Nucleon Eigenstates in Full QCD

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CSSM Lattice Collaboration

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Outline

- Two point correlation function:

$$G_{ij}(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}} \langle \Omega | T\{\chi_i(x)\bar{\chi}_j(0)\} | \Omega \rangle .$$

- Inserting completeness

$$\sum_{B,\vec{p}',s} |B,\vec{p}',s\rangle\langle B,\vec{p}',s| = I$$

- At $\vec{p} = 0$

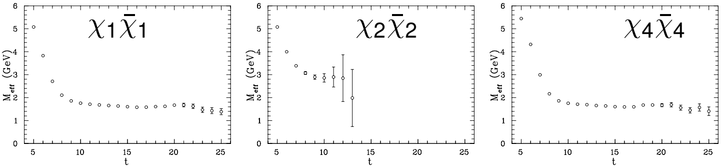
$$G^{\pm}_{ij}(t, \vec{0}) = \mathrm{Tr}_{\mathrm{sp}}[\Gamma_{\pm} G_{ij}(t, \vec{0})]$$

- Parity projection operator, $\Gamma_{\pm} = \frac{1}{2}(1 \pm \gamma_0)$
- In ensemble average, $G^{\pm}_{ij}(t) = G^{\pm}_{ji}(t)$
- $\frac{1}{2}[G^{\pm}_{ij}(t) + G^{\pm}_{ji}(t)]$ provides an improved unbiased estimator leads to use symmetric eigenvalue Eq.
- Effective mass, $M_{\mathrm{eff}}(t) = \ln\left(\frac{G(t)}{G(t+1)}\right)$

Interpolators

- Consider

$$\chi_1(x) = \epsilon^{abc}(u^{Ta}(x)\, C\gamma_5\, d^b(x))\, u^c(x)\,,$$
$$\chi_2(x) = \epsilon^{abc}(u^{Ta}(x)\, C\, d^b(x))\, \gamma_5\, u^c(x)\,,$$
$$\chi_4(x) = \epsilon^{abc}(u^{Ta}(x)\, C\gamma_5\gamma_4\, d^b(x))\, u^c(x).$$



Variational Method

- Consider N interpolating fields, then

$$\bar{\phi}^{\alpha} = \sum_{i=1}^{N} u_i^{\alpha} \bar{\chi}_i,$$

$$\phi^{\alpha} = \sum_{i=1}^{N} v_i^{\alpha} \chi_i,$$

- Then a two point correlation function matrix for $\vec{p} = 0$, right multiplied by u_j^α has the property

$$G_{ij}^{\pm}(t)\, u_j^\alpha = \Big(\sum_{\vec{x}} \mathrm{Tr}_{\mathrm{sp}}\{\Gamma_\pm \langle\Omega|\chi_i \bar{\chi}_j|\Omega\rangle\}\Big)\, u_j^\alpha$$
$$\propto e^{-m_\alpha t}.$$

(no sum over α)

- The t dependence is contained in the exponential term

- This provides a recurrence relation at time $(t_0 + \triangle t)$,

$$G_{ij}(t_0 + \triangle t)u_j^\alpha = e^{-m_\alpha \triangle t} G_{ij}(t_0)u_j^\alpha .$$

- Which leads to a right eigenvalue equation

$$[(G(t_0))^{-1} G(t_0 + \triangle t)]_{ij}\, u_j^\alpha = c^\alpha\, u_i^\alpha ,$$

- And a left eigenvalue equation

$$v_i^\alpha\, [G(t_0 + \triangle t)(G(t_0))^{-1}]_{ij} = c^\alpha\, v_j^\alpha .$$

- where $c^\alpha = e^{-m_\alpha \triangle t}$ is the eigenvalue.

- The projected correlator

$$v_i^\alpha G_{ij}^\pm(t) u_j^\alpha \equiv G_\pm^\alpha,$$

  is then analyzed to obtain masses of different states.
- Effective mass

$$M_{\text{eff}}^\alpha(t) = \ln\left(\frac{G_\pm^\alpha(t)}{G_\pm^\alpha(t+1)}\right).$$

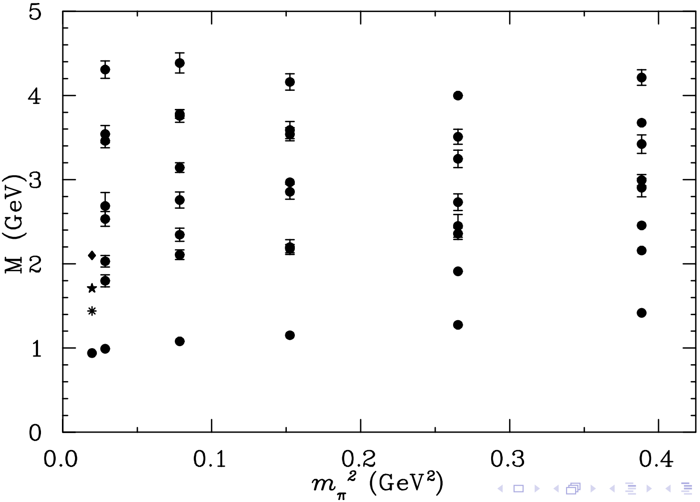
Simulation Details

- PACS-CS Collaboration: S. Aoki, et al., Phys. Rev. **D79** (2009) 034503.
- $2 + 1$ flavour dynamical-fermion QCD
- Lattice volume: $32^3 \times 64$
- $\beta = 1.9$ providing $a = 0.0907$ fm
- $K_{ud} = \{ 0.13700, 0.13727, 0.13754, 0.13770, 0.13781 \}$
- $K_s = 0.13640$
- Lightest pion mass is 156 MeV

- We use fermion source smearing, $N = \{16, 35, 100, 200\}$

Quark-mass flow of eigenstates

$N\frac{1}{2}^{+}$ states, 8×8 CM of χ_1, χ_2



Quark-mass flow of eigenstates

- M interpolating fields making an $M \times M$ correlation matrix $G(t)$
- We determine $\vec{u}^{\alpha}$ of $[(G(t_0))^{-1}\, G(t_0 + \triangle t)]$
- Matrix $[(G(t_0))^{-1}\, G(t_0 + \triangle t)]$ is not symmetric. So $\vec{u}^{\alpha}$ are not orthogonal
- We explore the extent to which the eigenvectors $\vec{u}^{\alpha}(m_q)$ are orthogonal, by $\vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_q)$
- By construction, $\vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_q)$ is 1 for $\alpha = \beta$.

$\vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_q)$

Table: The scalar product $\vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_q)$ for $\kappa = 0.13700$, for 8×8 correlation matrix of χ_1 and χ_2.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	0.02	-0.18	0.65	-0.07	0.10	-0.32	-0.09
0.02	**1.00**	0.02	0.07	0.15	0.06	0.42	0.03
-0.18	0.02	**1.00**	-0.10	0.36	-0.49	0.06	0.39
0.65	0.07	-0.10	**1.00**	-0.03	0.15	-0.57	-0.13
-0.07	0.15	0.36	-0.03	**1.00**	0.23	0.09	0.30
0.10	0.06	-0.49	0.15	0.23	**1.00**	-0.06	-0.61
-0.32	0.42	0.06	-0.57	0.09	-0.06	**1.00**	0.17
-0.09	0.03	0.39	-0.13	0.30	-0.61	0.17	**1.00**

Quark-mass flow of eigenstates continued · · ·

- This feature enables the use of the generalised measure

$$\mathcal{U}^{\alpha\beta}(m_q, m_{q'}) = \vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_{q'})$$

- Can be used to identify the states most closely related as we move from quark mass m_q to adjacent quark mass $m_{q'}$.

$\mathcal{U}^{\alpha\beta}(m_q, m_{q'})$

Table: $\vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_{q'})$, $\kappa = 0.13700$, $\kappa' = 0.13727$.

$\alpha \downarrow$	$\beta \longrightarrow$						
0.98	-0.29	-0.14	0.63	-0.07	0.10	-0.32	-0.08
-0.19	**-0.92**	0.08	-0.03	0.14	0.06	0.42	0.05
-0.16	0.07	**0.99**	-0.09	-0.04	-0.53	0.09	0.36
0.63	-0.44	-0.02	**0.99**	-0.05	0.13	-0.55	-0.12
-0.12	-0.11	0.40	0.00	**0.75**	0.00	0.08	0.36
0.05	-0.11	-0.42	0.17	0.76	**0.95**	-0.12	-0.53
-0.45	-0.17	0.03	-0.67	0.08	-0.05	**1.00**	0.18
-0.09	0.00	0.34	-0.14	-0.34	-0.82	0.21	**1.00**

Symmetric Correlation Matrix

- Recall, $G^{-1}(t_0)\, G(t_0 + \triangle t)\, |u_i\rangle \;=\; \lambda_i\, |u_i\rangle$
- Also, $G^{-1/2}(t_0)\, G^{+1/2}(t_0) \;=\; I$
- So, $G^{-1}(t_0)\, G(t_0 + \triangle t)\, G^{-1/2}(t_0)\, G^{+1/2}(t_0)\, |\, u_i\rangle \;=\; \lambda_i\, |u_i\rangle$
- Multiplying from the left by $G^{+1/2}(t_0)$ provides

$$G^{-1/2}(t_0) G(t_0 + \triangle t) G^{-1/2}(t_0) G^{+1/2}(t_0) \quad |u_i\rangle = \lambda_i G^{+1/2}(t_0) \quad |u_i\rangle$$

$$G^{-1/2}(t_0) G(t_0 + \triangle t) G^{-1/2}(t_0) \quad |w_i\rangle = \lambda_i \quad |w_i\rangle$$

where, $|w_i\rangle \;=\; G^{+1/2}(t_0)\, |u_i\rangle$

Symmetric Correlation Matrix

- $[G^{-1/2}(t_0)\, G(t_0 + \triangle t)\, G^{-1/2}(t_0)]$ is a real symmetric matrix with the same eigenvalue λ_i as before
- $\vec{w}^{\alpha}$ are orthogonal
- As in before, a scalar product of

$$\mathcal{W}^{\alpha\beta}(m_q, m_{q'}) = \vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'})$$

is constructed.

$\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_q)$

Table: $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_q)$ for $\kappa = 0.13700$ and for an 8×8 "symmetric" correlation matrix of χ_1 and χ_2.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	**1.00**	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	**1.00**	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	**1.00**	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	**1.00**	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	**1.00**	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	**1.00**	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	**1.00**

$\mathcal{W}^{\alpha\beta}(m_q, m_{q'})$

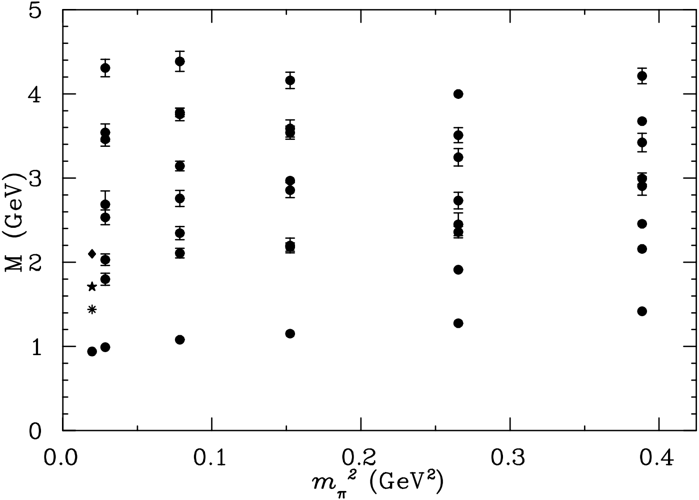
Table: $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'})$, $\kappa = 0.13700$, $\kappa' = 0.13727$.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	-0.09	0.00	0.00	0.01	0.00	0.01	0.00
0.09	**0.99**	-0.07	0.13	-0.01	0.00	0.01	0.00
0.01	0.07	**1.00**	-0.01	0.00	-0.01	0.00	0.00
-0.01	-0.13	0.02	**0.98**	-0.09	0.02	0.07	0.00
0.01	0.01	0.00	-0.09	**-0.97**	0.21	-0.01	0.03
0.00	0.00	0.01	0.00	0.20	**0.95**	-0.07	-0.23
-0.01	0.00	0.00	-0.07	0.01	0.07	**0.99**	-0.01
0.00	0.00	0.00	-0.01	-0.08	-0.21	0.01	**-0.97**

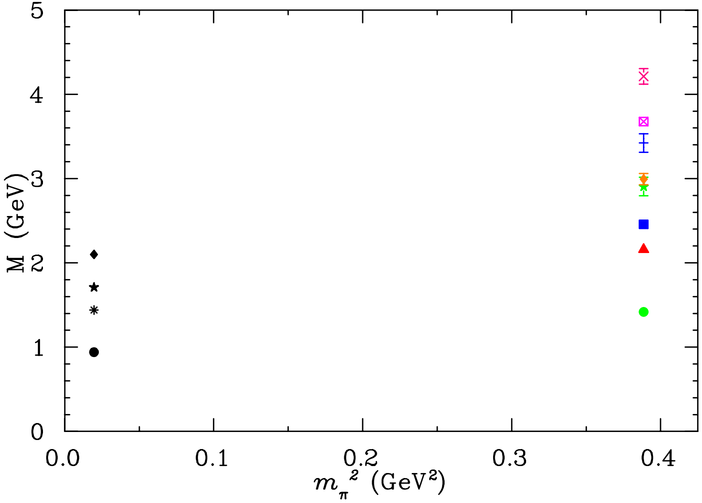
$\mathcal{U}^{\alpha\beta}(m_q, m_{q'})$

Table: $\vec{u}^{\alpha}(m_q) \cdot \vec{u}^{\beta}(m_{q'})$, $\kappa = 0.13700$, $\kappa' = 0.13727$.

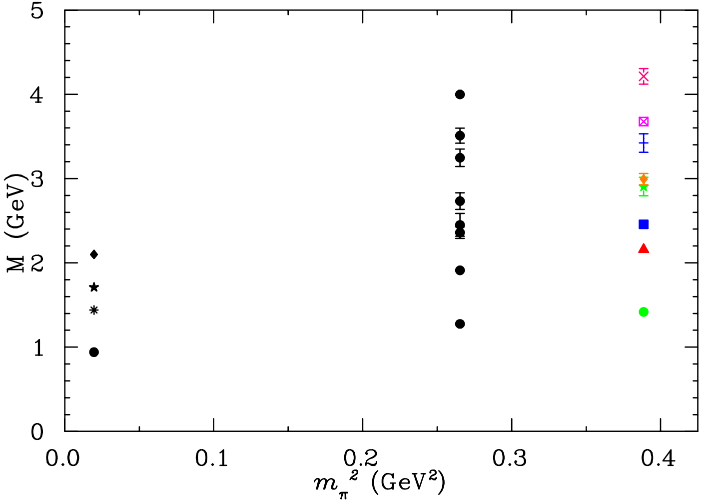
$\alpha \downarrow$	$\beta \longrightarrow$						
0.98	-0.29	-0.14	0.63	-0.07	0.10	-0.32	-0.08
-0.19	**-0.92**	0.08	-0.03	0.14	0.06	0.42	0.05
-0.16	0.07	**0.99**	-0.09	-0.04	-0.53	0.09	0.36
0.63	-0.44	-0.02	**0.99**	-0.05	0.13	-0.55	-0.12
-0.12	-0.11	0.40	0.00	**0.75**	0.00	0.08	0.36
0.05	-0.11	-0.42	0.17	0.76	**0.95**	-0.12	-0.53
-0.45	-0.17	0.03	-0.67	0.08	-0.05	**1.00**	0.18
-0.09	0.00	0.34	-0.14	-0.34	-0.82	0.21	**1.00**


5
4
3
2
1
0
M (GeV)
0.0
0.1
0.2
0.3
0.4
m_π^2 (GeV²)

Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors

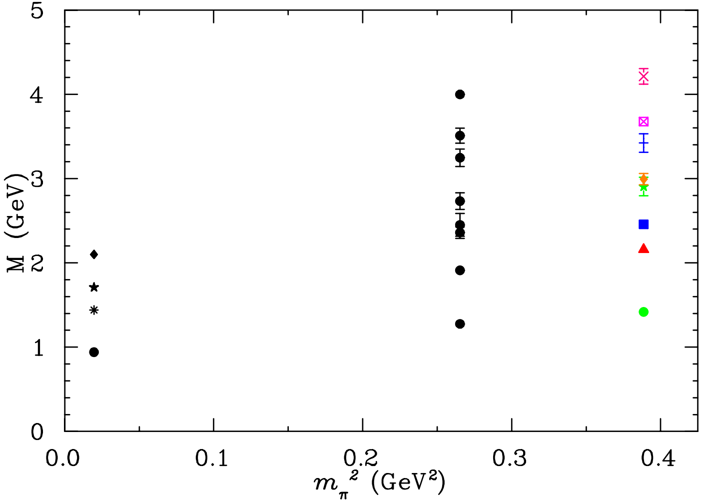


$\mathcal{W}^{\alpha\beta}(m_q, m_{q'})$

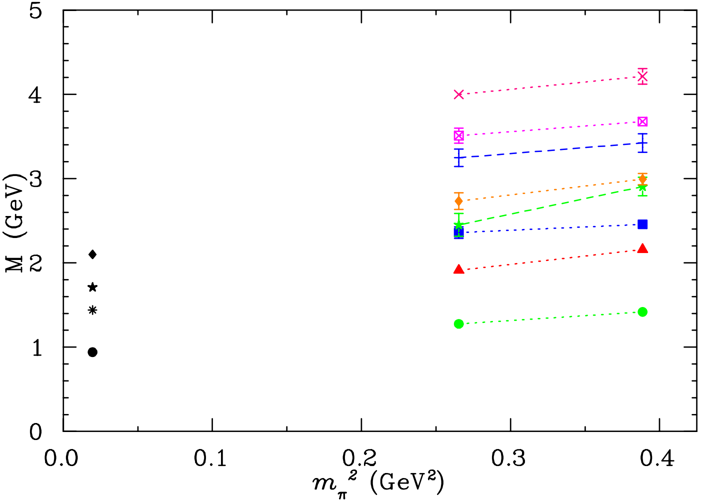
Table: $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'})$, $\kappa = 0.13700$, $\kappa' = 0.13727$.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	-0.09	0.00	0.00	0.01	0.00	0.01	0.00
0.09	**0.99**	-0.07	0.13	-0.01	0.00	0.01	0.00
0.01	0.07	**1.00**	-0.01	0.00	-0.01	0.00	0.00
-0.01	-0.13	0.02	**0.98**	-0.09	0.02	0.07	0.00
0.01	0.01	0.00	-0.09	**-0.97**	0.21	-0.01	0.03
0.00	0.00	0.01	0.00	0.20	**0.95**	-0.07	-0.23
-0.01	0.00	0.00	-0.07	0.01	0.07	**0.99**	-0.01
0.00	0.00	0.00	-0.01	-0.08	-0.21	0.01	**-0.97**

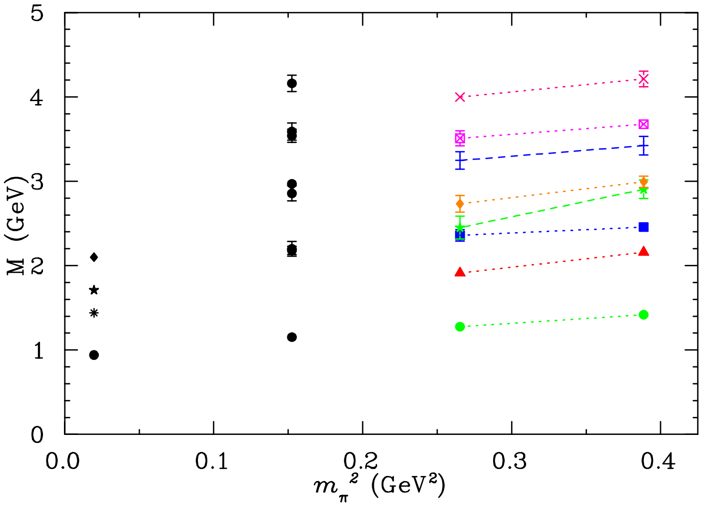
Tracking of $N\frac{1}{2}^+$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors

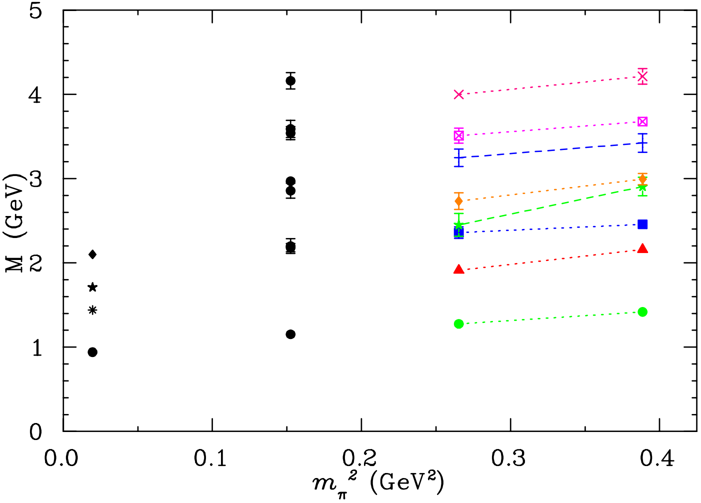


$\mathcal{W}^{\alpha\beta}(m_q, m_{q'})$

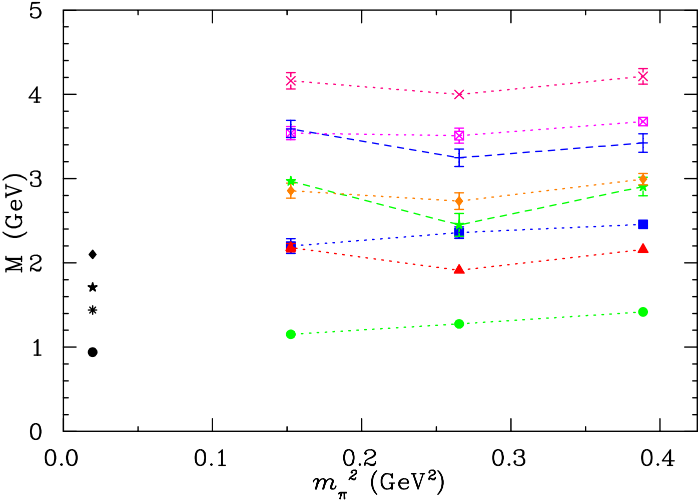
Table: $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'})$, $\kappa = 0.13727$, $\kappa' = 0.13754$.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	-0.08	0.01	-0.01	0.01	0.01	0.00	0.00
0.08	**0.98**	0.12	-0.03	0.09	0.01	0.00	0.00
-0.02	-0.12	**0.99**	-0.08	0.00	0.00	0.00	-0.01
-0.01	-0.09	-0.01	0.03	**0.99**	-0.10	0.00	0.00
0.01	0.02	0.08	**0.99**	-0.02	0.01	0.07	0.05
0.00	0.00	-0.01	-0.08	0.00	-0.08	**0.99**	0.07
0.01	0.02	0.00	0.01	-0.10	**-0.99**	-0.08	0.03
0.00	0.00	0.00	-0.04	0.00	0.03	-0.08	**1.00**

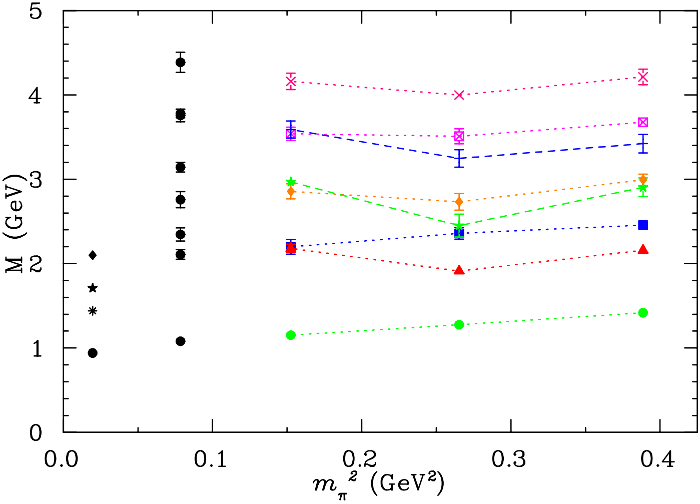
Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors

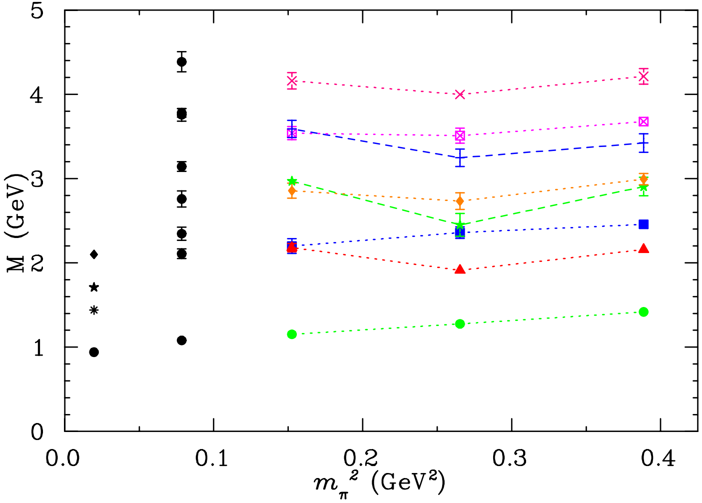


$\mathcal{W}^{\alpha\beta}(m_q, m_{q'})$

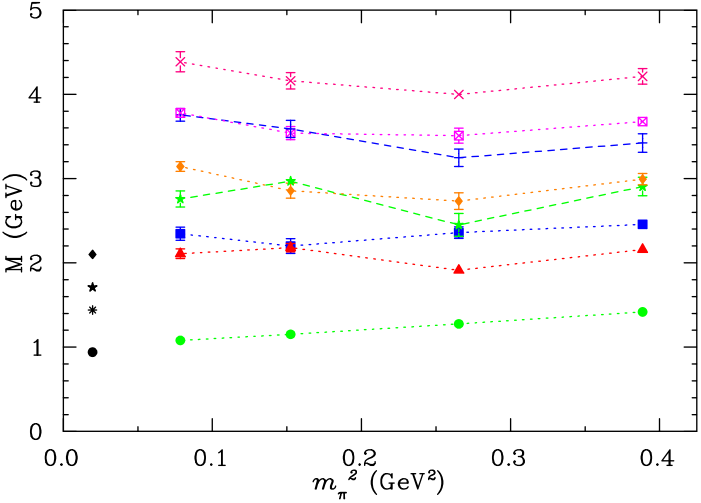
Table: $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'})$, $\kappa = 0.13754$, $\kappa' = 0.13770$.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	-0.04	-0.02	0.04	0.01	0.00	0.00	0.00
0.03	**0.98**	-0.21	0.04	-0.01	0.00	-0.03	0.00
0.02	0.21	**0.97**	0.01	0.14	0.04	-0.02	-0.04
0.01	-0.01	-0.13	-0.37	**0.92**	0.08	-0.03	-0.03
-0.04	-0.04	-0.05	**0.93**	0.36	0.02	0.00	-0.01
0.00	-0.03	-0.01	0.01	-0.01	-0.25	**-0.97**	-0.03
0.00	-0.01	-0.04	0.01	-0.10	**0.95**	-0.24	-0.16
0.00	-0.01	-0.02	0.00	-0.02	-0.15	0.07	**-0.99**

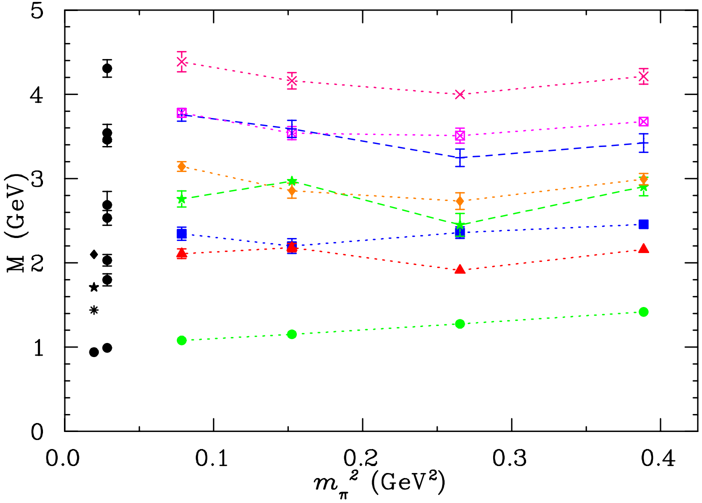
Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors

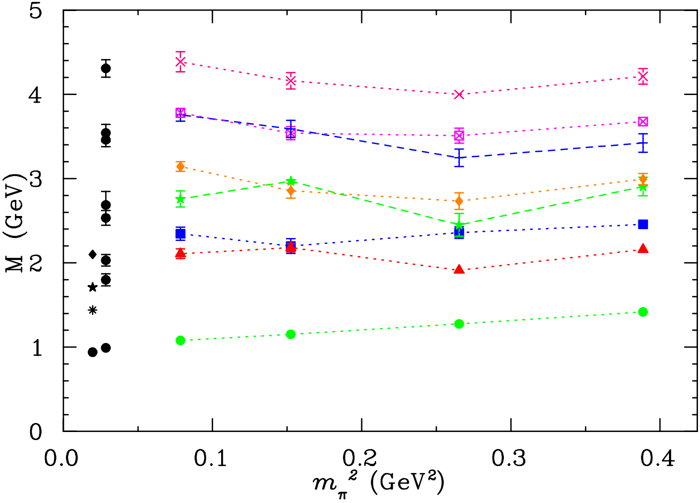


$\mathcal{W}^{\alpha\beta}(m_q, m_{q'})$

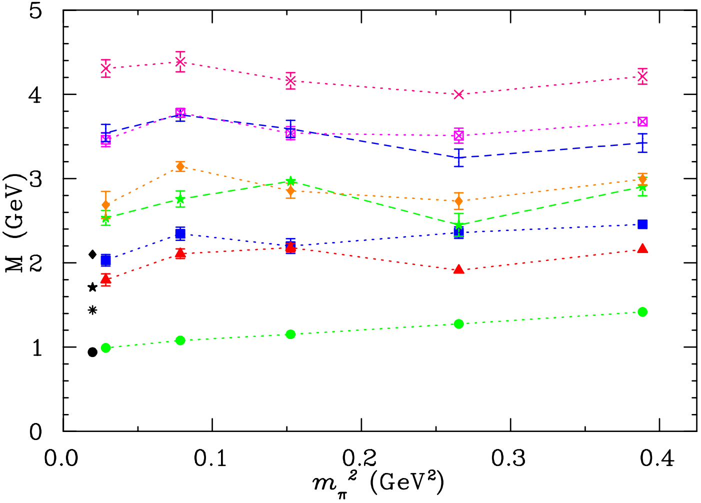
Table: $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'})$, $\kappa = 0.13770$, $\kappa' = 0.13781$.

$\alpha \downarrow$	$\beta \longrightarrow$						
1.00	-0.04	0.03	-0.02	0.01	0.00	0.01	0.00
0.03	**0.97**	0.25	0.06	-0.02	-0.01	-0.01	-0.01
-0.03	-0.24	**0.94**	-0.07	-0.21	0.00	-0.03	-0.02
0.02	-0.06	-0.03	**0.93**	-0.36	-0.06	0.02	0.00
-0.01	-0.05	0.20	0.35	**0.89**	-0.03	-0.21	-0.01
-0.01	-0.01	0.06	0.04	0.18	-0.23	**0.93**	-0.20
0.01	0.00	-0.02	-0.08	-0.04	**-0.97**	-0.22	0.05
0.01	0.00	-0.04	-0.02	-0.04	0.00	-0.20	**-0.98**

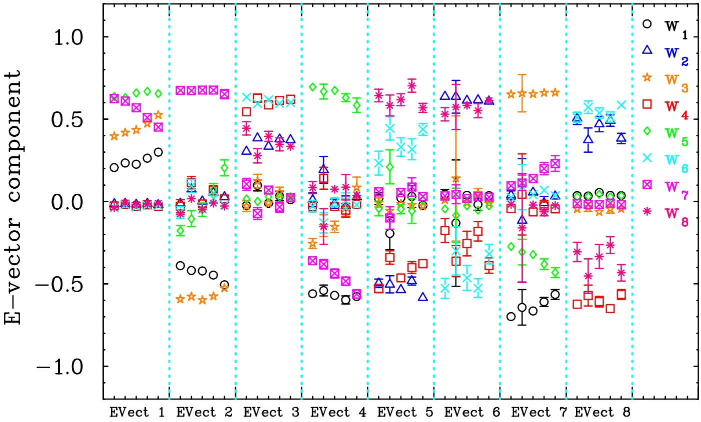
Tracking of $N\frac{1}{2}^{+}$ eigenvectors



Tracking of $N\frac{1}{2}^{+}$ eigenvectors



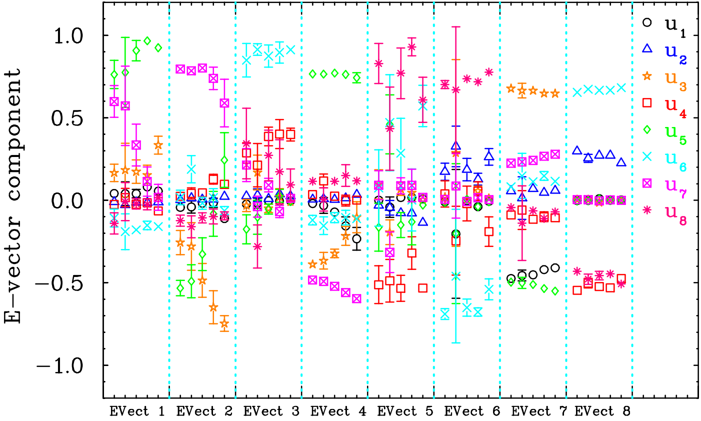
Eigenvectors, $|w_i\rangle$



Observations: $|w_i\rangle$

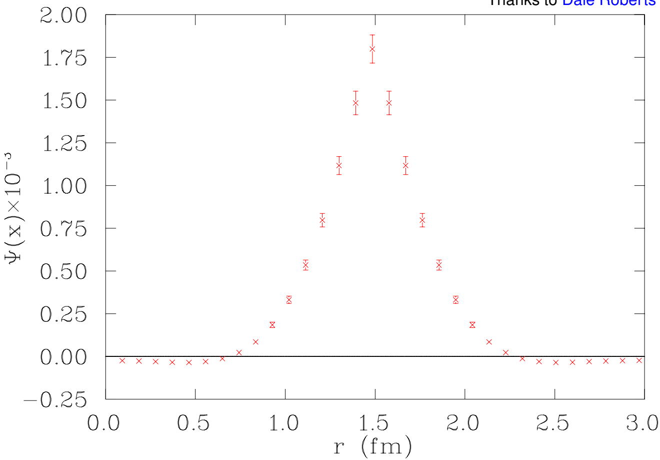
- The variations of eigenvectors is smooth and slowly varying.
- The tracking of eigenvectors is robust, with $\vec{w}^{\alpha}(m_q) \cdot \vec{w}^{\beta}(m_{q'}) \sim 1$.
- Each eivencector has unique flow

Eigenvectors, $|\,u_i\,\rangle \;=\; G^{-1/2}(t_0)\,|w_i\rangle$



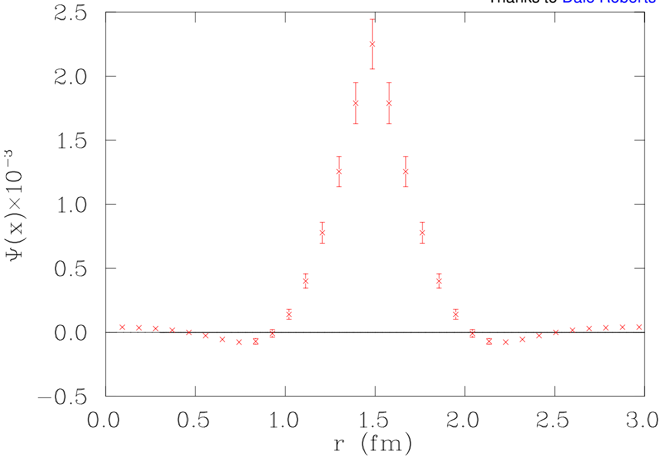
Roper wave function

Thanks to Dale Roberts



3rd excited-state wave function

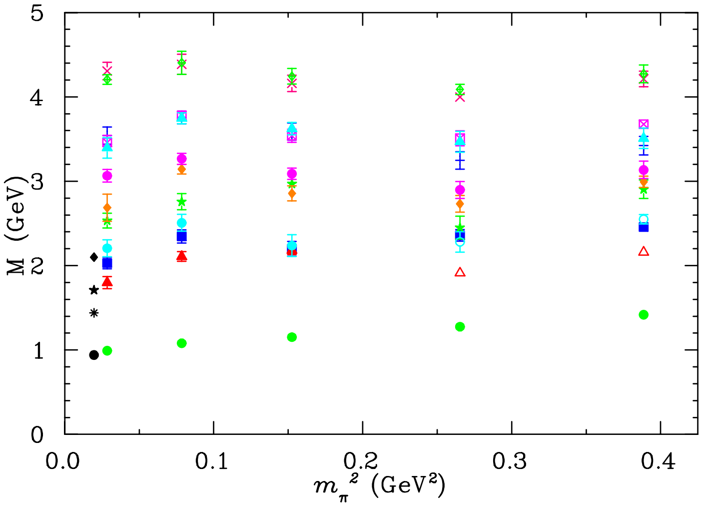
Thanks to Dale Roberts



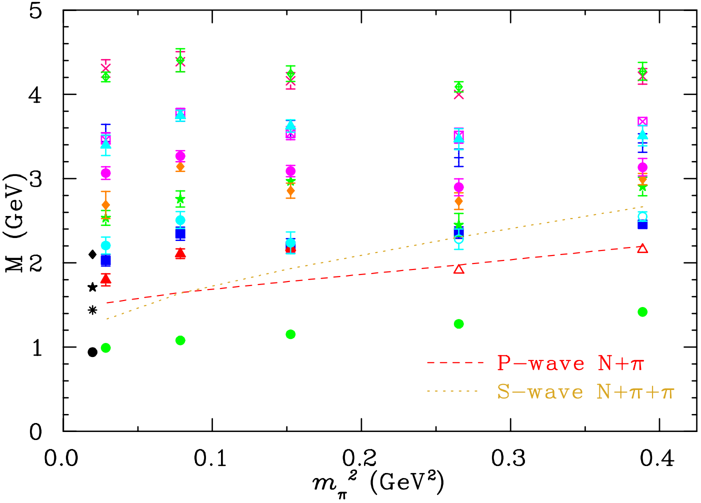
Observations: $|u_i\rangle$

- Ground and first excited states are dominated by χ_1 interpolator
- The second excited state is χ_2 dominated
- As the quark mass become lighter, the larger source and sink smearings contribution increase
- Superposition of Gaussian sources with various widths resemble the wave functions of the states. For instance, Roper state has a single node.

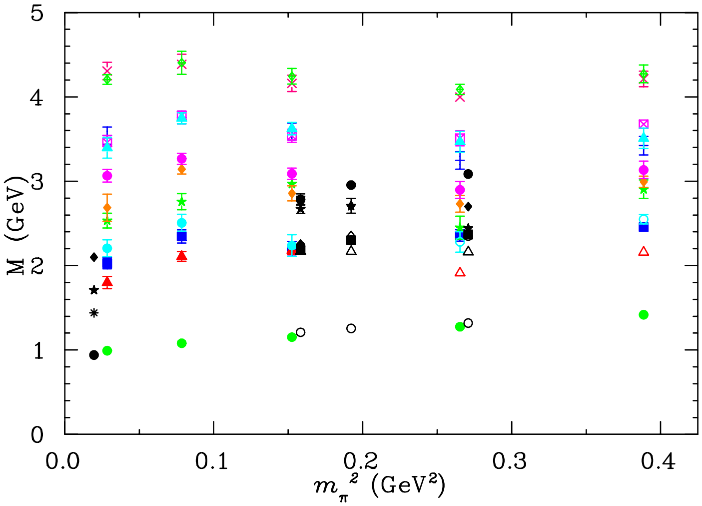
m_π^2 dependence of the N^+ Spectrum



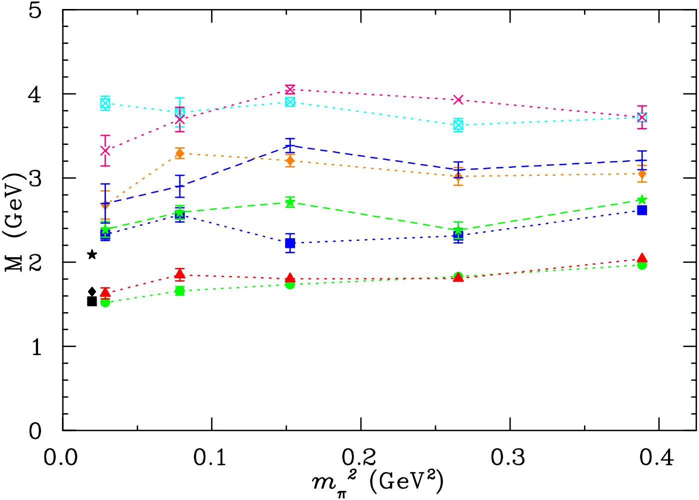
N^+ Spectrum: S and P-wave $N\pi$ thresholds



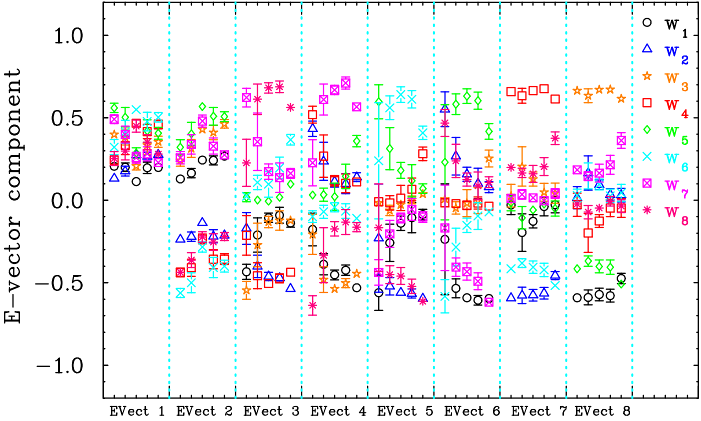
N^+ Spectrum: with HSC [PRD84(2011)074508]



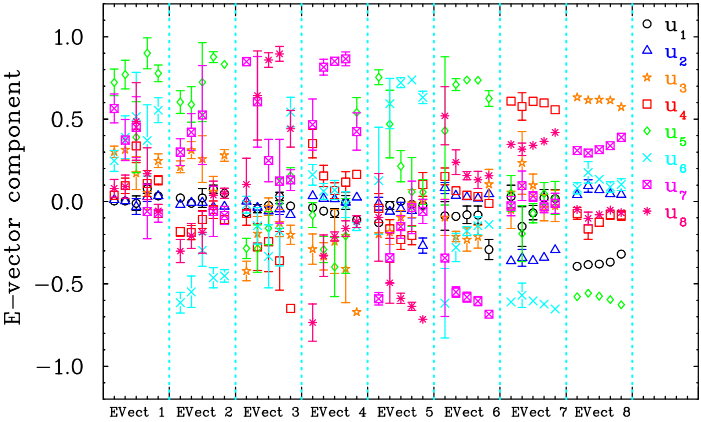
Tracking of the $N\frac{1}{2}^-$ eigenvectors



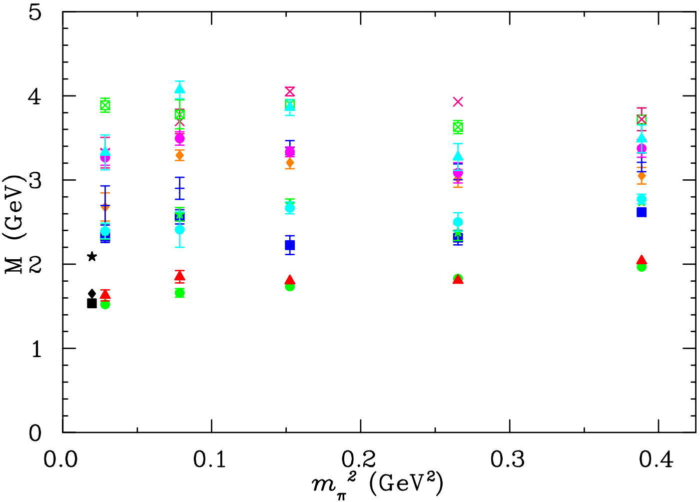
Eigenvectors, $|w_i\rangle$



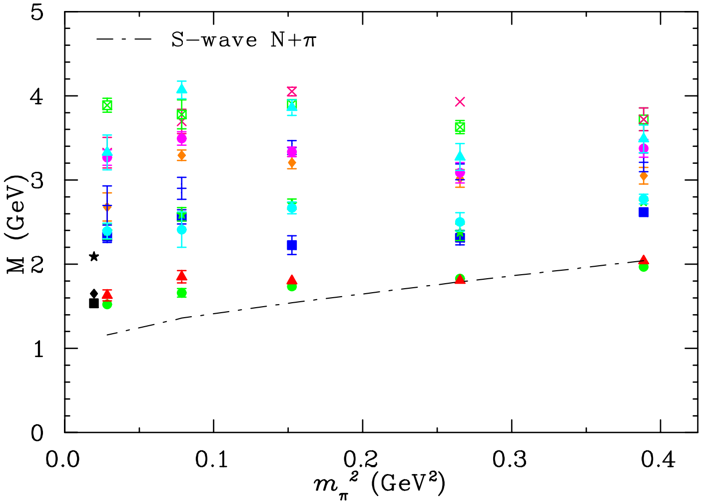
Eigenvectors, $|\, u_i \,\rangle \;=\; G^{-1/2}(t_0)\,|w_i\rangle$



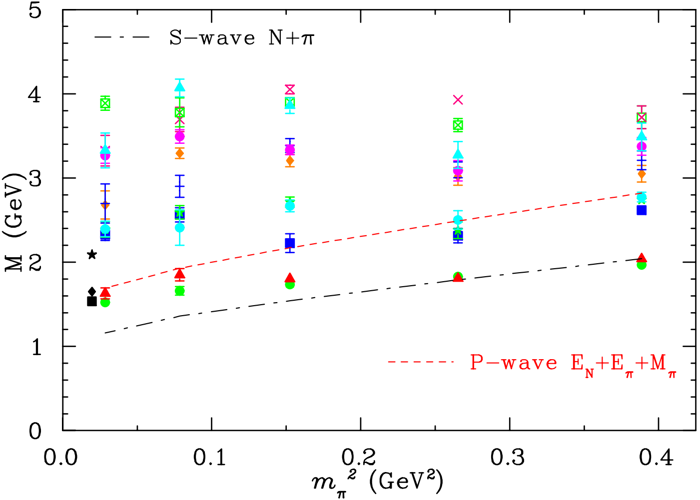
m_π^2 dependence of the $N\frac{1}{2}^-$ Spectrum



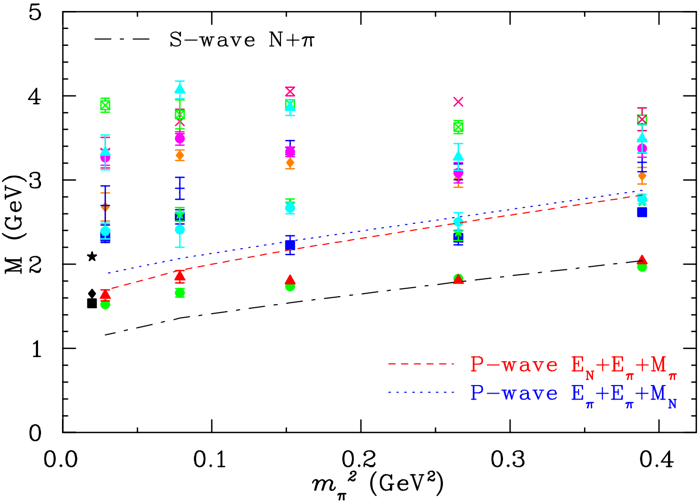
$N\frac{1}{2}^-$ Spectrum: S-wave $N\pi$ threshold



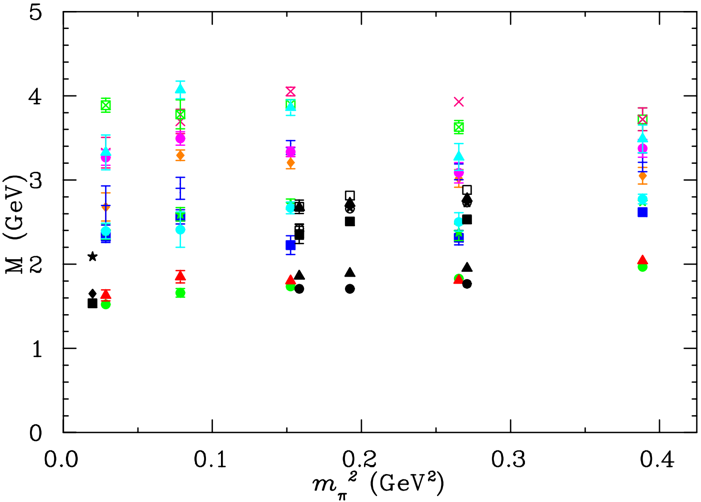
$N\frac{1}{2}^-$ Spectrum: S and P-wave $N\pi$ thresholds



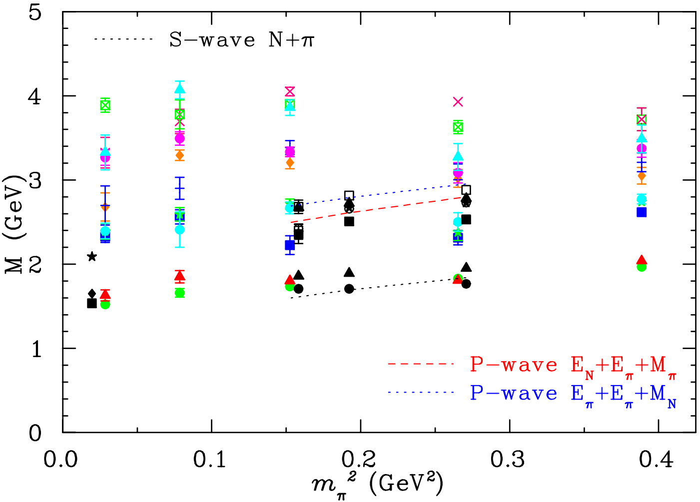
$N\frac{1}{2}^-$ Spectrum: S and P-wave $N\pi$ thresholds



$N\frac{1}{2}^-$ Spectrum: with HSC [PRD84(2011)074508]



$N\frac{1}{2}^-$ Spectrum: with HSC [PRD84(2011)074508]



Summary

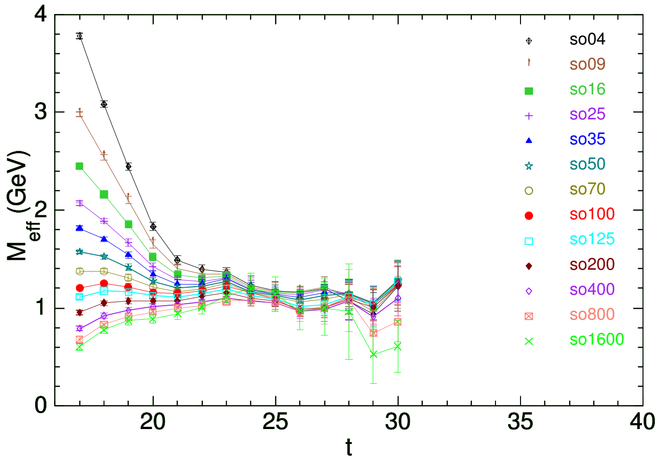
- Using correlation matrix method, propagation of states from heavy to light quark mass region can be tracked using eigenvectors.
- Method can be used to track states that are involved in an avoided level crossing
- States must be tracked using eigenvectors $\vec{w}^{\alpha}$ of symmetric eigenvalue equation.

Summary continued...

- The approach of the extracted two lowest $N\frac{1}{2}^{-}$ energy-states to the physical value is remarkable.
- Challenges remain to link the extracted lattice $N\frac{1}{2}^{+}$ excitations to its physical value.
- Doing finite volume analysis and multi-particle interpolators are needed to understand scattering states.

Smeared Source - Point Sink Effective Masses

For second lightest quark mass and 50 configurations



An illustration of avoided level crossing