Random Matrix Models for the Hermitian Wilson-Dirac operator of QCD-like theories

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Motivation

$N_c = 2$ **QCD fundamental**

- Lattice simulations (non-zero chemical potential $+$ even number of pairwise degenerate quarks)
  $\rightarrow$ No sign problem
- Technicolor: $6 \leq N_f \leq 10$ (within the conformal window)
- Compare to $\mathrm{SU}(3)$: $8 \leq N_f \leq 16$

(Appelquist et al (2007),Bursa et al (2010),Hasenfratz (2010), Rummukainen (2011))

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Any-color QCD with quarks in the adjoint

- Lattice SUSY YM
  Rigorous definition and Non-perturbative control
- Technicolor : $SU(2)$ with two adjoint fermions most likely in the conformal window
  Studies with unimproved Wilson fermions large $O(a)$ errors
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Introduction of the Model

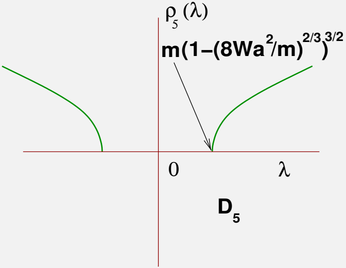
Random Matrix Theory- Dyson's threefold way

- Classification according to antiunitary symmetries of QCD and QCD like theories

  Verbaarschot (1994)
- Three distinct classes
  - Real $\beta = 1$
  - Complex $\beta = 2$
  - Quaternion $\beta = 4$

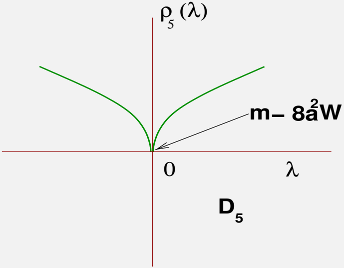
The Goal

- Effect of lattice spacing on the low-lying Dirac eigenvalues
- Behavior of the spectral gap of $D_5 + m\gamma_5 = \gamma_5(D_W + m)$ at finite mass with different lattice spacings



Mean field result for ρ^5 of $\beta = 2$ at $a = 0$

Akemann et al (2010)



Mean field result for ρ^5 of $\beta = 2$ at the onset of the Aoki phase

Akemann et al (2010)

$N_c = 2$ *fundamental* (*arbitrary* N_c *adjoint*)

- Partition function of D_5 with N_f flavors :

  $Z_{N_f}^{RMT,\nu} = \int dD_5 det^{N_f}(D_5 + m\gamma_5 + z)P(D_5)$
- $P(D_5) \rightarrow$ is a Gaussian
- $D_5 = \begin{pmatrix} aA & W \\ W^\dagger & aB \end{pmatrix}$ (Damgaard et al (2010),Akemann et al (2010), Kieburg et al (2011)
- A : $n \times n$ Real Symmetric (Quaternion self dual)
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- W : $n \times (n+\nu)$ Real (Quaternion)
- At $a = 0$: D_W has ν generic zero modes
- At finite a : definition of the index through spectral flow lines or equivalently $\nu = \sum_{\lambda_k^W \in \mathbb{R}} \mathrm{sign}(\langle k|\gamma_5|k\rangle)$ Itoh et al (1987)

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Chiral Lagrangians

Microscopic limit : $\widehat{m} = mV\Sigma, \widehat{z} = zV\Sigma, \widehat{a}^2 = a^2 V$ fixed
$V \to \infty$

Solution:

- $$\begin{aligned} Z^{\nu}_{N_f} &= \int_{\mathcal{M}} d\mu(U) det^{\kappa} U \\ &\times \exp\left[\mathrm{tr}\,\frac{\widehat{m}}{2}(U + U^{-1})\right] \\ &\times \exp\left[\mathrm{tr}\,\frac{\widehat{z}}{2}(U - U^{-1}) - \widehat{a}^2 \mathrm{tr}\,(U^2 + U^{-2})\right] \end{aligned}$$
- $\mathcal{M} = U(2N_f)/Sp(2N_f)$ or $\mathcal{M} = U(2N_f)/O(2N_f)$
- $\kappa = \nu/2$ or $\kappa = \nu$
- Double trace terms have not been considered

SUSY generating functional

- Using SUSY for partially quenched partition function of D_5
- By adding one pair of bosonic and fermionic quarks and replacing $U \to iU$

$$
\begin{aligned}
\mathcal{Z}^{\nu}_{N_{\mathrm{f}}+1|1} &= \int_{\mathcal{M}_s} d\mu(U) \mathrm{Sdet}^{\kappa} U \exp\left[\frac{i}{2}\mathrm{Str}\,\widehat{M}(U-U^{-1})\right] \\
&\times \exp\left[\frac{i}{2}\mathrm{Str}\,\widehat{Z}(U+U^{-1}) + \widehat{a}^2 \mathrm{Str}\,(U^2+U^{-2})\right]
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- The same Goldstone manifold $\mathcal{M}_s = \mathrm{U}\,(\mathrm{N_f}+2|2)/\mathrm{UOSp}\,(\mathrm{N_f}+2|2)$ as for $a=0$
- $\hat{M} = \mathrm{diag}\,(\widehat{m}+J_m, \widehat{m}+J_m, \widehat{m}, \widehat{m})$
- $\hat{Z} = \mathrm{diag}\,(\widehat{z}+J_z, \widehat{z}+J_z, \widehat{z}, \widehat{z})]$
- $J_m = J_z = 0 \to N_{\mathrm{f}}$ flavor partition function

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Quenched microscopic spectral density of D_5

- Generating function for the D_5 spectrum:
$\mathcal{Z}^{\nu}_{N_f+1|1}(\widehat{m}, \widehat{z}, \widehat{z'}; \widehat{a}) =$
$$\left\langle \det{}^{N_f}(\gamma_5(D_W + \widehat{m})) \frac{\det(\gamma_5(D_W + \widehat{m}) + \widehat{z})}{\det(\gamma_5(D_W + \widehat{m}) + \widehat{z'})} \right\rangle_{\nu}$$
- Quenched resolvent :
$$G^{\nu}(\widehat{z}, \widehat{m}; \widehat{a}) = \lim_{J_z \to 0} \partial_{J_z} \mathcal{Z}^{\nu}_{1|1} = \left\langle \mathrm{tr}\, \frac{1}{D_5 + \widehat{z}} \right\rangle$$
- Quenched spectral density :
$\rho_5^{\nu}(\widehat{\lambda}^5, \hat{m}; \hat{a}) = \frac{1}{\pi} \mathrm{Im}\left[G^{\nu}(\widehat{z} = \hat{\lambda}^5, \widehat{m}; \widehat{a})\right]$

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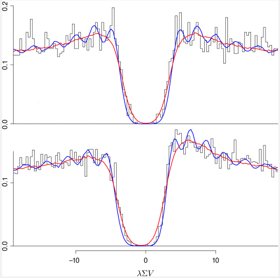
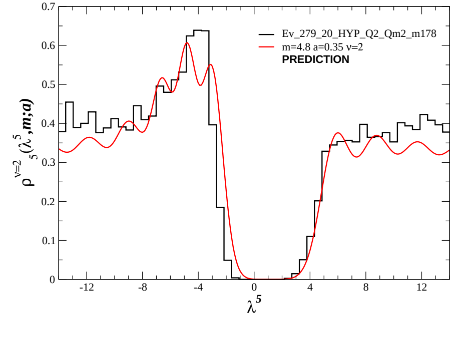
The Eigenvalue Densities

Lattice results of ρ_5 of $\beta = 2$





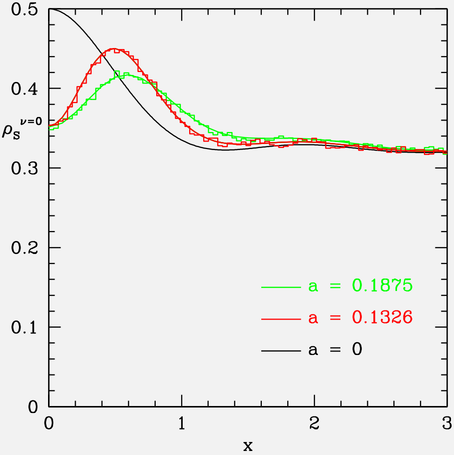
$\widehat{a_6} = \widehat{a_7} = 0.25$, $\widehat{a_8} = 0.7$
$\widehat{m} = 5.3$
$\nu = 0$ (top) and
$\nu = 1$ (bottom)

(Deuzeman,Wenger and Wuilloud(2011))

$\hat{m} = 4.8$, $\nu = 2$

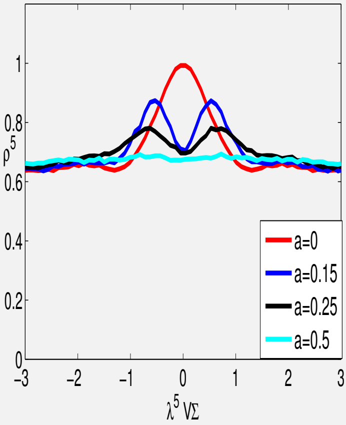
(Damgaard,Heller and Splittorff (2011))

ρ_5 of $\beta = 1$ with $\nu = 0$ and $\hat{m} = 0$

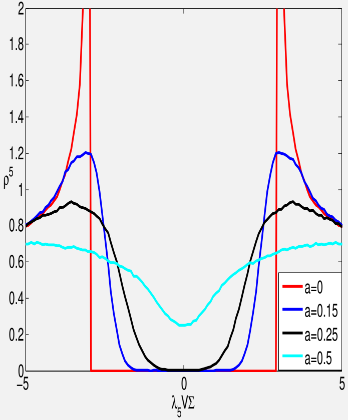


analytical result (continuous curves) vs Monte Carlo simulation (histograms)

ρ_5 of $\beta = 1$ with $\nu = 0$

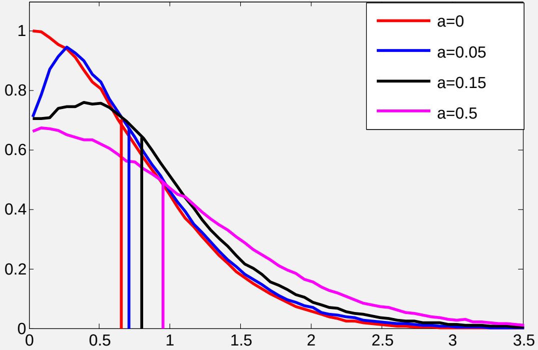


$\widehat{m} = 0$



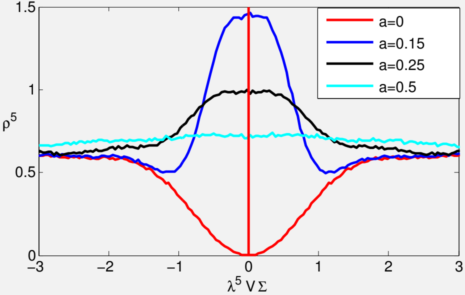
$\widehat{m} = 3$

Distribution of the first eigenvalue of the $\beta = 1$ D_5



$P_{min}(\lambda^5)$ with $\widehat{m} = 0$, $\nu = 0$

ρ_5 of $\beta = 1$ with $\nu = 2$



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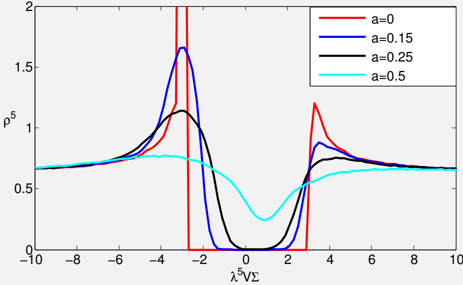
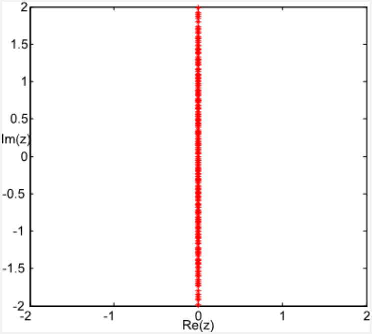
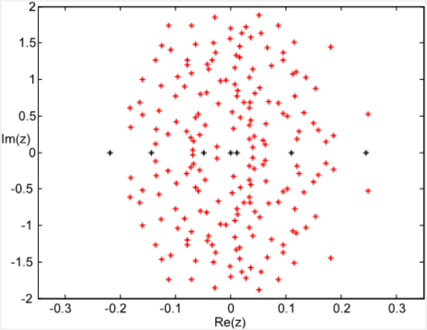


$\widehat{m} = 3$

Eigenvalues of D_W with $\nu = 5$ for $\beta = 1$

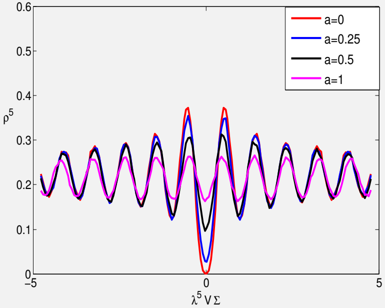


$\hat{a} = 0$

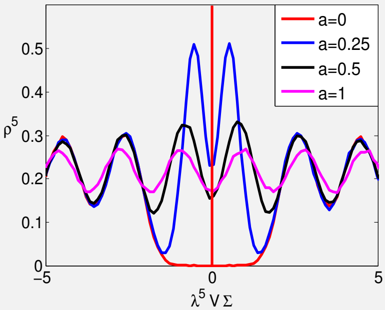


$\hat{a} = 1$

ρ_5 of $\beta = 4$ with $\hat{m} = 0$



$\nu = 0$



$\nu = 2$

Conclusions

- Random Matrix Theory for QCD-like theories with Wilson fermions
  - N_f flavor partition function
  - Spectral density of D_5
- The max at zero virtuality of $\beta = 1$ becomes a minimum as $a \neq 0$!
  $\rho^5(0)$ jumps discontinuously
- Oscillations tend to vanish for increasing a as for $\beta = 2$
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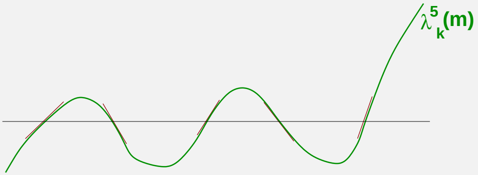
for upcoming results . . .

Thank you for your attention!

Collaborators:
Mario Kieburg
Jacobus J. M. Verbaarschot

Appendix

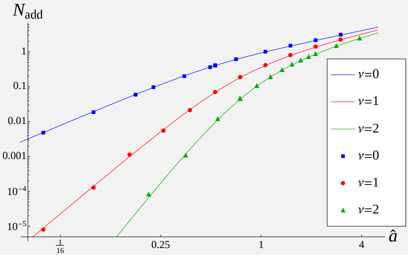
Eigenvalue flow



Eigenvalue flow (Splittorff and Verbaarschot (2011))

Appendix

Additional Real Modes

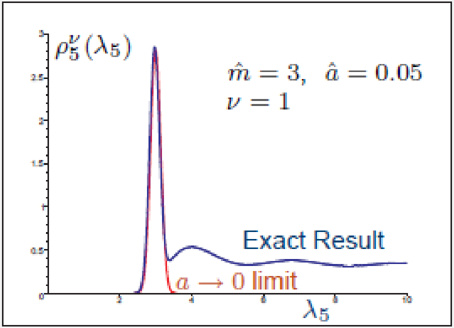


Log-log plot of the average number of additional real modes over $\hat{a}$ for $\beta = 2$

(Kieburg, Verbaarschot and S.Z. (2011))

Appendix

Fitting for W_8



ρ^5 of the $\beta = 2$ D_5 for $\nu = 1$ (blue),distribution of the smallest eigenvalue for $\nu = 1$ (red) in the saddle point $\sigma/\Delta\Lambda \propto a\sqrt{W_8 V}$

(Akemann,Damgaard,Splittorff and Verbaarschot . (2011))