Taste non-Goldstone Pion Decay Constants in SχPT

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Pion decay constants

- **Definition**

$$\langle 0|A_\mu|\pi(p)\rangle = -i f_\pi p_\mu$$

- **Experimental values**

$$f_\pi = 130.41 \pm 0.3 \pm 0.20 \text{ MeV}$$
$$f_K = 156.1 \pm 0.2 \pm 0.8 \pm 0.2 \text{ MeV}$$

- **Implications**

$\chi PT \implies$ Gasser-Leutwyler low energy constants

$$f_K/f_\pi \implies |V_{us}|$$
$$f_\pi \implies r_1$$

Pion decay constants in SχPT

- Pions are well described by χPT
- Extrapolation to physical quark mass requires fitting function
- SχPT provides us a systematic way to obtain the fitting function including lattice artifacts (taste-violation) of staggered fermions

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- Aubin and Bernard have calculated the decay constants for taste Goldstone pions at NLO [Aubin and Bernard, 2003]
- We extend the calculation to **taste non-Goldstone pions**

Staggered chiral perturbation theory

- **Exponential parameterization of PGB fields**

$$\Sigma = \exp(i\phi/f)$$

- **Chiral transformation for 3 flavor case**

$$SU(12)_L \times SU(12)_R : \ \Sigma \rightarrow L\Sigma R^\dagger$$

where $L \in SU(12)_L$, $R \in SU(12)_R$

- **PGB fields,** ϕ

$$\phi = \sum_a \phi^a \otimes T^a,$$

$$\phi^a = \begin{pmatrix} U_a & \pi_a^+ & K_a^+ \\ \pi_a^- & D_a & K_a^0 \\ K_a^- & \bar{K}_a^0 & S_a \end{pmatrix},$$

$$T^a \in \{\xi_5,\ i\xi_{\mu 5},\ i\xi_{\mu\nu}(\mu < \nu),\ \xi_\mu, \xi_I\}.$$

Staggered chiral perturbation theory

- **Power counting**

$$\mathcal{O}(a^2\Lambda_\chi^2) \approx \mathcal{O}(p^2/\Lambda_\chi^2) \approx \mathcal{O}(m_\pi^2/\Lambda_\chi^2) \approx \mathcal{O}(m_q/\Lambda_\chi)$$

- **Lee and Sharpe Lagrangian for multiple flavors**

[Aubin and Bernard, 2003]

$$\mathcal{L}_{\mathrm{LO}} = \frac{f^2}{8}\mathrm{Tr}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger) - \frac{1}{4}\mu f^2 \mathrm{Tr}(M\Sigma + M\Sigma^\dagger) + \frac{2m_0^2}{3}(U_I + D_I + S_I)^2 + a^2\mathcal{V}$$

- $M = \mathrm{diag}(m_u, m_d, m_s) \otimes \xi_I$
- $\mathcal{V}$: taste-violating potential [Lee and Sharpe, 1999]

$$SO(4) \times SU(4)_T \xrightarrow{a\neq 0} SW_{4,\mathrm{diag}} \underset{p\ll\Lambda_\chi}{\subset} SO(4) \times SO(4)_T$$

Axial currents and decay constants

- **Decay constant** : $\langle 0 | A^P_{\mu,a} | P^+_a(p) \rangle = -i f_{P^+_a} p_\mu$

- **Axial current**

$$A^P_{\mu,a} = -i\frac{f^2}{8}\mathrm{Tr}\left[T^{a(3)}\mathscr{P}^P(\partial_\mu\Sigma\Sigma^\dagger + \Sigma^\dagger\partial_\mu\Sigma)\right]$$

- $P = \pi^+, K^+, K^0$
- $T^{a(3)} \equiv I_3 \otimes T^a$
- $\mathscr{P}^P$: chooses a flavor from Σ

  (e.g.) $\mathscr{P}^{\pi^+}_{ij} = \delta_{i1}\delta_{j2}$

  (e.g.) $\mathscr{P}^{K^+}_{ij} = \delta_{i1}\delta_{j3}$

  Generally, $\mathscr{P}^P_{ij} = \delta_{ix}\delta_{jy}$

$$\phi^a = \begin{pmatrix} U_a & \pi^+_a & K^+_a \\ \pi^-_a & D_a & K^0_a \\ K^-_a & \bar{K}^0_a & S_a \end{pmatrix}$$

Axial currents and decay constants

- **Axial current**

$$A^P_{\mu,a} = -i\frac{f^2}{8}\mathrm{Tr}\left[T^{a(3)}\mathscr{P}^P(\partial_\mu\Sigma\Sigma^\dagger + \Sigma^\dagger\partial_\mu\Sigma)\right]$$

$$\partial_\mu\Sigma\Sigma^\dagger + \Sigma^\dagger\partial_\mu\Sigma = \frac{2i}{f}\partial_\mu\phi - \frac{i}{3f^3}\left(\partial_\mu\phi\phi^2 - 2\phi\partial_\mu\phi\phi + \phi^2\partial_\mu\phi\right) + \cdots$$

$\mathcal{O}(\phi)$ term $\Longrightarrow$
- ·Leading order term, f
- ·NLO wave function renormalization correction, δf^Z

$\mathcal{O}(\phi^3)$ terms $\Longrightarrow$
- ·NLO loop correction, $\delta f^{\mathrm{current}}$

- **Decay constant**

$$f_{P_a^+} = f\left[1 + \frac{1}{16\pi^2 f^2}\left(\delta f^Z + \delta f^{\mathrm{current}}\right) + \delta f^{\mathrm{anly}}\right]$$

Wave function renormalization correction

- **Axial current up to $\mathcal{O}(\phi)$**

$$A^{P,\phi}_{\mu,a} = f\left(\mathscr{P}^{P}_{ij}\partial_\mu \phi^a_{ji}\right)$$

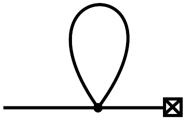
- **Decay constant from the axial current up to $\mathcal{O}(\phi)$**

$$\langle 0|A^{P,\phi}_{\mu,a}|P^+_a(p)\rangle = f(-ip_\mu)\langle 0|\phi^a_{yx}|P_a(p)\rangle$$

$$\langle 0|\phi^a_{yx}|P_a(p)\rangle = \sqrt{Z_{P_a}} \equiv \sqrt{1+\delta Z_{P_a}} \simeq 1 + \frac{1}{2}\delta Z_{P_a}$$

- **Wave function renormalization correction**

$$\delta f^Z_{P_a} \equiv \frac{16\pi^2 f^2}{2}\delta Z_{P_a} = -\frac{16\pi^2 f^2}{2}\frac{d\Sigma(p^2)}{dp^2},$$



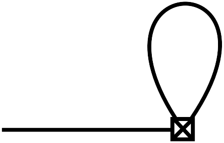
Current loop correction

- **Axial current at** $\mathcal{O}(\phi^3)$

$$A^{P,\phi^3}_{\mu,a} = -\frac{1}{96f}\mathrm{Tr}(T^aT^bT^cT^d)\Big[\mathscr{P}_{ij}\partial_\mu\phi^b_{jk}\phi^c_{kl}\phi^d_{li} - 2\mathscr{P}_{ij}\phi^b_{jk}\partial_\mu\phi^c_{kl}\phi^d_{li} + \mathscr{P}_{ij}\phi^b_{jk}\phi^c_{kl}\partial_\mu\phi^d_{li}\Big]$$

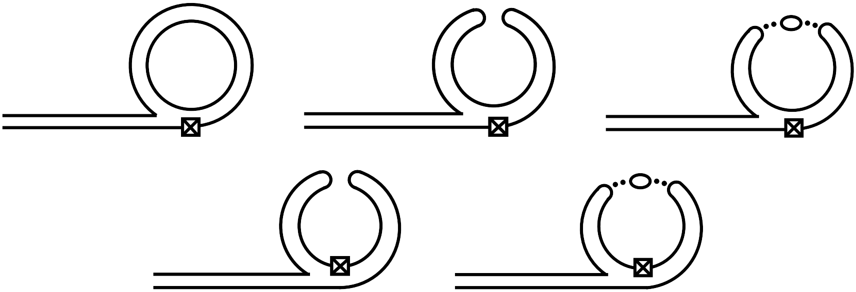
- **Loop contribution of** $A^{P,\phi^3}_{\mu,a}$

$$\mathscr{P}_{ij}\partial_\mu\phi^a_{jk}\overbrace{\phi^b_{kl}\phi^c_{li}} \quad\rightarrow\quad -ip_\mu\delta_{ta}\delta_{bc}\delta_{ix}\delta_{jy}\delta_{kx}\int\frac{d^4q}{(2\pi)^4}\langle\phi^b_{kl}\phi^b_{li}\rangle$$



Current loop correction

- **Quark flow diagram**



Current loop correction

- **Current loop correction (NLO)**

$$\delta f_{P_a}^{\text{current}} = -\frac{1}{6} \sum_b \left[\sum_Q l(Q_b) + 16\pi^2 \int \frac{d^4 q}{(2\pi)^4} \left(D^b_{xx} + D^b_{yy} - 2\theta^{ab} D^b_{xy} \right) \right]$$

  - Q : runs over six flavor combinations, xi and yi for $i \in \{u, d, s\}$
  - Q_b : squared tree-level meson mass with flavor Q and taste b
  - $\theta^{ab} \equiv \frac{1}{4} \mathrm{Tr}(T_a T_b T_a T_b)$
  - D^b_{xy} : disconnected propagator

- **NLO correction to the pion decay constant**

$$\delta f^{\text{current}} = -4 \delta f^Z$$

$$\delta f = \delta f^{\text{current}} + \delta f^Z$$

Correction to the pion decay constant

- **Correction to the pion decay constant**

$$\delta f_{P_F} = \delta f_{P_F}^{\text{con}} + \delta f_{P_F}^{\text{disc}}$$

$$\delta f_{P_F}^{\text{con}} = -\frac{1}{8} \sum_{Q,B} g_B \; l(Q_B)$$

$$\delta f_{P_F}^{\text{disc}} = -2\pi^2 \int \frac{d^4 q}{(2\pi)^4} \Big(D^I_{xx} + D^I_{yy} - 2D^I_{xy} + 4D^V_{xx} + 4D^V_{yy} - 2\Theta^{VF} D^V_{xy} + 4D^A_{xx} + 4D^A_{yy} - 2\Theta^{AF} D^A_{xy} \Big)$$

- B and F represent the $SO(4)_T$ irreps, $B, F \in \{I, V, T, A, P\}$
- $a \in F$, $\Theta^{BF} \equiv \sum_{b \in B} \theta^{ab}$, $g_B \equiv \sum_{b \in B} 1$

Analytic contributions

- **NLO terms in the SχPT Lagrangian**
  - n_{p^2} : Number of derivative pairs
  - n_m : Powers in quark mass
  - n_{a^2} : Powers in a^2

  (e.g.) $\mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma)\mathrm{Tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger)$: $(n_{p^2}, n_m, n_{a^2}) = (1,1,0)$

  ⋆ At NLO, $n_{p^2} + n_m + n_{a^2} = 2$

(n_{p^2}, n_m, n_{a^2})	Form	Comment
(2,0,0)	$\mathcal{O}(p^4)$	Do not contribute to f_π at NLO ($(\partial_\mu \phi)^4$)
(0,2,0)	$\mathcal{O}(m^2)$	Do not contribute to f_π (Constant)
(0,1,1)	$\mathcal{O}(a^2 m)$	Do not contribute to f_π (No derivatives)
(0,0,2)	$\mathcal{O}(a^4)$	Do not contribute to f_π (Constant)
(1,1,0)	$\mathcal{O}(mp^2)$	Gasser-Leutwyler terms
(1,0,1)	$\mathcal{O}(a^2 p^2)$	Sharpe and Van de Water terms [PRD, 2005]

Analytic contributions

- $\mathcal{O}(mp^2)$ **Gasser-Leutwyler terms**

$$\mathcal{L}_{\rm GL} = L_4 \mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma)\mathrm{Tr}(\chi^\dagger \Sigma + \chi \Sigma^\dagger) + L_5 \mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial_\mu \Sigma(\chi^\dagger \Sigma + \Sigma^\dagger \chi)), \quad \chi = 2\mu M$$

$$\delta f_{P_a}^{\textsf{GL}} = \frac{64}{f^2} L_4 \mu(m_u + m_d + m_s) + \frac{8}{f^2} L_5 \mu(m_x + m_y)$$

- $\mathcal{O}(a^2p^2)$ **Sharpe and Van de Water terms**

$$a^2 \mathrm{Tr}(\partial_\mu \Sigma^\dagger \xi_5 \partial_\mu \Sigma \xi_5), \; a^2 \mathrm{Tr}(\partial_\mu \Sigma^\dagger \xi_{\nu 5})\mathrm{Tr}(\partial_\mu \Sigma \xi_{5\nu}), \; \cdots$$

$$\delta f_{P_a}^{\textsf{SV}} = a^2 \mathscr{F}_a$$

where $\mathscr{F}_a$ are constants, degenerate within the lattice symmetry

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- **Analytic contributions to the pion decay constant**

$$\delta f_{P_a}^{\textsf{anly}} = \delta f_{P_a}^{\textsf{GL}} + \delta f_{P_a}^{\textsf{SV}}$$

Results in 4+4+4 theory → Results in 1+1+1 theory

- **We use replica method to formulate the results in rSχPT:**

[Bernard, Golterman and Shamir]

1. Replicate each sea-quark flavor n_r times (total $n_r N_F$ flavors)
2. Calculate with $n_r N_F$ flavors
3. Set $n_r = \frac{1}{4}$ $\Longrightarrow$ Cancels the taste d.o.f

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- **Results in 1+1+1 theory**

$$\delta f_{P_F}^{\text{con}} = -\mathbf{\frac{1}{4}} \times \frac{1}{8} \sum_{Q,B} g_B \, l(Q_B)$$

$$\delta f_{P_F}^{\text{disc}} = -2\pi^2 \int \frac{d^4 q}{(2\pi)^4} \Big(\tilde{D}^I_{xx} + \tilde{D}^I_{yy} - 2\tilde{D}^I_{xy} + 4\tilde{D}^V_{xx} + 4\tilde{D}^V_{yy} - 2\Theta^{VF}\tilde{D}^V_{xy} \quad + (V \to A) \Big)$$

$$\delta f_{P_a}^{\text{anal}} = \mathbf{\frac{1}{4}} \times \frac{64}{f^2} L_4 \mu (m_u + m_d + m_s) + \frac{8}{f^2} L_5 \mu (m_x + m_y) + a^2 \mathcal{F}_a$$

Pion decay constant

- **Performing integrals gives the decay constant**
- **Example:**
  - Pion decay constant for fully dynamical case ($xy = ud$)
  - SU(2) chiral perturbation theory ($m_u, m_d \ll m_s$)
  - 2+1 flavors ($m_u = m_d \neq m_s$)

$$f_{\pi_F} = f\Bigg\{1 + \frac{1}{32\pi^2 f^2}\Bigg[-\frac{1}{4}\sum_B g_B l(\pi_B) + (4-\Theta^{VF})\Big\{l(\pi_V) - l(\eta_V)\Big\} + (V \to A)\Bigg] + L_4\frac{16\mu}{f^2}(2m_l + m_s) + L_5\frac{16\mu}{f^2}m_l + a^2\mathcal{F}_F\Bigg\}$$

Summary and Future Work

Summary

- We calculated the pion decay constants of taste non-Goldstone pions in staggered chiral perturbation theory at NLO

Future work

- Decay constants in Mixed SχPT (AsqTad sea + HYP valence)
- Data analysis