Baryon resonances in a finite volume: improved phase shift extraction

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Overview

- Introduction:
  - Goals
  - Lattice QCD
  - Scattering phase shifts
- Testing Lüscher's method
  - Introducing a well-defined Hamiltonian model
  - Applying Lüscher's formula to the model
  - Features and challenges of Lüscher's method
- Improved method of phase shift extraction
  - Analysis of model-dependence
- Conclusion

Goals

- We would like to have a way of relating lattice QCD results to phase shifts, as measured in experiments.
- One particular way is Lüscher's method: what assumptions are made in deriving the formula in this method?
- We would like to test Lüscher's method in the context of a model that can be solved exactly.
- We want to develop an improved method for identifying resonance parameters.

Lattice QCD

- Lattice QCD is performed at finite volume.
- Multi-particle interactions cannot be resolved easily in very large volumes.
  - Resonances and multi-particle states appear as a tower of discrete energy levels
- It is essential to be able to compare with experiment, and relate the energy levels to asymptotic states of the S-matrix in the continuum.

Scattering phase shifts

- In scattering theory, the cross-section is related to the phase shift $\delta(k)$ of the wavefunction:

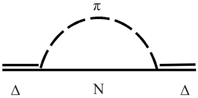
$$\sigma(k) = \int \mathrm{d}\Omega |f(\theta)|^2 = 4\pi \frac{3\sin^2\delta(k)}{k^2} \tag{1}$$

- The external momentum k of a particle (mass m) is related to the external energy by: $k = \sqrt{E^2 - m^2}$.
- The phase shift passes through 90° at the resonance energy: E_{res}.

Scattering phase shifts

- Consider the phase shift from $N\pi$-scattering off a Δ-baryon.
- The phase shift can be expressed in terms of the on-shell *t*-matrix by solving the Lippmann-Schwinger equation:

$$T = \frac{g^2_{\Delta N}(k)}{E - \mathring{\Delta} - \Sigma_{\Delta N}(k)} = -\frac{1}{\pi k E} e^{i\delta(k)} \sin \delta(k), \tag{2}$$



- where $g_{\Delta N}$ the $\Delta N \pi$ coupling, and $\Sigma_{\Delta N}$ is a one-pion loop integral.
- $\mathring{\Delta}$ is the bare resonance energy, which gets renormalized by the loop integral $\Sigma_{\Delta N}(k)$. We can choose $\mathring{\Delta}$ so that $E_{\mathrm{res}} = 292$ MeV relative to the nucleon mass.

The infinite-volume phase shift

- The phase shift can be calculated directly from the t-matrix– a standard result, re: heavy-baryon chiral perturbation theory:

$$\frac{g_{\Delta N}^2(k)}{\omega_\pi(k) - \overset{\circ}{\Delta} - \Sigma_{\Delta N}(k)} = -\frac{1}{\pi k\, \omega_\pi(k)}\, e^{i\delta(k)} \sin\delta(k), \tag{3}$$

$$\text{using} \quad \Sigma_{\Delta N}(k) = \mathcal{P}\int_0^\infty \mathrm{d}k' \frac{k'^{\,2} g_{\Delta N}^2(k')}{\omega_\pi(k) - \omega_\pi(k')}, \tag{4}$$

$$\text{with} \quad g_{\Delta N}^2(k) = \frac{2}{\pi}\chi_\Delta \frac{k^2\, u^2(k)}{\omega_\pi(k)}. \tag{5}$$

- Here, the external energy E is relative to the nucleon mass– $E = \omega_\pi(k) = \sqrt{k^2 + m_\pi^2}$.
- $u(k)$ regularizes the integral, cutting off the divergence.
- It is straightforward to solve for $\delta(k)$!

Scattering phase shifts

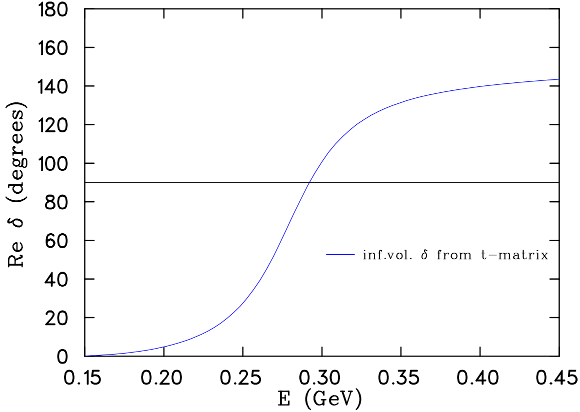


Figure: The phase shift associated with $N\pi$-scattering with a Δ-baryon intermediate, plotted against E, the external energy. $M_\Delta = M_N + E_{\mathrm{res}}$.

Lüscher's method

Lüscher's formula

- Lüscher's formula describes a fixed relationship between the scattering phase shift in the continuum, and the rth energy level in a finite volume L^3:

$$\delta(k_r\,;\,L) = r\,\pi - \phi(k_r L). \tag{6}$$

- $k_r = \sqrt{E_r^2 - m_\pi^2}$ for the rth energy level E_r.
- $\phi(k_r L)$ is a known geometric function related to the Zeta function.
- Assumptions:
  - two particles scatter from a finite-range interaction ($V(r) = 0$ for range $R < r$),
  - the non-relativistic Helmholtz wavefunction is in the asymptotic region: $R < r < L/2$, i.e. the two particles are well-separated,
  - finite-size corrections from vacuum polarization effects are exponentially suppressed,
  - virtual particle exchange across the periodic boundary have exponentially suppressed amplitudes.

A test: the Hamiltonian model

- We would like to test Lüscher's method for a Hamiltonian model that can be solved exactly.
- We define a system of hadronic interactions on a finite volume and solve for the energy levels.
- We can use Lüscher's formula to convert these energy levels into phase shifts.
- The Hamiltonian model is designed so that the interaction matches that of the t-matrix calculation.
- We can also calculate the infinite-volume phase shift directly from the t-matrix, and compare it with the result from Lüscher's formula.

Hamiltonian matrix

- Consider a heavy-baryon model in a finite volume box (length L), where $H = H_0 + H_I$ includes the interactions between the Δ-baryon and the pion-nucleon system:

$$H_0 = \begin{pmatrix} \mathring{\Delta} & 0 & 0 & \cdots \\ 0 & \sqrt{k_1^2 + m_\pi^2} & 0 & \cdots \\ 0 & 0 & \sqrt{k_2^2 + m_\pi^2} & \\ \vdots & \vdots & & \ddots \end{pmatrix}. \quad (7)$$

- The rows and columns of H represent the momentum states of the pion relative to the nucleon.
- The momentum-squared values available in a finite volume are: $k_n^2 = \left(\frac{2\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2) \equiv \left(\frac{2\pi}{L}\right)^2 n$.

Hamiltonian matrix

- Allowing only interactions that include the Δ-baryon, H_I takes the form:

$$H_I = \begin{pmatrix} 0 & g^{\mathrm{fin}}_{\Delta N}(k_1) & g^{\mathrm{fin}}_{\Delta N}(k_2) & \cdots \\ g^{\mathrm{fin}}_{\Delta N}(k_1) & 0 & 0 & \cdots \\ g^{\mathrm{fin}}_{\Delta N}(k_2) & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{8}$$

- $g^{\mathrm{fin}}_{\Delta N}$ is the finite-volume version of the coupling $g_{\Delta N}$, requiring the appropriate dimensional factors, etc.
- Recall:

$$g_{\Delta N}(k_n) = \sqrt{\frac{2}{\pi}\,\chi_\Delta\,\frac{k_n^2\,u^2(k_n)}{\omega_\pi(k_n)}}. \tag{9}$$

Loop regularization

- A regulator function $u(k)$ is introduced to regulate the loop integral, and give a finite range to the couplings $g_{\Delta N}$ & $g_{\Delta N}^{\text{fin}}$.

- We shall choose a dipole form, with cutoff $\Lambda = 0.8$ GeV:

$$u(k) = \left(1 + \frac{k^2}{\Lambda^2}\right)^{-2} . \qquad (10)$$

- However, it will be shown that the form of the regulator has little impact on the consistency of the final result: extracting the resonance position accurately.

Solving the eigenvalue problem

- In the Hamiltonian model, the eigenvalue problem, $\det(H - \lambda\mathbb{I}) = 0$, can be solved exactly. We find:

$$\lambda = \mathring{\Delta} - \sum_{n=1}^{\infty} \frac{\left(g_{\Delta N}^{\mathrm{fin}}(k_n)\right)^2}{\omega_\pi(k_n) - \lambda} \tag{11}$$

$$= \mathring{\Delta} - \frac{\chi_\Delta}{2\pi^2}\left(\frac{2\pi}{L}\right)^3 \sum_{n=1}^{\infty} C_3(n) \frac{k_n^2 u^2(k_n)}{\omega_\pi(k_n)[\omega_\pi(k_n) - \lambda]}. \tag{12}$$

- In the limit $\lambda \to \mathring{\Delta}$, the eigenvalue equation and the finite-volume one-loop calculation are the same (by construction).
  - Recalling $k_r = \sqrt{\lambda^2 - m_\pi^2}$, and using Lüscher's formula:
    $\delta(k_r\,;\,L) = r\pi - \phi(k_r L)$,

  the eigenvalues can be converted into phase shifts.

Solving the eigenvalue problem

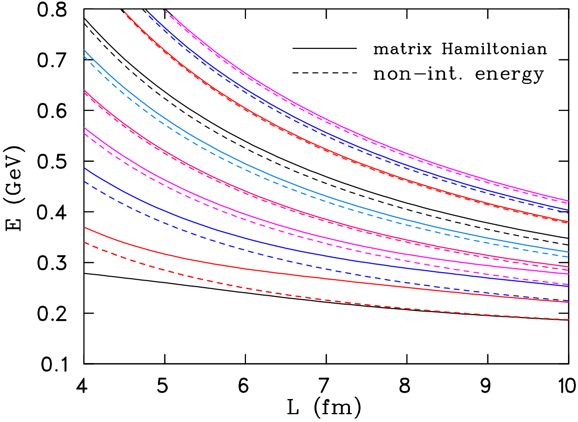


Figure: The ten lowest energy levels ($\lambda = E$) from the matrix Hamiltonian model (solid lines), and the non-interacting energies (dotted lines) vs. L.

Comparison with the exact phase shift

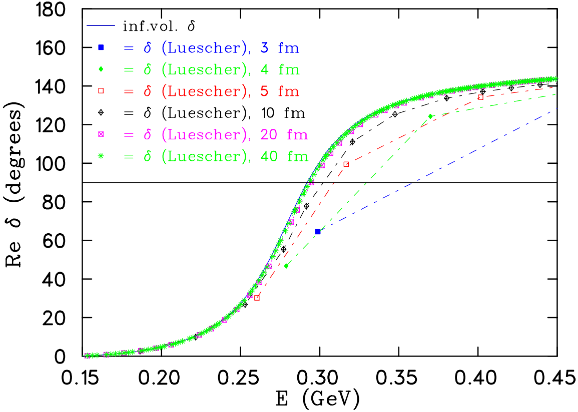


Figure: The infinite-volume phase shift (blue), and the finite-volume estimates of the phase shift using Lüscher's formula, plotted against the external energy E.

The L-dependence of E_{res}

- The finite-volume resonance position correctly approaches the infinite-volume resonance value as $L \to \infty$ — a good sign!
- But even at currently ambitious box sizes of 5 fm, the resonance position E_{res} has an error of ~ 20 MeV.
- This may indicate that the criteria for which Lüscher's formula was derived are not realized until much larger values of L.
- So: check how $E_{\mathrm{res}} = E(90^\circ)$ behaves as a function of $1/L$.

The L-dependence of $E_{\rm res}$

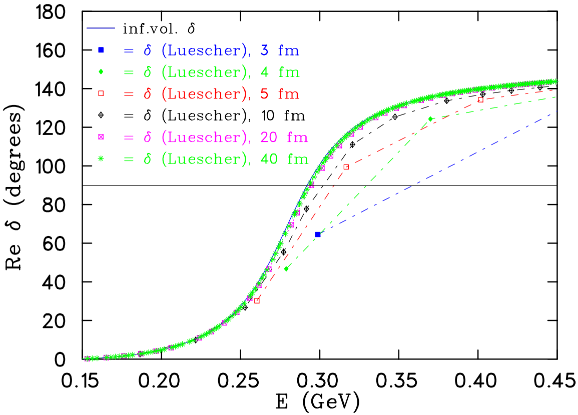


Figure: Use a Linear ($\delta = a\,E + b$) [shown] or a Breit-Wigner ($\frac{\Gamma/2}{E - E_{\rm res} - i\Gamma/2} = e^{i\delta}\sin\delta$) interpolation from the two closest values of δ to 90° to obtain $E_{\rm res}$.

The L-dependence of E_{res}

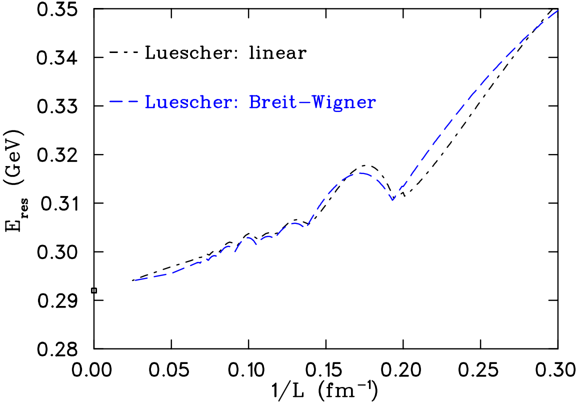


Figure: The resonance position E_{res} obtained using Lüscher's method, plotted against $1/L$. Both a linear and a Breit-Wigner interpolation are used. The experimental value is marked with a square.

Improved phase shift extraction

New method for obtaining a phase shift

- We propose a new method for obtaining a phase shift from the energy levels of lattice QCD.
- Free parameters ($\overset{\circ}{\Delta}$ & χ_{Δ}) in the Hamiltonian model can be constrained by lattice results, by matching the energy levels.
- Then use the infinite-volume t-matrix to calculate $\delta(k)$.
- Note: Lüscher's formula is not used at all!

New method for obtaining a phase shift

- But what about the model-dependence in choosing, say, a dipole with $\Lambda = 0.8$ GeV in both the t-matrix and the Hamiltonian?
- As a test, try using a Gaussian function for $u(k)$ for a suitable range of Λ values, and match its energy levels (with constraint $E_r < 500$ MeV) to those of the previous dipole model.
- We are essentially treating the previously established energy levels of the dipole model as pseudodata. Thus, without use of lattice QCD results, the model-dependence can be tested.
- With new values of $\overset{\circ}{\Delta}$ & χ_Δ, use the t-matrix to obtain $\delta(k)$, and thus $E_{\rm res}$.

The model-dependence of E_{res}

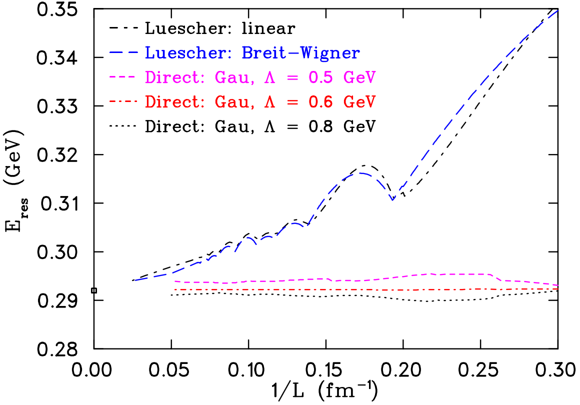


Figure: The resonance position E_{res} plotted against $1/L$. The result of the new method is shown, fitting a Gaussian model to pseudodata. e.g. at 4 fm, the model-dependence is ~ 5 MeV $\ll 40$ MeV!

Conclusion

Conclusion

- Lüscher's formula exhibits a $1/L$ dependence in the resonance position. $L > 10$ fm boxes are required to extrapolate reliably to the infinite-volume limit.
- Care must be taken in matching the asymptotic states of elastic scattering to the finite-volume energy spectrum using Lüscher's method.
- A new method was developed in which a Hamiltonian model was constructed, and the energy levels matched with those of a lattice QCD calculation. The phase shift is then computed from the t-matrix.
- The results of a pseudodata analysis show that the resonance position of the new phase shift estimate is accurate, largely independent of L, and relatively insensitive to model-dependence.

Appendix

The model-dependence of E_{res}

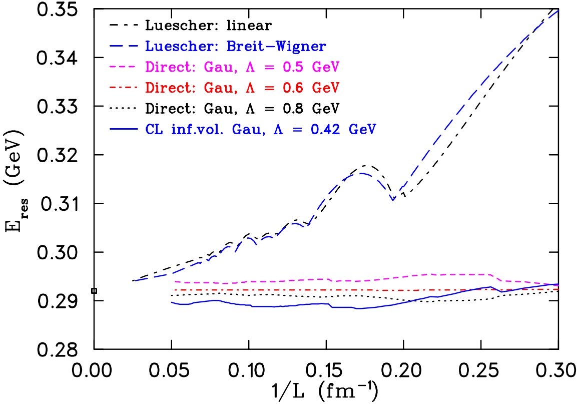


Figure: The resonance position E_{res} plotted against $1/L$, fitting a Gaussian model (without off-diagonal terms) to pseudodata (with off-diagonal Chew-Low terms).