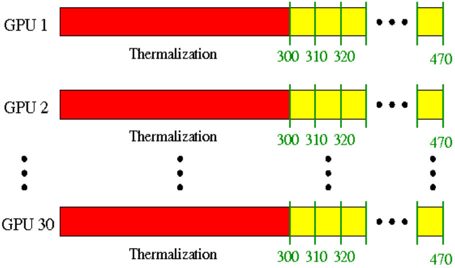
$$\Sigma_{\rm eff}(m_q) = \Sigma(0) \left[1 - \frac{3 M_\pi^2}{32 \pi^2 F^2} \ln \frac{M_\pi^2}{\mu_{\rm sub}^2} + \frac{32 L_6 M_\pi^2}{F^2} \right] \tag{6}$$

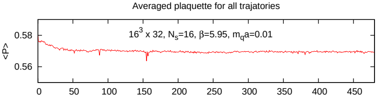
Simulation setup (see Ting-Wai Chiu's talk in this session)

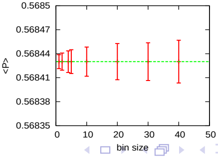
In this work, we perform the HMC simulations of two flavors QCD with ODWF on the $16^3 \times 32$, $N_s = 16$ lattice, for 8 sea-quark masses $m_q a = 0.01, 0.02, \ldots, 0.08$, and the plaquette gauge action at $\beta = 5.95$, with lattice spacing $a = 0.1032(2)$ fm ($1/a = 1.909(1)$ GeV).

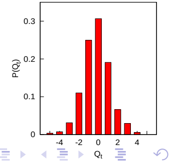
Mathematically, ODWF is a theoretical framework which preserves the chiral symmetry optimally for any N_s [Chiu, PRL (2003)] The 4D effective Dirac operator of massless ODWF is

$$D = m_0[1 + \gamma_5 S_{\rm opt}(H_w)], \quad S_{\rm opt}(H_w) = \frac{1 - \prod_{s=1}^{N_s} T_s}{1 + \prod_{s=1}^{N_s} T_s}, \quad T_s = \frac{1 - \omega_s H_w}{1 + \omega_s H_w}$$









Low-mode projection of the Dirac operator

For each sampled gauge conf., we project zero modes plus 80+80 conjugate pairs of low-lying eigenmodes of the effective 4-D operator of ODWF (i.e., the overlap Dirac operator with Zolotarev optimal approximation), with 128 poles in order to keep high precision of the chiral symmetry.

$$D_{ov} = m_0(1+V), \qquad V \equiv \gamma_5 H_w R_Z(H_w) \stackrel{N_s\to\infty}{\longrightarrow} \gamma_5 \, \mathrm{sign}(H_w) \tag{7}$$

Thus the eigen-problem of D_{ov} can be parameterized as:

$$D_{ov}|\theta\rangle = \lambda(\theta)|\theta\rangle, \qquad \lambda(\theta) = m_0(1+e^{i\theta}), \tag{8}$$

Since $[D_{ov}D_{ov}^\dagger, \gamma_5] = 0$, one can decompose the eigen-problem to

$$S_\pm|\theta\rangle_\pm \equiv P_\pm H_w R_Z(H_w) P_\pm |\theta\rangle_\pm = \pm\cos\theta|\theta\rangle_\pm \tag{9}$$

$$|\theta\rangle = |\theta\rangle_+ + |\theta\rangle_- = \frac{1}{i\sin\theta}(V - e^{-i\theta})|\theta\rangle_+ \qquad \theta \neq 0, \pm\pi, \ldots \tag{10}$$

Thus, one can perform the eigenmode projection on the operator $S_\pm$.

Thick-Restart Lanczos algorithm (TRLan)

Basic Lanczos iteration:

$$AQ_m = Q_m T_m + \beta_m q_{m+1} e_m^T$$

- T_m: $m \times m$ tridiagonal.
- $Q_m = [q_1, q_2, \ldots, q_m]$, ($m$-dim. complete set).
- $\beta_m q_{m+1} e_m^T$: residule vec.

algorithm

Input: r_0, $\beta_0 = ||r_0||$, $q_0 = 0$

For: $i = 1, 2, \ldots$

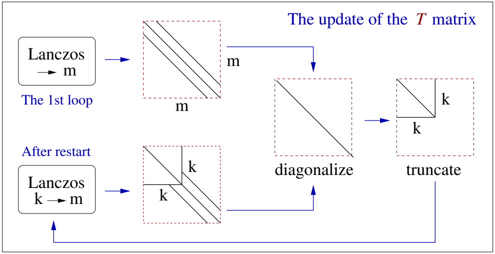
- $q_i = r_{i-1}/\beta_{i-1}$
- $p = Aq_i$
- $\alpha_i = q_i^H p$
- $r_i = p - \alpha_i q_i - \beta_{i-1} q_{i-1}$
- $\beta_i = ||r_i||$

The Ritz pair $(\hat{\lambda}_i, \hat{x}_i)$:

$$\hat{T}_m = U_m^\dagger T_m U_m, \qquad X_m = Q_m U_m$$

where $\hat{T}_m$ is diagonal with eigenvalues $\hat{\lambda}_i$, U_m is unitary, and X_m has columns $\hat{x}_i$. When $m \to \infty$, $(\hat{\lambda}_i, \hat{x}_i) \to (\lambda_i, x_i)$

Thick Restart:



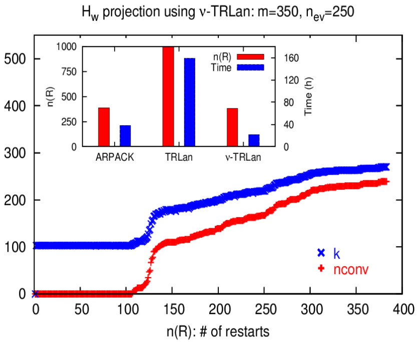
The schematic diagram for the Thick-Restart Lanczos process. The non-zero values of the T matrix are illustrated as black lines.

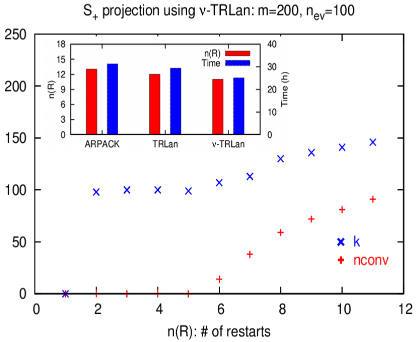
Adaptive Thick-Restart Lanczos algorithm (ν-TRLan)

To maximize the performance, which depends on the truncated dim. k and the dim. of the Krylov subpace m, one search for the optimal k to maximize the object function: [Yamazaki *et al*, ACM Trans. Math. Software (2010)]

$$f_j^{(l)}(k) = d_j^{(l)} / \, (\# \text{ of FPO}) \qquad (11)$$

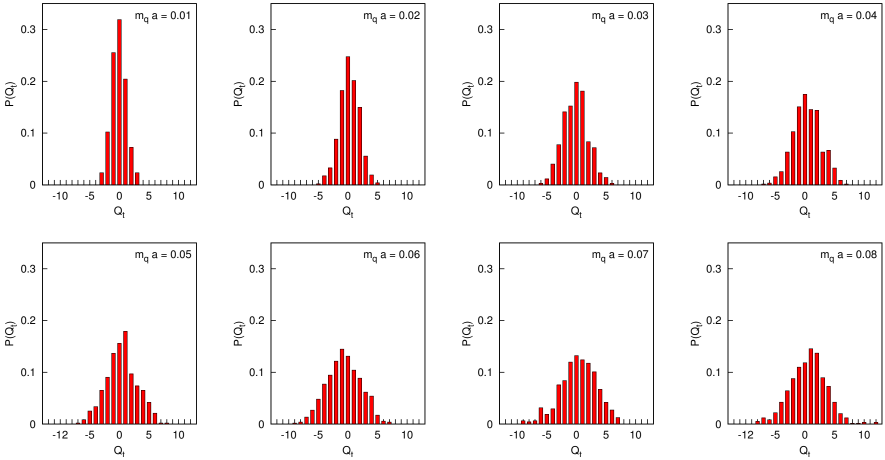
$d_j^{(l)}$: the reduction factor of j-th non-converged Ritz pair at l-th restart.



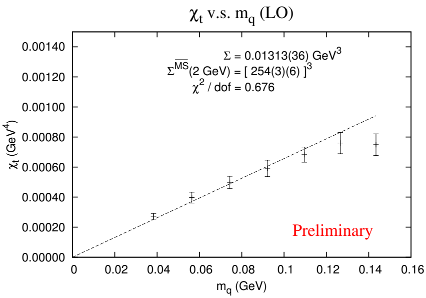


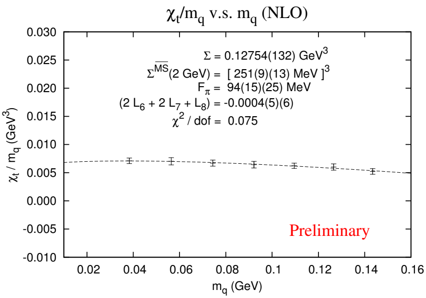
The projection of $16^3 \times 32$, $N_s = 16$, $\beta = 5.95$, $m_q a = 0.01$, $Q_t = 3$ using Intel Xeon E5530 @ 2.4GHz, 8 cores, 24GB memory

Topological charge distribution ($N_{\mathrm{conf}} = 500$)



Fitting χ_t to ChPT ($N_{\rm conf} = 500$)





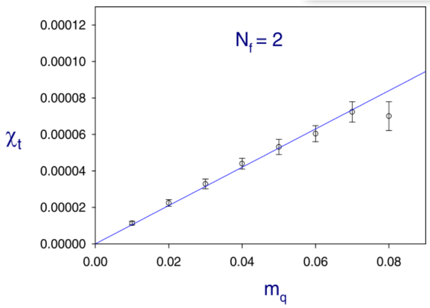
$$\text{LO}: \quad \chi_t = 2\Sigma / m_q \qquad (\text{where } m_u = m_d = m_q)$$

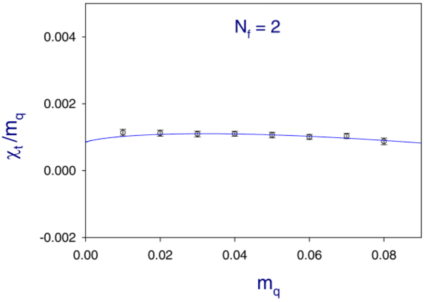
$$\text{NLO}: \quad \frac{\chi_t}{m_q} = \frac{\Sigma}{2}\left[1 - 3\left(\frac{\Sigma m_q}{16\pi^2 F_\pi^4}\right)\ln\left(\frac{2\Sigma m_q}{F_\pi^2 \mu_{sub}^2}\right) + 32\left(\frac{\Sigma}{F_\pi^4}\right)(2L_6 + 2L_7 + L_8)m_q\right]$$

- $a^{-1} = 1.909(1)$ GeV.
- $Z^{\overline{\rm MS}}(2\ \text{GeV}) = 1.244(18)(39)$, $\mu_{\rm sub} = 770$ MeV.
- $\Sigma^{\overline{\rm MS}}(2\ \text{GeV}) = [251(9)(13)\ \text{MeV}]^3$, $F_\pi = 94(15)(25)$ MeV

Preliminary !

Fitting χ_t to ChPT ($N_{\rm conf} = 300$)



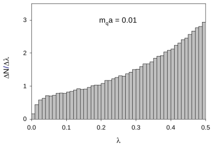


[Chiu, Hsieh, Mao (TWQCD), PLB(2011)]

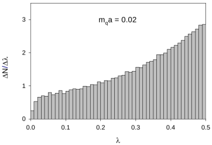
$$\frac{\chi_t}{m_q} = \frac{\Sigma}{2}\left[1 - 3\left(\frac{\Sigma m_q}{16\pi^2 F_\pi^4}\right)\ln\left(\frac{2\Sigma m_q}{F_\pi^2 \mu_{sub}^2}\right) + 32\left(\frac{\Sigma}{F_\pi^4}\right)(2L_6 + 2L_7 + L_8)m_q\right]$$

- $a^{-1} = 1.911(4)(6)$ GeV.
- $Z^{\overline{\rm MS}}(2\ {\rm GeV}) = 1.244(18)(39)$, $\mu_{\rm sub} = 770$ MeV.
- $\Sigma a^3 = 0.0020(2)$, $F_\pi a = 0.048(7)$, $2L_6 + 2L_7 + L_8 = -0.0001(3)$.
- $\Sigma^{\overline{\rm MS}}(2\ {\rm GeV}) = [259(6)(7)\ {\rm MeV}]^3$, $F_\pi = 92(12)(2)$ MeV

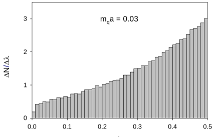
Near-zero mode density: $\Delta N/\Delta\lambda$



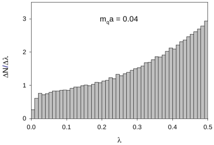
$m_q a = 0.01$



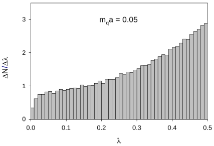
$m_q a = 0.02$



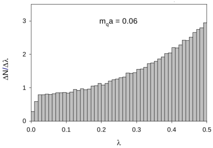
$m_q a = 0.03$



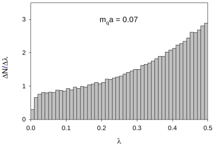
$m_q a = 0.04$



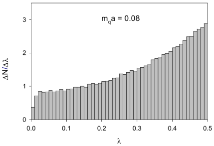
$m_q a = 0.05$



$m_q a = 0.06$

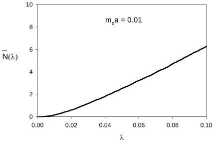


$m_q a = 0.07$

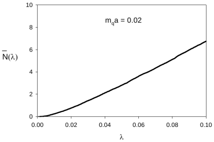


$m_q a = 0.08$

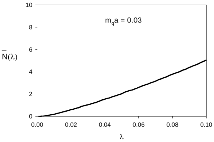
Near-zero mode density: $N(\lambda)$



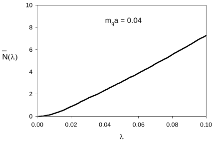
$m_q a = 0.01$



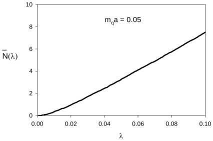
$m_q a = 0.02$



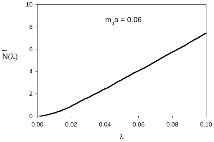
$m_q a = 0.03$



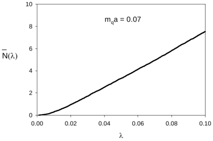
$m_q a = 0.04$



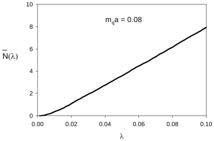
$m_q a = 0.05$



$m_q a = 0.06$



$m_q a = 0.07$



$m_q a = 0.08$

Chiral condensate

- $\rho(0) = \frac{1}{\Omega} \frac{1}{N_{\rm conf}} \left. \frac{dN}{d\lambda} \right|_0$
- $\Sigma = \lim_{m_q \to 0} \pi \rho(0)$
- $Z_s^{\overline{\rm MS}}(2 \text{ GeV}) = 1.244(18)(39)$
- $\Sigma^{\overline{\rm MS}}(2 \text{ GeV}) = Z_s \Sigma = [226(5)(10) \text{ MeV}]^3$

preliminary !

Summary

- We have performed $N_f = 2$ lattice QCD simulation with ODWF:
  - lattice size: $16^3 \times 32$, $N_s = 16$, with $m_{\rm res} a \leq 0.0004$.
  - lattice spacing: $a \simeq 0.1$ fm, $1/a = 1.909(1)$ GeV.
  - 8 sea quark masses: $0.01 \leq m_q a \leq 0.08$, with statistics $N_{\rm conf} = 500$.
- We project zero modes plus 80+80 conjugate pairs of low-lying eigenmodes of the effective 4-D operator of ODWF, using the Adaptive Lanczos Thick-Restart algorithm.
- Fitting χ_t to NLO ChPT:

$N_{\rm conf}$	$[\Sigma^{\overline{\rm MS}}(2~{\rm GeV})]^{1/3}$	F_π (MeV)	$(2L_6 + 2L_7 + L_8)$
300	259(6)(7)	92(12)(2)	-0.0001(3)
500	251(9)(13)	94(15)(25)	-0.0004(5)

- We measured the near-zero mode density $\Delta N/\Delta\lambda$, and $N(\lambda)$, in order to extract Σ via Banks-Casher relation.
- Fitting Σ to NLO ChPT is in progress.