

# Probing the $N_f = 2$ chiral dynamics with topological zero-modes in the mixed chiral regime

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## Ingredients : Mixed-actions setup

- valence : Neuberger fermions
- sea :  $N_f = 2$   $\mathcal{O}(a)$ -improved Wilson fermions  
→ use of CLS gauge configurations

Giusti *et al.* (2003-2006)

Del Debbio *et al.* (2006, 2007)

What about observables when sea and valence masses don't belong to the same chiral regime ?

## Matching of ChPT and Lattice QCD

Benefit from testing different chiral regimes

→ reduce systematic uncertainties.

## Advantages

- + Chiral symmetry preserved in the valence sector
- + Overlap  $\rightarrow$  well-defined topological charge
- + Wilson  $\rightarrow$  all topological sectors are sampled
- + Renormalization easier (no operator mixing)

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## Disadvantages

- Overlap  $\rightarrow$  numerical cost much more higher than Wilson
- Possible large unitarity violations

Dürr *et al.* PoSLAT 2007

### p-regime

Applicable if :  $M_\pi L \gg 1$ ,  $4\pi F \gg L^{-1}$

Standard ChPT in finite volume :  $m \sim p^2$ ,  $L^{-1} \sim T^{-1} \sim p$

Gasser and Leutwyler (1983, 1987)

Mass and volume effects are relevant.

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### $\epsilon$ -regime

Applicable if :  $m\Sigma V \sim \mathcal{O}(1)$ ,  $M_\pi L \sim 1$ ,  $4\pi F \gg L^{-1}$

Reordering of the chiral expansion :  $m \sim p^4 \sim \epsilon^4$ ,  $L^{-1} \sim T^{-1} \sim \epsilon$

Hansen (1990) – Hansen and Leutwyler (1991)

Volume effects are relevant, **mass effects are suppressed.**

Pion zero-momentum modes become non-perturbative. Factorization :

$$U = U_0 e^{2i\xi/F} \text{ with } \int d^4x \xi(x) = 0.$$

$\Rightarrow$  "Exact" factorization of the lagrangian :

$$S_{\text{LO}}(\xi, U_0) = \underbrace{\int d^4x \text{Tr} [\partial_\mu \xi \partial^\mu \xi]}_{\text{usual perturbative treatment}} - \underbrace{\frac{\Sigma V}{2} \text{Tr} [MU_0 + U_0^\dagger M]}_{\text{non-perturbative integration}}.$$

The integration over the zero-modes manifold can be done exactly via master integrals.

Topology plays here an important role : it becomes here an extra scaling variable.

$\rightarrow$  compute observables at fixed topology.

# Pseudoscalar correlator in ChPT

$N_v$  valence quarks with mass  $m_v$  and  $N_s$  sea quarks with mass  $m_s$

## Pseudoscalar correlator

$$\text{Tr}[T^a T^b] C(t) = \int d^3 \vec{x} \langle P^a(x) P^b(0) \rangle, \quad P^a \equiv \bar{\psi}_v T^a i \gamma_5 \psi_v$$

$T^a$  generator of the valence group  $SU(N_v)$

→ ChPT predictions at NLO in finite volume  $V = L^3 T$ .

## Propagators

$$\begin{aligned} G(x, M^2) &= \frac{1}{V} \sum_p \frac{e^{ip \cdot x}}{p^2 + M^2}, \\ E(x, M_a^2, M_b^2) &= \frac{1}{V} \sum_p \frac{e^{ip \cdot x}}{(p^2 + M_a^2)(p^2 + M_b^2) \left( \frac{N_v}{p^2 + M_{vv}^2} + \frac{N_s}{p^2 + M_{ss}^2} \right)}, \\ \bar{G}(x) &= \frac{1}{V} \sum_{p^*} \frac{e^{ip \cdot x}}{p^2}. \end{aligned}$$



Take all masses in the p-regime

$$m_{s,v} \Sigma V \gg 1$$

$$C(t) = \frac{2\Sigma_{\text{eff}}}{F^2} \frac{\cosh [M_{vv,\text{eff}} (t - T/2)]}{2M_{vv,\text{eff}} \sinh [M_{vv,\text{eff}} T/2]} \quad \text{with} \quad \begin{cases} M_{vv,\text{eff}}^2 &= M_{vv}^2 (1 + \Delta_M) \\ \Sigma_{\text{eff}} &= \Sigma (1 + \Delta_\Sigma) \end{cases}$$

NLO chiral corrections on the mass and chiral condensate

$$\begin{aligned} \Delta_M &= \frac{E(0, M_{vv}^2, M_{vv}^2)}{F^2} - \frac{8}{F^2} [(N_v M_{vv}^2 + N_s M_{ss}^2) (L_4 - 2L_6) \\ &\quad + M_{vv}^2 (L_5 - 2L_8)] \\ \Delta_\Sigma &= -\frac{N_v}{F^2} G(0, M_{vv}^2) - \frac{N_s}{F^2} G(0, M_{ss}^2) + \frac{E(0, M_{vv}^2, M_{vv}^2)}{F^2} \\ &\quad + \frac{16}{F^2} [(N_v M_{vv}^2 + N_s M_{ss}^2) L_6 + M_{vv}^2 L_8] \end{aligned}$$

## Spectral decomposition of the propagator

For correlators computed in fixed topological sectors, exact poles in  $1/m^n$  may appear when some propagators are saturated by zero-modes :

$$D_{xy}^{-1} = \sum_{i, \text{zero-modes}} \frac{v_i(x) v_i^\dagger(y)}{mV} + \sum_{i, \text{non zero-modes}} \frac{v_i(x) v_i^\dagger(y)}{(\lambda_i + m) V}$$

→ One can match the residues instead of the correlators themselves in the  $\epsilon$ -regime.

$$\begin{aligned} C_\nu(x-z, y-z) &= \frac{\text{Res}_n}{(mV)^n}, \\ \text{Res}_n &= \lim_{m \rightarrow 0} (mV)^n C_\nu(x-z, y-z) \end{aligned}$$

Giusti, Hernández, Laine, Weisz, Wittig (2004).

Take all masses in the  $\epsilon$ -regime. With the spectral representation of the quark propagator :

$$\lim_{m \rightarrow 0} (mV)^2 C_\nu^{ab}(x) = \text{Tr} [T^a T^b] A_\nu(x) + \text{Tr} [T^a] \text{Tr} [T^b] \tilde{A}_\nu(x)$$



“Connected” (left) and “disconnected” (right) external flavours contractions.

Damgaard, Hernández, Jansen, Laine and Lellouch (2003).

## Amplitudes

$$A_\nu(x-y) = \left\langle \sum_{i,j} v_j^\dagger(x) v_i(x) v_i^\dagger(y) v_j(y) \right\rangle_\nu,$$

$$\tilde{A}_\nu(x-y) = - \left\langle \sum_i v_i^\dagger(x) v_i(x) \sum_j v_j^\dagger(y) v_j(y) \right\rangle_\nu$$

$$m_{s,v}\Sigma V \sim 1, N = N_v + N_s$$

$$C_\nu^{aa}(x) = C_\nu + \alpha_a \bar{G}(x) + \beta_a [\bar{G}(x)]^2 + \gamma_a \int d^4y \bar{G}(x-y) \bar{G}(y) + \varepsilon_a \delta(x)$$

Time derivative of the residues in  $1/m^2$

$$\lim_{m \rightarrow 0} (mV)^2 \frac{d}{dt} \int d^3\vec{x} C_\nu^{aa}(x) = \frac{1}{2} A'(t),$$

$$\lim_{m \rightarrow 0} (mV)^2 \frac{d}{dt} \int d^3\vec{x} C_\nu^{00}(x) = N A'(t) + N^2 \tilde{A}'(t).$$

Let  $\tau = t/T$  :

$$F^2 A'(t) = \frac{2|\nu|}{N} \left[ (1 + N|\nu|) h'_1(\tau) + \frac{T^2}{NF^2V} H_2(\tau) \right],$$

$$F^2 \tilde{A}'(t) = -\frac{2|\nu|}{N} \left[ (N + |\nu|) h'_1(\tau) + \frac{T^2}{NF^2V} \tilde{H}_2(\tau) \right].$$

$$H_2(\tau) = -(1 + N|\nu|) N^2 \frac{\beta_1 \sqrt{V}}{T^2} h'_1(\tau) + [N(6 - N^2)|\nu| + 4 + N^2(2\nu^2 - 1)] h'_2(\tau) + \left[ N \left( 2 - \frac{N^2}{2} \right) |\nu| + 1 + \frac{N^2}{2} \right] g'_1(\tau),$$

$$\tilde{H}_2(\tau) = -(N + |\nu|) N^2 \frac{\beta_1 \sqrt{V}}{T^2} h'_1(\tau) + [(4 + N^2)|\nu| + 2N(2 + \nu^2) - N^3] h'_2(\tau) + \left[ \left( 1 + \frac{N^2}{2} \right) |\nu| + 2N - \frac{N^3}{2} \right] g'_1(\tau).$$

$$h_1(\tau) = \frac{1}{2} \left[ \left( \tau - \frac{1}{2} \right)^2 - \frac{1}{12} \right], \quad h_2(\tau) = \frac{1}{24} \left[ \tau^2 (\tau - 1)^2 - \frac{1}{30} \right]$$

$$g_1(\tau) = [h_1(\tau)]^2 + \sum_{\vec{n} \neq \vec{0}} \left\{ \frac{\cosh [|\vec{p}| (\tau - 1/2)]}{2|\vec{p}| \sinh (|\vec{p}|/2)} \right\}^2, \quad |\vec{p}| = 2\pi \frac{T}{L} \left[ \sum_{i=1}^3 n_i^2 \right]^{1/2}$$

⇒ Only depends on F at NLO!!

Expansion around  $z = \tau - 1/2 = 0$

$$\begin{aligned}\frac{1}{L^2} A'(t) &= D_\nu z + C_\nu z^3 + \mathcal{O}(z^5), \\ \frac{1}{L^2} \tilde{A}'(t) &= \tilde{D}_\nu z + \tilde{C}_\nu z^3 + \mathcal{O}(z^5).\end{aligned}$$

$$\begin{aligned}D_\nu &= \frac{2|\nu|}{N(FL)^2} \left\{ (1 + N|\nu|) \left( 1 - N \frac{\beta_1}{F^2 \sqrt{V}} \right) + \frac{T^2}{F^2 V} \left[ \gamma_1 \left( \frac{2 + N^2}{2N} \right. \right. \right. \\ &\quad \left. \left. + \frac{4 - N^2}{2} |\nu| \right) + \zeta_2 \left( (6 - N^2) |\nu| + \frac{4}{N} + N(2\nu^2 - 1) \right) \right] \right\}, \\ \tilde{D}_\nu &= -\frac{2|\nu|}{N(FL)^2} \left\{ (N + |\nu|) \left( 1 - N \frac{\beta_1}{F^2 \sqrt{V}} \right) + \frac{T^2}{F^2 V} \left[ \gamma_1 \left( \frac{4 - N^2}{2} \right. \right. \right. \\ &\quad \left. \left. + \frac{2 + N^2}{2N} |\nu| \right) + \zeta_2 \left( 4 + 2\nu^2 - N^2 + \frac{4}{N} |\nu| + N|\nu| \right) \right] \right\}\end{aligned}$$

with

$$\zeta_2 = -\frac{1}{24} \text{ and } \gamma_1 = -\frac{1}{12} + \frac{1}{2} \sum_{\vec{n} \neq \vec{0}} \frac{1}{\sinh^2(|\vec{p}|/2)}.$$

Let's consider now  $N_v$  valence quark in the  $\epsilon$ -regime and  $N_s$  sea quarks in the p-regime  $\Rightarrow$  **partial-quenching**.

Valence pion zero-momentum modes still non-perturbative but factorization becomes :

$$U = \begin{pmatrix} \bar{U}_0 & 0 \\ 0 & e^{-i\frac{\bar{\eta}}{N_s} \mathbb{1}_s} \end{pmatrix} e^{i\frac{2\xi}{F}} \text{ with } \int d^4x \text{Tr} [T^a \xi(x)] = 0, \begin{cases} T^a \text{ generator of } SU(N_v) \\ T^\eta = \begin{pmatrix} \frac{1}{N_v} & 0 \\ 0 & -\frac{1}{N_s} \end{pmatrix} \end{cases}$$

### Power counting

$$\begin{aligned} p &\sim \partial_\mu \sim \xi \sim \epsilon \\ L^{-1} &\sim T^{-1} \sim \epsilon \\ m_v &\sim \epsilon^4 \\ m_s &\sim \epsilon^2 \end{aligned}$$

Bernardoni and Hernández (2007),  
Bernardoni, Damgaard, Fukaya, Hernández (2008).

$$m_v \Sigma V \sim 1, m_s \Sigma V \gg 1, M_\eta^2 = \frac{N_v}{N} M_{ss}^2$$

$$\begin{aligned} C_\nu^{aa}(x) = & C_\nu + \alpha_a^{(1)} \bar{G}(x) + \alpha_a^{(2)} \bar{G}(x, M_\eta^2) + \alpha_a^{(3)} \frac{d}{dM_{ss}^2} [M_{ss}^2 \bar{G}(x, M_\eta^2)] \\ & + \beta_a^{(1)} [\bar{G}(x)]^2 + \beta_a^{(2)} [\bar{G}(x, M_\eta^2)]^2 + \beta_a^{(3)} [\bar{G}(x, M_{ss}^2)]^2 \\ & + \beta_a^{(4)} \bar{G}(x) \bar{G}(x, M_\eta^2) + \int d^4 y \left[ \gamma_a^{(1)} \bar{G}(x-y) \bar{G}(y) \right. \\ & + \gamma_a^{(2)} \bar{G}(x-y, M_\eta^2) \bar{G}(y, M_\eta^2) + \gamma_a^{(3)} \bar{G}(x-y) \bar{G}(y, M_\eta^2) \left. \right] \\ & + \varepsilon_a \delta(x) \end{aligned}$$

Time derivative of the residues in  $1/m^2$

$$\begin{aligned} \lim_{m \rightarrow 0} (mV)^2 \frac{d}{dt} \int d^3 \vec{x} C_\nu^{aa}(x) &= \frac{1}{2} A'(t), \\ \lim_{m \rightarrow 0} (mV)^2 \frac{d}{dt} \int d^3 \vec{x} C_\nu^{00}(x) &= N_v A'(t) + N_v^2 \tilde{A}'(t). \end{aligned}$$



$$m_v \Sigma V \sim 1, m_s \Sigma V \gg 1, M_\eta^2 = \frac{N_v}{N} M_{ss}^2$$

$$\begin{aligned} C_\nu^{aa}(x) = & C_\nu + \alpha_a^{(1)} \bar{G}(x) + \alpha_a^{(2)} \bar{G}(x, M_\eta^2) + \alpha_a^{(3)} \frac{d}{dM_{ss}^2} [M_{ss}^2 \bar{G}(x, M_\eta^2)] \\ & + \beta_a^{(1)} [\bar{G}(x)]^2 + \beta_a^{(2)} [\bar{G}(x, M_\eta^2)]^2 + \beta_a^{(3)} [\bar{G}(x, M_{ss}^2)]^2 \\ & + \beta_a^{(4)} \bar{G}(x) \bar{G}(x, M_\eta^2) + \int d^4 y \left[ \gamma_a^{(1)} \bar{G}(x-y) \bar{G}(y) \right. \\ & + \gamma_a^{(2)} \bar{G}(x-y, M_\eta^2) \bar{G}(y, M_\eta^2) + \gamma_a^{(3)} \bar{G}(x-y) \bar{G}(y, M_\eta^2) \left. \right] \\ & + \varepsilon_a \delta(x) \end{aligned}$$

## Effective couplings

$$\begin{aligned} \tilde{\Sigma} &= \Sigma \left\{ 1 - \frac{1}{F^2} [N_v \bar{G}(0) + N_s G(0, M_{ss}^2/2) - \bar{E}(0, 0, 0) - 16L_6 N_s M_{ss}^2] \right\} \\ \tilde{F} &= F \left\{ 1 - \frac{1}{2F} [N_v \bar{G}(0) + N_s G(0, M_{ss}^2/2) - 8L_4 N_s M_{ss}^2] \right\} \end{aligned}$$

### How to get quenched results from the full ones ?

- **Supersymmetric method** : add  $N_v$  spin 1/2 bosons with mass  $m_v$  to cancel the fermionic loops. Zero-modes integrals performed now with the supergroup  $U(N|N)$ .
- **Replica limit** : keep  $N_v$  dependence explicitly and take the limit  $N_v \rightarrow 0$  add the very end of the computation. Zero-mode integrals as for the SUSY method.

→ both methods equivalent at any order. We checked the consistency of the results in our computation.

### Quenched propagator

$$\begin{aligned}\bar{E}(x, 0, 0) &= \frac{\alpha}{2N_c} \bar{G}(x) + \frac{m_0^2}{2N_c} \bar{F}(x), \\ \bar{F}(x) &= \frac{1}{V} \sum_{p^*} \frac{e^{ip \cdot x}}{p^4}.\end{aligned}$$

→ double poles as expected due to unitary violations.

## Overall form of the correlator

$$\begin{aligned}
 C_\nu^{aa}(x) = & \tilde{C} + \tilde{\alpha}_a^{(1)} \tilde{G}(x) + \tilde{\alpha}_a^{(2)} \tilde{E}(x) + \tilde{\beta}_a^{(1)} [\tilde{G}(x)]^2 + \tilde{\beta}_a^{(2)} [\tilde{E}(x)]^2 \\
 & + \tilde{\beta}_a^{(3)} [\tilde{G}(x, M_{ss}^2/2)]^2 + \tilde{\beta}_a^{(4)} \tilde{G}(x) \tilde{E}(x) \\
 & + \int d^4y \left[ \tilde{\gamma}_a^{(1)} \tilde{G}(x-y) \tilde{G}(y) + \tilde{\gamma}_a^{(2)} \tilde{G}(x-y) \tilde{E}(y) \right. \\
 & \left. + \tilde{\gamma}_a^{(3)} \tilde{E}(x-y) \tilde{E}(y) \right] + \tilde{\varepsilon}_a \delta(x)
 \end{aligned}$$

## Effective couplings

$$\begin{aligned}
 F_{\text{eff}}^2 &= F^2 \left\{ 1 - \frac{N_s}{F^2} [G(0, M_{ss}^2/2) - 8L_4 M_{ss}^2] \right\} \\
 \frac{\alpha_{\text{eff}}}{2N_c} &= \frac{1}{N_s} \left\{ 1 - \frac{1}{F^2} [N_s G(0, M_{ss}^2/2) - 8L_5 M_{ss}^2] \right\} \\
 \frac{m_{0,\text{eff}}^2}{2N_c} &= \frac{M_{ss}^2}{N_s} \left\{ 1 - \frac{1}{F^2} \left[ \frac{N_s^2 - 1}{N_s} G(0, M_{ss}^2) - 16M_{ss}^2 (N_s L_6 \right. \right. \\
 &\quad \left. \left. + N_s L_7 + L_8) \right] \right\}
 \end{aligned}$$

## Result

$$\begin{aligned}
 D_\nu = & \frac{2|\nu|}{(F_{\text{eff}}L)^2} \left\{ |\nu| + \frac{\alpha_{\text{eff}}}{2N_c} + \frac{\rho}{(F_{\text{eff}}L)^2} \left[ -\beta_1 \rho^{-3/2} + \left( \frac{5}{N_s^2} + \frac{8|\nu|}{N_s} + 3 \right. \right. \right. \\
 & + \left. \left. 2\nu^2 - 2\langle \nu^2 \rangle_{\text{eff}} \right) \zeta_2 + \left( \frac{1}{N_s^2} + \frac{2|\nu|}{N_s} + \frac{1}{2} \right) \gamma_1 \right. \\
 & + \left. 2 \frac{M_{\text{ss}}^2}{N_s} T^2 \left( \frac{1}{N_s} + |\nu| \right) (\gamma_2 - 4\zeta_3) + \left( \frac{M_{\text{ss}}^2}{N_s} \right)^2 T^4 (\gamma_3 + 4\zeta_4) \right. \\
 & \left. \left. - \frac{N_s}{2} |\nu| \gamma_4 (M_{\text{ss}}^2/2) \right] \right\}
 \end{aligned}$$

Using the Witten-Veneziano formula at LO :  $m_0^2 F^2 = 4N_c \frac{\langle \nu^2 \rangle}{V}$ .

To compare with Giusti, Hernández, Laine, Weisz, Wittig (2004) :  $\frac{\alpha}{2N_c} \sim \mathcal{O}(\epsilon^2)$ ,  $\frac{m_0^2}{2N_c} \sim \mathcal{O}(\epsilon^4)$

$$\begin{aligned}
 D_\nu = & \frac{2|\nu|}{(FL)^2} \left\{ |\nu| + \frac{\alpha}{2N_c} - \frac{2KF}{\Sigma} + \frac{\rho}{(FL)^2} \left[ -\beta_1 \rho^{-3/2} \right. \right. \\
 & \left. \left. + (3 + 2\nu^2 - 2\langle \nu^2 \rangle) \zeta_2 + \frac{\gamma_1}{2} \right] \right\}
 \end{aligned}$$

### Statistics

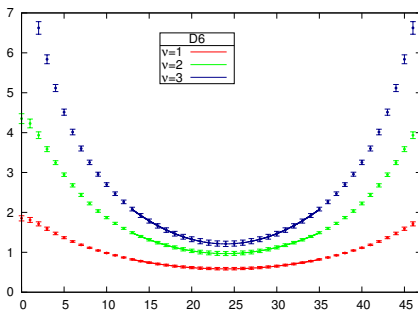
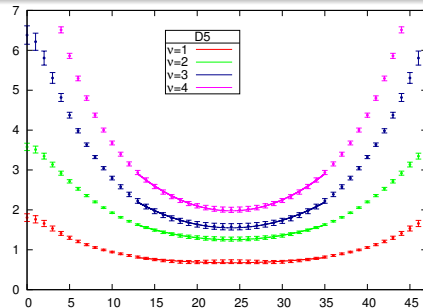
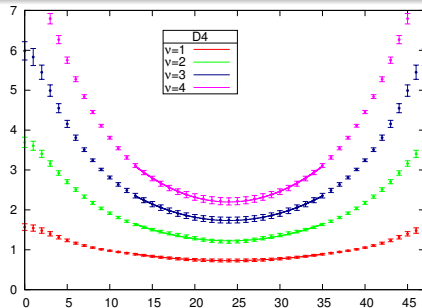
| Lattice | Total # conf. | # of conf. by topological sector |           |           |           |
|---------|---------------|----------------------------------|-----------|-----------|-----------|
|         |               | $\nu = 1$                        | $\nu = 2$ | $\nu = 3$ | $\nu = 4$ |
| D4      | 117           | 43                               | 18        | 31        | 25        |
| D5      | 129           | 40                               | 43        | 23        | 23        |
| D6      | 153           | 85                               | 43        | 25        | —         |

$$a \sim 0.065 \text{ fm},$$

$$V = 24^3 \times 48.$$

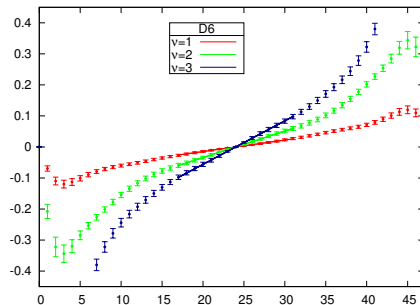
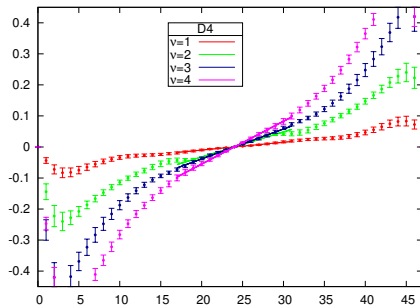
# Pseudoscalar correlator in ChPT

Mixed regime - Quenched case



# Pseudoscalar correlator in ChPT

Mixed regime - Quenched case



Temporal dependance of the residue of  $\langle PP \rangle$

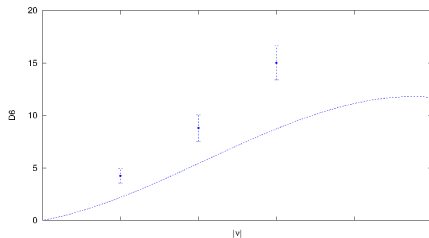
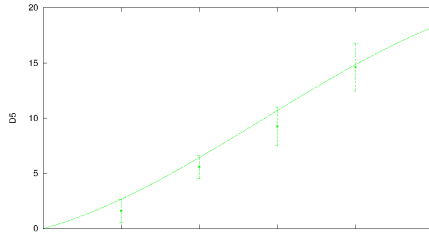
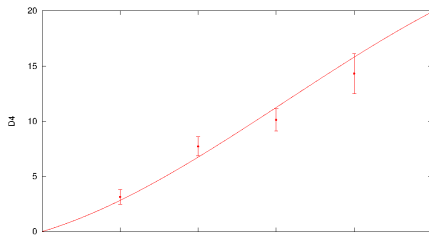
$$\lim_{m \rightarrow 0} m^2 C_\nu(t) = \alpha_\nu + 2\beta_1 h_1(t) = \alpha_\nu - \frac{\beta_\nu}{12} + \beta_\nu \left( \frac{t}{T} - \frac{1}{2} \right)^2 + \mathcal{O} \left( \frac{t}{T} - \frac{1}{2} \right)^3$$

→ fit of the time derivative of  $\lim_{m \rightarrow 0} m^2 C_\nu(t)$  with functional form from mixed regime computation.



# Pseudoscalar correlator in ChPT

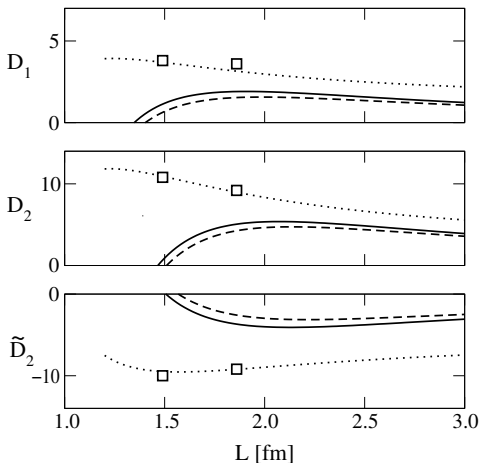
Mixed regime - Quenched case



→ Bad convergence domain in the mixed regime.

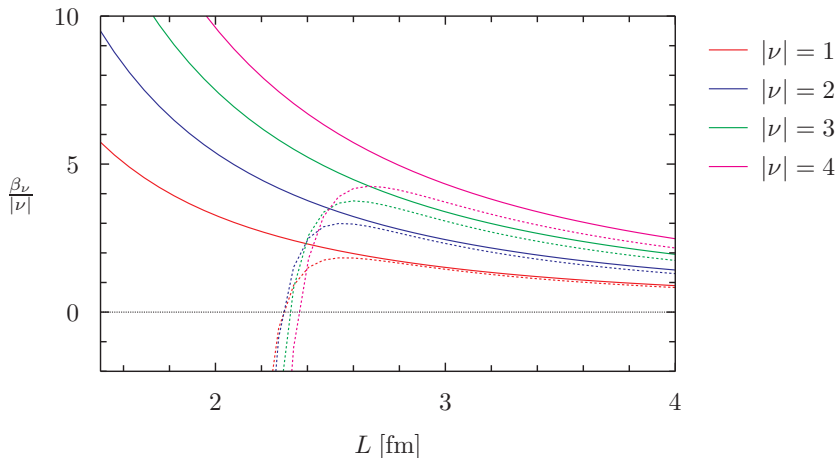
To compare with results from the  $\epsilon$ -regime :

Giusti, Hernández, Laine, Weisz, Wittig (2004)



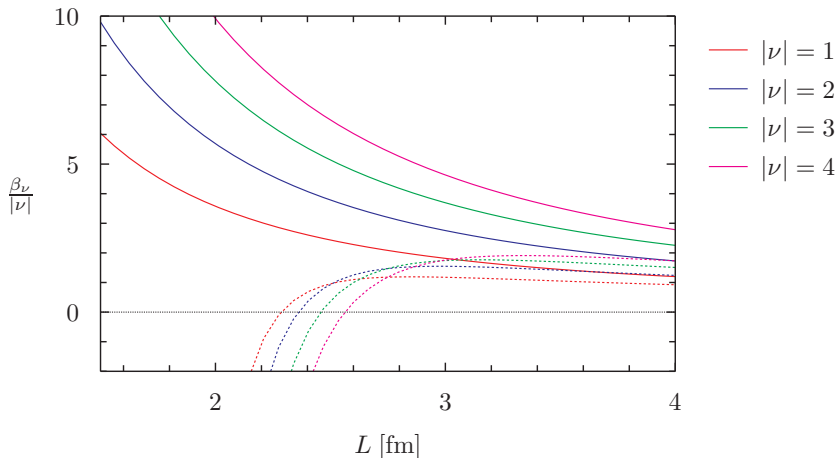
Solid : full  $N_f = 3$  - Dashed : full  $N_f = 2$  - Dotted : quenched  $N_f = 0$ .

$\beta_\nu$  vs.  $L$  for different values of  $|\nu| - M_{ss} = 150$  MeV,  $\rho = 1$



Solid : LO - Dashed : NLO.

$\beta_\nu$  vs.  $L$  for different values of  $|\nu| - M_{ss} = 150$  MeV,  $\rho = 2$



Solid : LO - Dashed : NLO.

- **Mixed action simulation** is a tool to investigate QCD at low energy  
→ **Exact chiral symmetry** with less numerical cost
- Explore “unusual” chiral regime : **mixed regime**
- **Lattice artifacts** can be very easily checked : use of **ChPT at finite lattice spacing**
- Allow to extract more precisely  $\Sigma, F$  and  $L_i$ 's
- Role of the charm in  $\Delta I = 1/2$  rule
- Weak LEC's :  $K \rightarrow \pi\pi$  decay