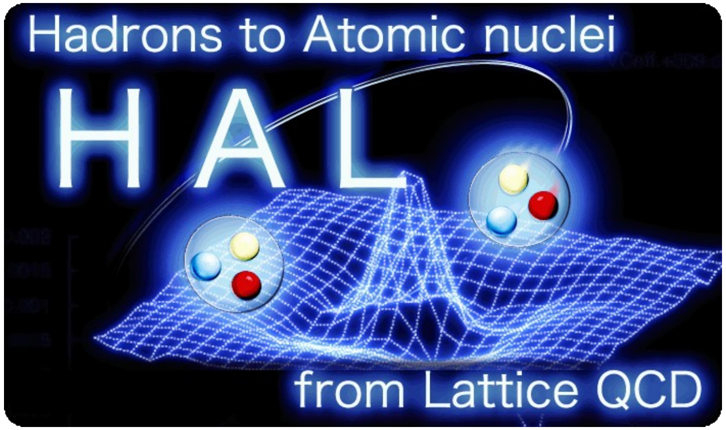
Quark mass dependence of hyperonic interactions from lattice QCD

Kenji Sasaki (*University of Tsukuba*)

for HAL QCD collaboration



HAL (Hadrons to Atomic nuclei from Lattice) QCD Collaboration

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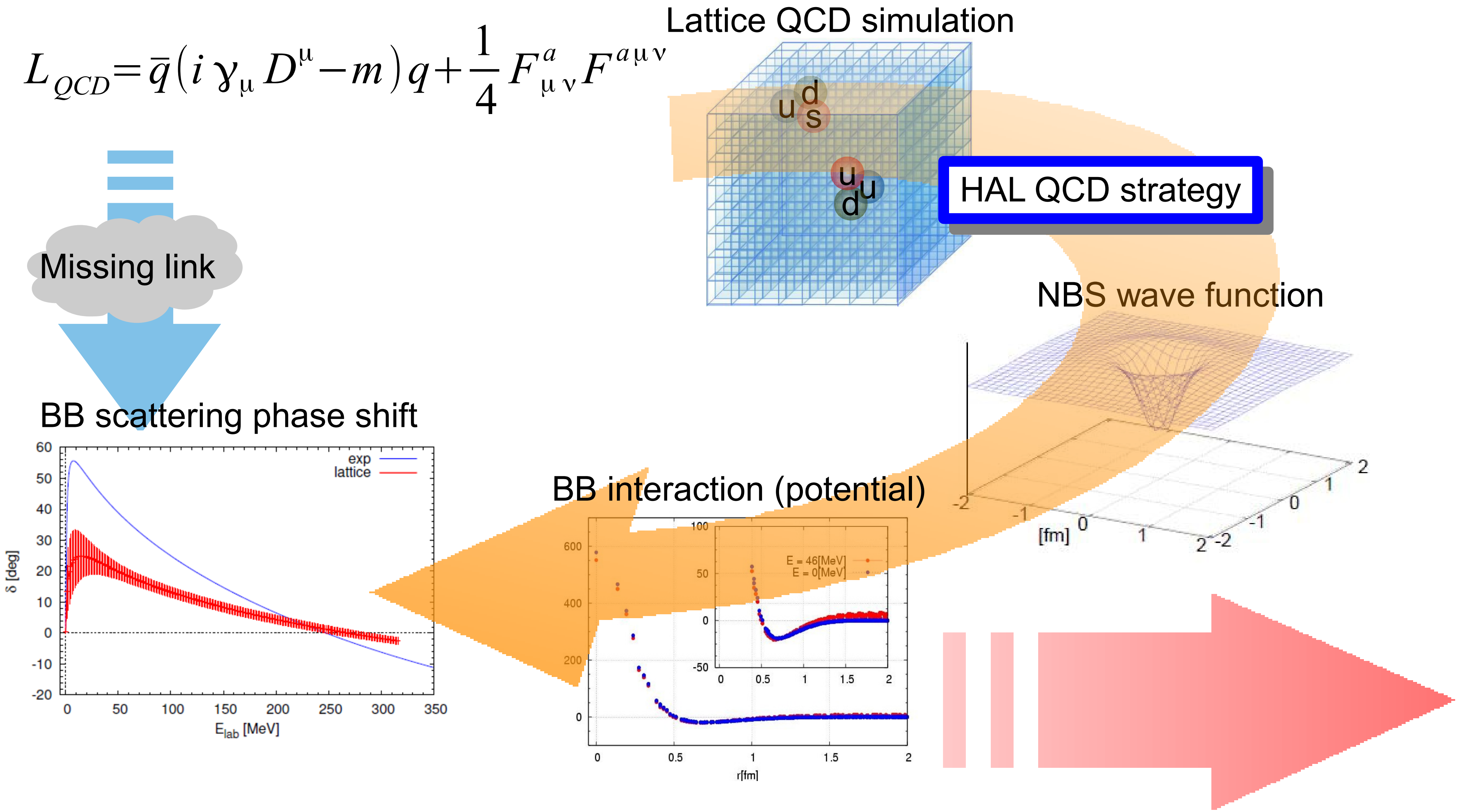
Introduction

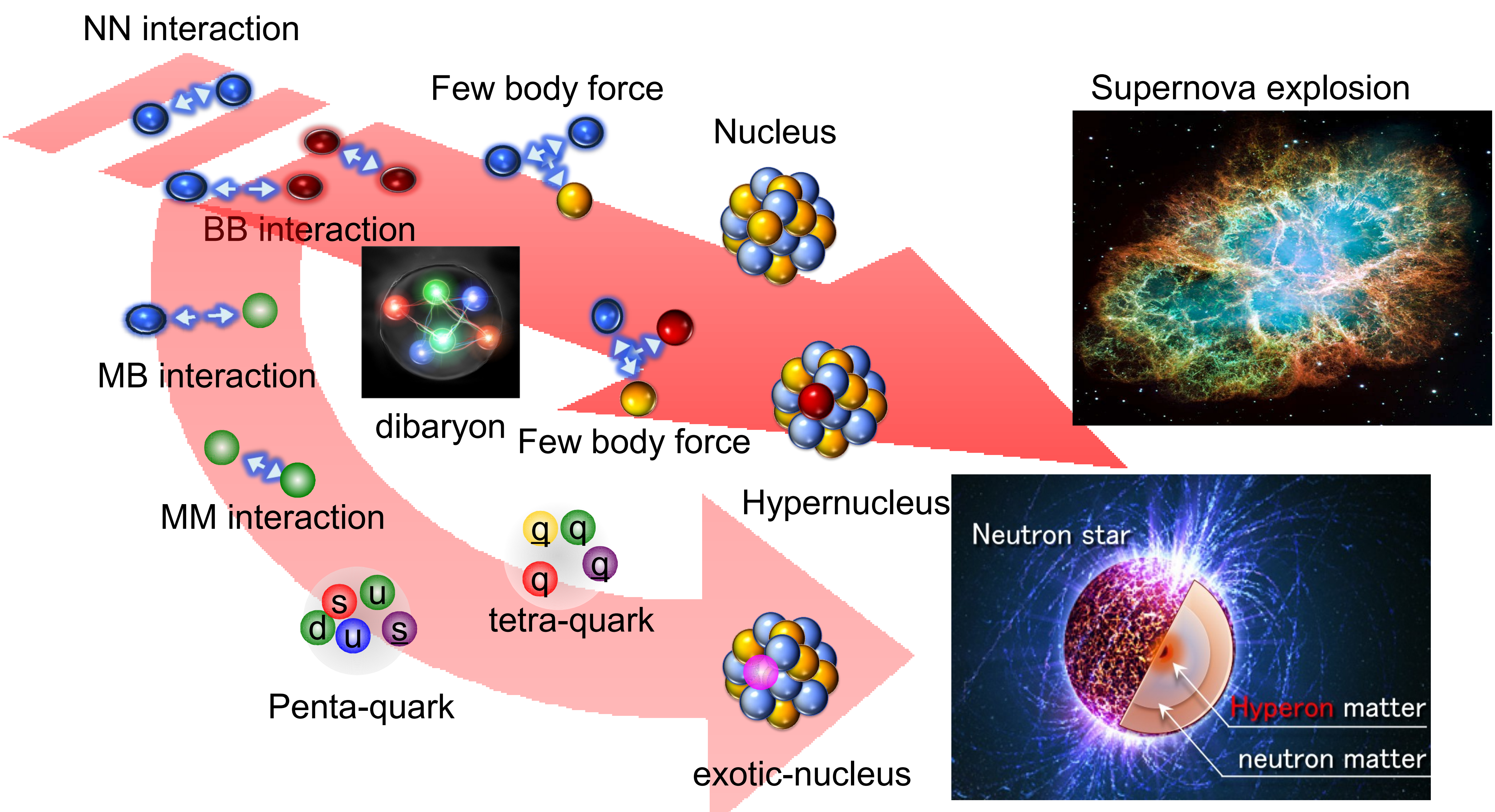
Lattice QCD simulation connects the fundamental QCD with nuclear physics

$$L_{QCD}=\bar{q}(i\gamma_\mu D^\mu - m)q + \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$



Introduction

Prospects



Introduction

Study the hyperon-nucleon (YN) and hyperon-hyperon (YY) interactions

Impact on the (hyper-) nuclear physics and astrophysics

Theory

Phenomenological model

Shortage of scattering data

Lattice QCD simulation

High performance for more data

Massively parallel super computer

Experiment

Difficult to perform collision experiment

Collision data are scarce

More intensity for more data

Huge experimental facility

Complete knowledge of generalized BB interaction

Aim of this work

Study of baryon-baryon interactions with strangeness S=-2

- Quark mass dependence of BB potential
- The SU(3) breaking effects in BB interaction.
  - Structures of double-Λ hypernuelei and Ξ-hypernuclei.
  - Fate of “H-dibaryon“ at physical point.

Recent Lattice QCD studies

- HAL QCD: SU(3) limit

  BE = 30MeV mπ = 470MeV

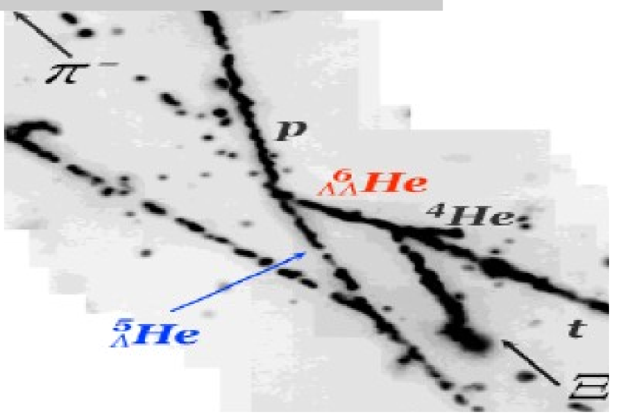
- NPLQCD: SU(3) breaking

  BE = 13MeV mπ = 390MeV

Conclusions of the “NAGARA Event”

K.Nakazawa and KEK-E176 & E373 collaborators

Λ–N attraction
Λ–Λ weak attraction
$m_H \geq 2m_\Lambda$ - 6.9MeV

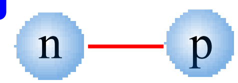


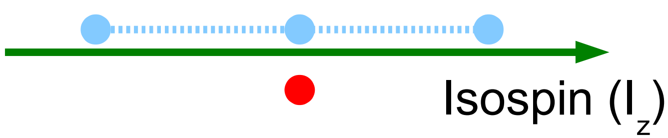
What happens on the physical point?

SU(3) extension of B-B interaction

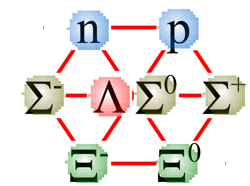
Two flavor (u, d) world

$2 \times 2 = 1 + 3$





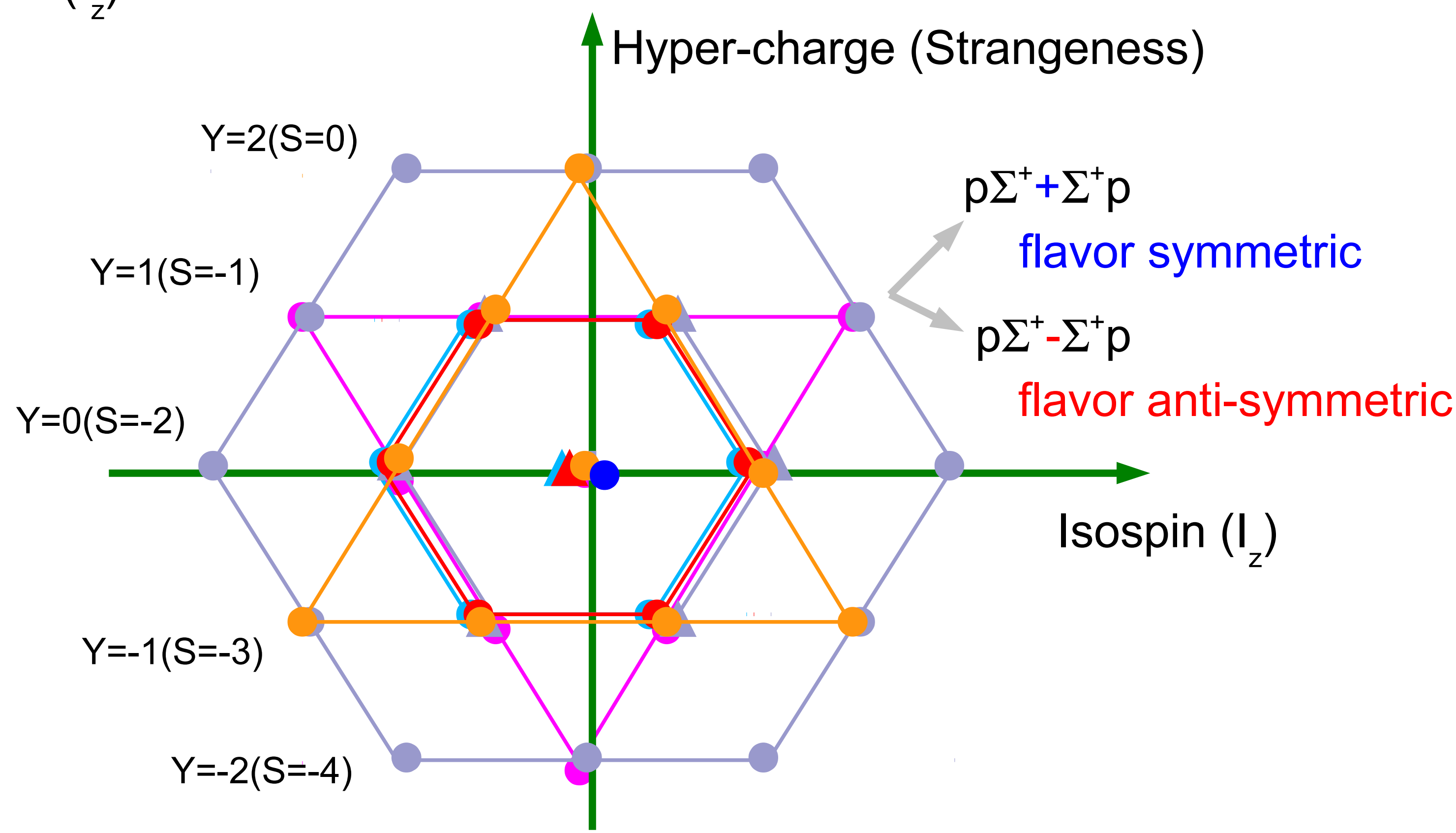
Three flavor (u, d, s) world



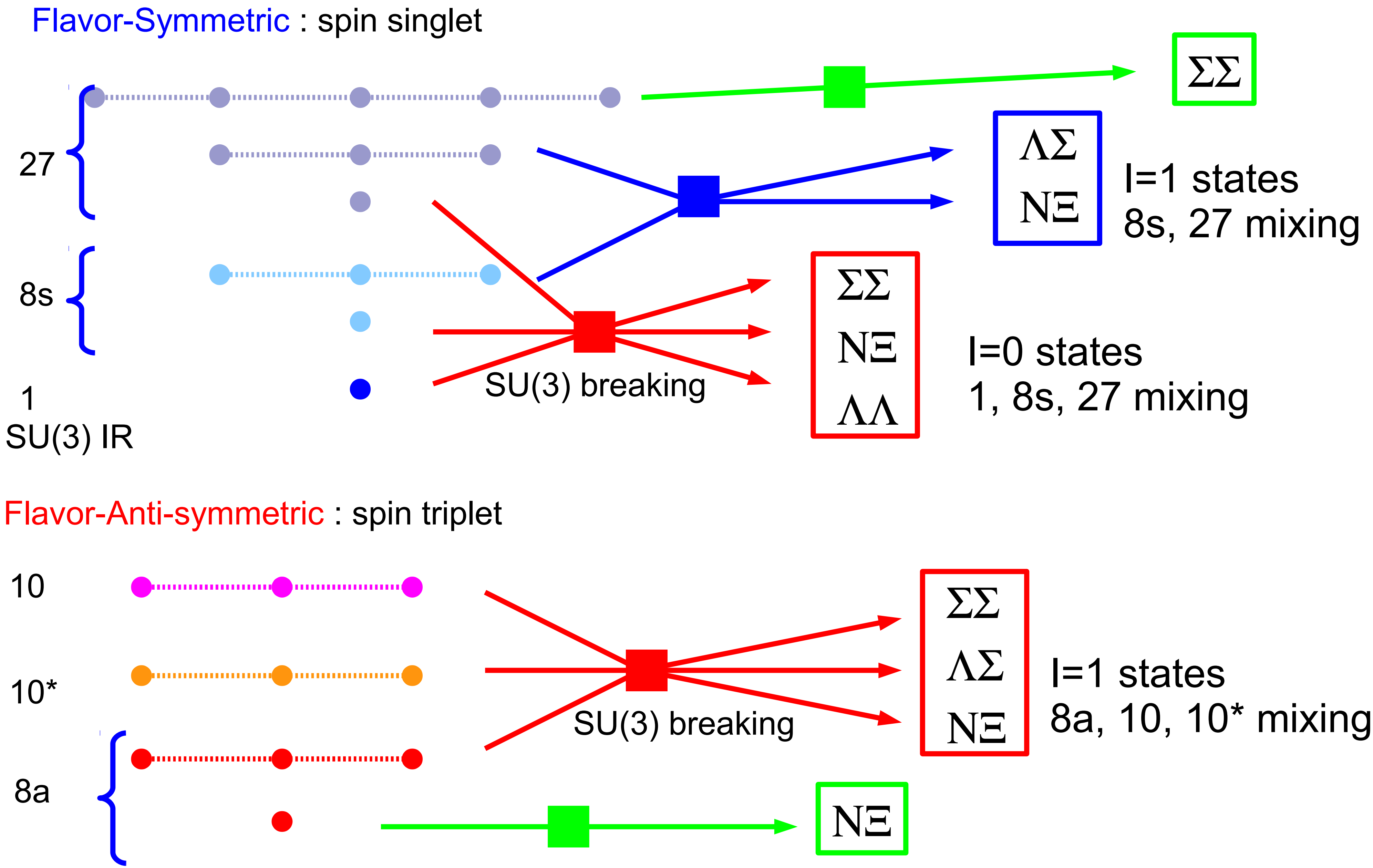
$8 \times 8 = 1 + 8_S + 27 + 8_A + 10 + 10^*$

$1 + 8_S + 27$: Flavor symmetric, Spin singlet

$8_A + 10 + 10^*$: Flavor anti-symmetric, Spin triplet



Classification of B-B states with S=-2



Coupled channel Schrödinger equation

The region inside the interaction range

Two-channel coupling case

In the *leading order of velocity expansion* of non-local potential,

Coupled channel Schrödinger equation.

$$\left(\frac{p_\alpha^2}{2\mu_\alpha}+\frac{\nabla^2}{2\mu_\alpha}\right)\psi^\alpha(\vec{x},E)=V^\alpha{}_\alpha(\vec{x})\psi^\alpha(\vec{x},E)+V^\alpha{}_\beta(\vec{x})\psi^\beta(\vec{x},E)$$

We replace ψ to R defined below

$$R_\alpha(\vec{x},E) \equiv \frac{A\Psi_\alpha(\vec{x},E)e^{-Et}}{e^{-m_A t}e^{-m_B t}} \propto \exp\left(-\frac{p_\alpha^2}{2\mu_\alpha}t\right)$$

The asymptotic momentum are given as the time derivative of R

$$\partial_t R_\alpha(\vec{x},E) = -\frac{p_\alpha^2}{2\mu_\alpha}R_\alpha(\vec{x},E)$$

$$\begin{pmatrix}(\partial_t+\nabla^2)R^\alpha(\vec{r},E)\\(\partial_t+\nabla^2)R^\beta(\vec{r},E)\end{pmatrix}=\begin{pmatrix}U^\alpha{}_\alpha(\vec{r}) & U^\alpha{}_\beta(\vec{r})\\U^\beta{}_\alpha(\vec{r}) & U^\beta{}_\beta(\vec{r})\end{pmatrix}\begin{pmatrix}R^\alpha(\vec{r},E)\\R^\beta(\vec{r},E)\end{pmatrix}$$

$$U^\delta{}_\epsilon\equiv 2\mu_\delta V^\delta{}_\epsilon$$

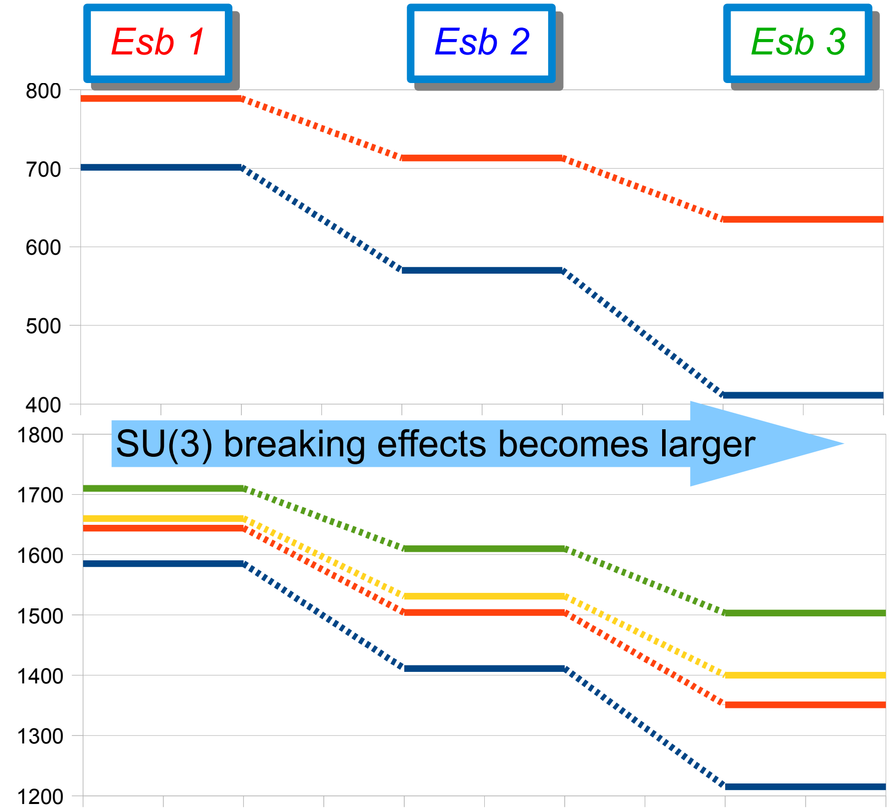
Here, unknown variables are transition potentials having 4 components without considering the hermiticity of transition potentials.

Numerical setup

- 2+1 flavor gauge configurations by PACS-CS collaboration.
  - RG improved gauge action & O(a) improved clover quark action
  - β = 1.90, a^{-1} = 2.176 [*GeV*], 32^3x64 lattice, *L* = 2.902 [*fm*].
  - κ_s = 0.13640 is fixed, κ_{ud} = 0.13700, 0.13727 and 0.13754 are chosen.
- Flat wall source is considered to produce S-wave B-B state.
- The KEK computer system A resources are used.

In unit of MeV	Esb 1	Esb 2	Esb 3
π	701±1	570±2	411±2
K	789±1	713±2	635±2
m_π/m_K	0.89	0.80	0.65
N	1585±5	1411±12	1215±12
Λ	1644±5	1504±10	1351± 8
Σ	1660±4	1531±11	1400±10
Ξ	1710±5	1610± 9	1503± 7

u,d quark masses lighter



Lists of channels

I=0 states

Spin	BB channels			SU(3) representation		
$^1S_0$	$\Lambda\Lambda$	$N\Xi$	$\Sigma\Sigma$	1	8s	27
$^3S_1$	--	$N\Xi$	--	8a	--	--

Strong attraction (H-dibaryon)

Aattraction

Strong repulsion

I=1 states

Spin	BB channels			SU(3) representation		
$^1S_0$	$N\Xi$	--	$\Lambda\Sigma$	--	8s	27
$^3S_1$	$N\Xi$	$\Sigma\Sigma$	$\Lambda\Sigma$	8a	10	10*

Similar to
The NN potential

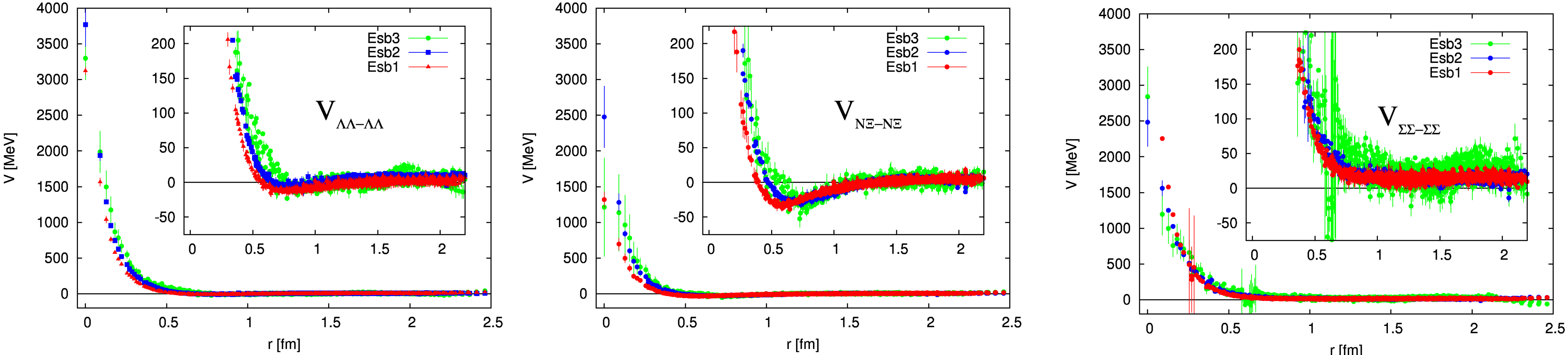
Repulsion

I=2 states

Spin	BB channels			SU(3) representation		
$^1S_0$	$\Sigma\Sigma$			--	--	27
~~$^3S_1$~~						

ΛΛ, NΞ, ΣΣ (I=0) $^1S_0$ channel

Esb1
Esb2
Esb3

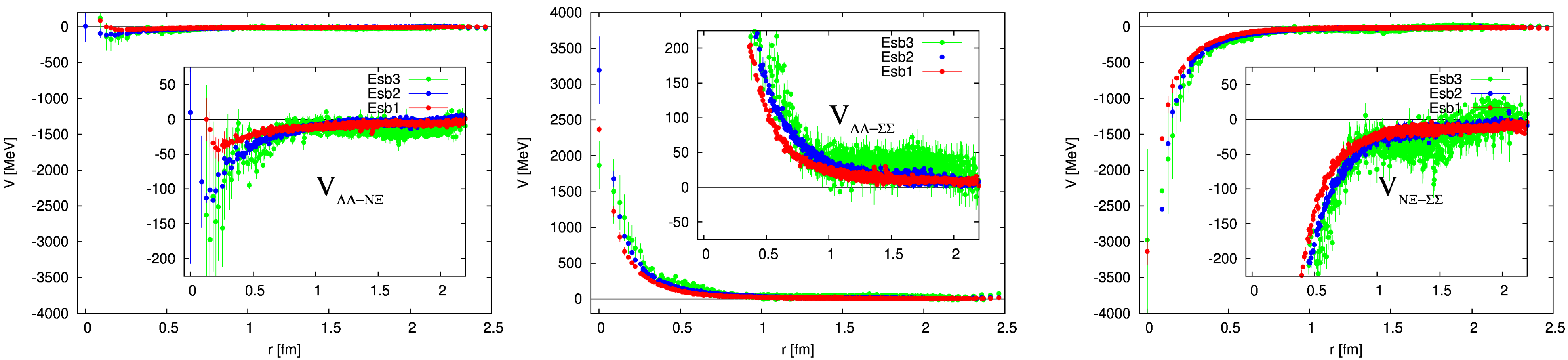


shallow attractive pocket

Deeper attractive pocket

Strongly repulsive

All channels have repulsive core

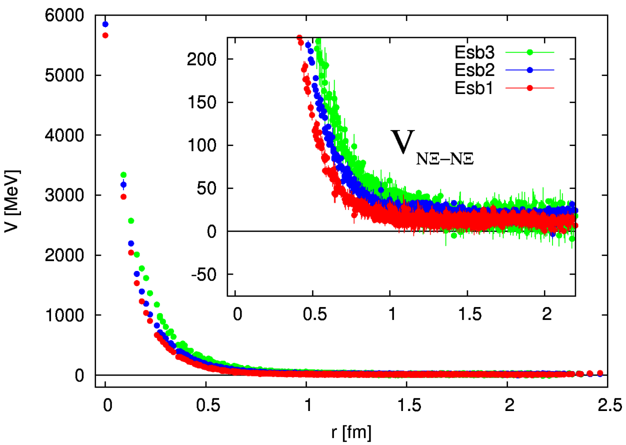


Relatively weaker than the others

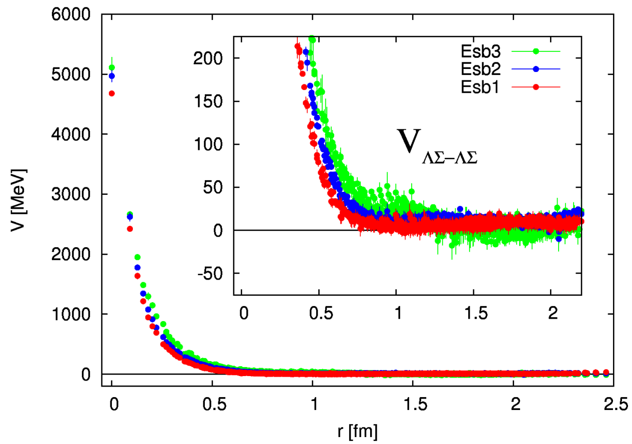
In this channel, our group found the "H-dibaryon" in the SU(3) limit.

$N\Xi$, $\Lambda\Sigma$ ($I=1$) $^1S_0$ channel

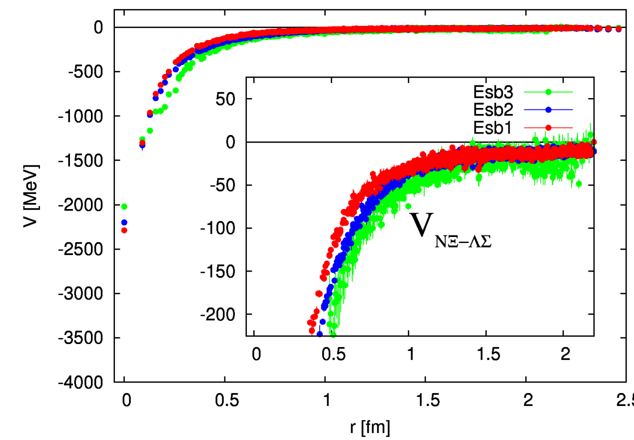
Esb1
Esb2
Esb3



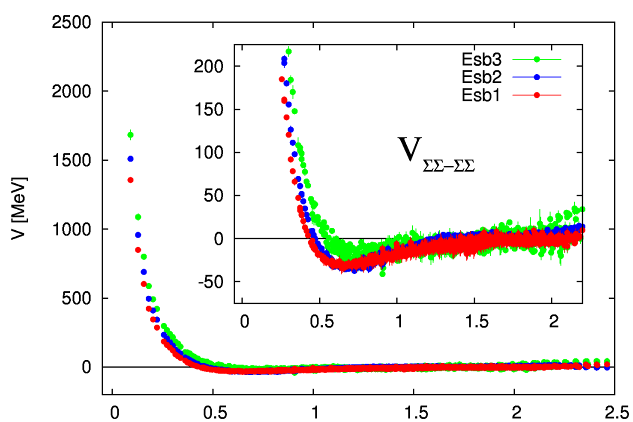
repulsive



repulsive

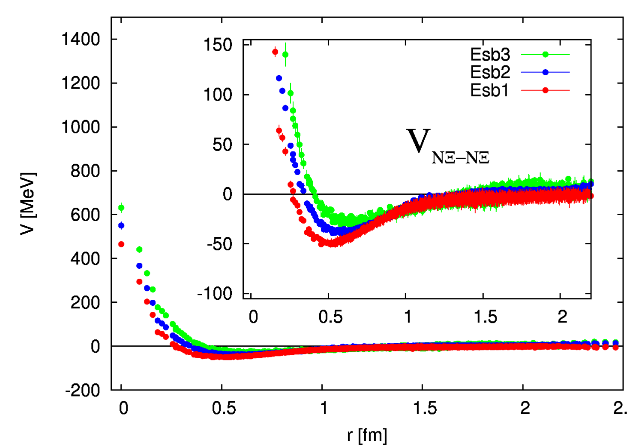


$\Sigma\Sigma$ ($I=2$) $^1S_0$ channel



Potential shape is similar to NN potential

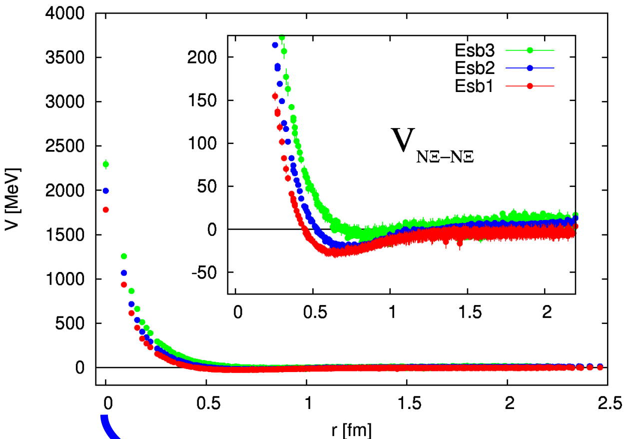
$N\Xi$ ($I=0$) $^3S_1$ channel

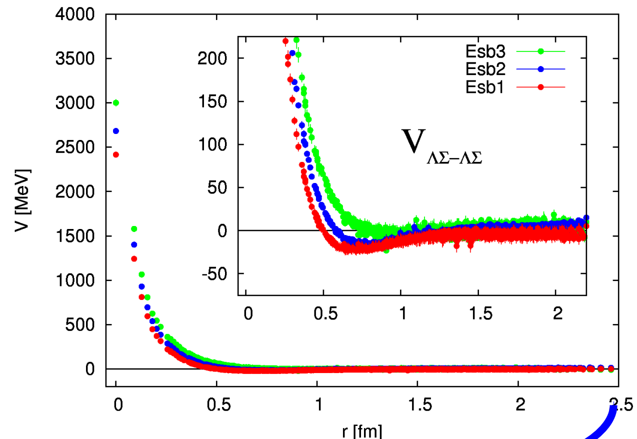


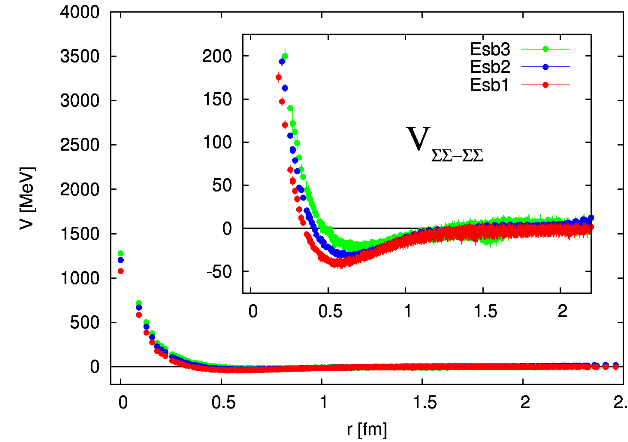
Small repulsive core
Deep attractive pocket

$N\Xi$, $\Sigma\Sigma$, $\Lambda\Sigma$ (I=1) $^3S_1$ channel

Esb1
Esb2
Esb3

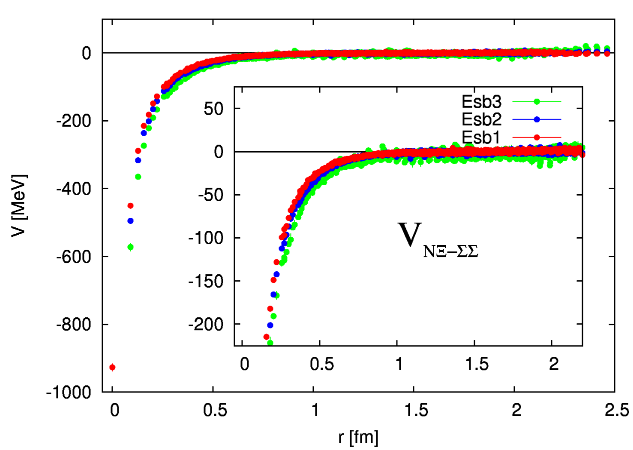


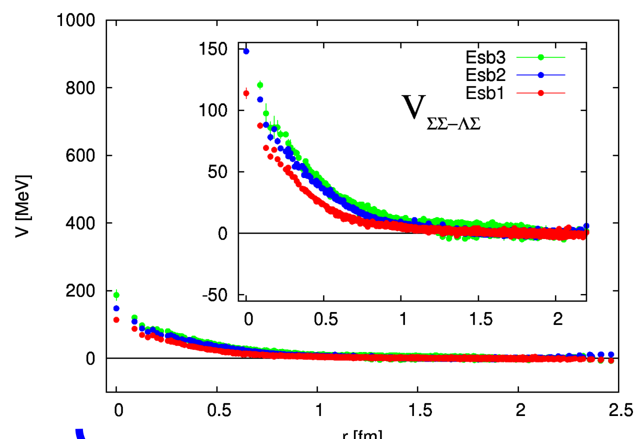


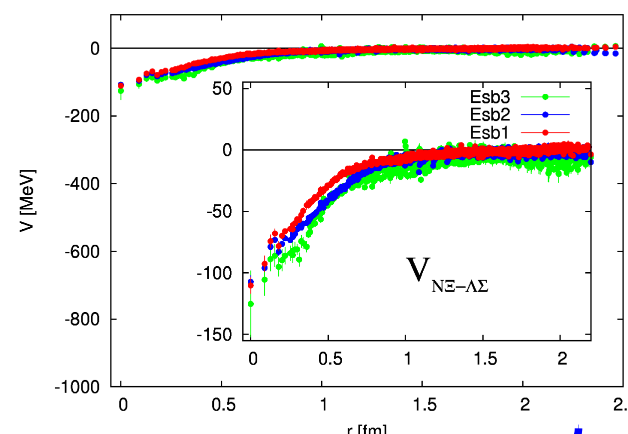


Most attractive

Attractive pocket becomes shallower as a lighter quark mass







Transitions from/to $\Lambda\Sigma$ channel is quite small

Comparison of potential matrices

Transformation of potentials
from the particle basis to the SU(3) irreducible representation (IR) basis.

SU(3) Clebsh-Gordan coefficients

$$\begin{pmatrix} |1\rangle \\ |8\rangle \\ |27\rangle \end{pmatrix} = U \begin{pmatrix} |\Lambda\Lambda\rangle \\ |N\Xi\rangle \\ |\Sigma\Sigma\rangle \end{pmatrix}, \quad U \begin{pmatrix} V^{\Lambda\Lambda} & V^{\Lambda\Lambda}_{N\Xi} & V^{\Lambda\Lambda}_{\Sigma\Sigma} \\ V^{N\Xi}_{\Lambda\Lambda} & V^{N\Xi} & V^{N\Xi}_{\Sigma\Sigma} \\ V^{\Sigma\Sigma}_{\Lambda\Lambda} & V^{\Sigma\Sigma}_{N\Xi} & V^{\Sigma\Sigma} \end{pmatrix} U^t \rightarrow \begin{pmatrix} V_1 & & \\ & V_8 & \\ & & V_{27} \end{pmatrix}$$

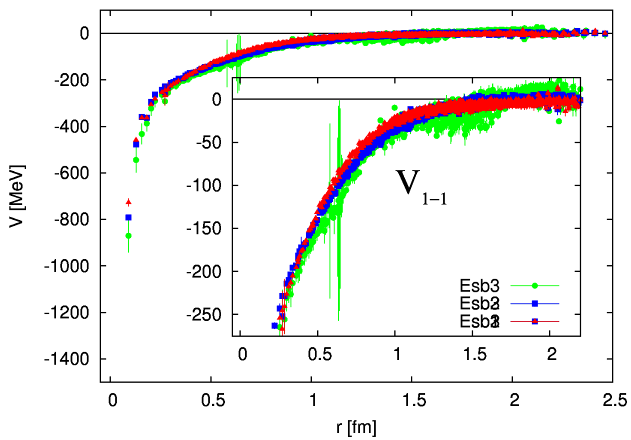
In the SU(3) irreducible representation basis,
the potential matrix should be diagonal in the SU(3) symmetric configuration.

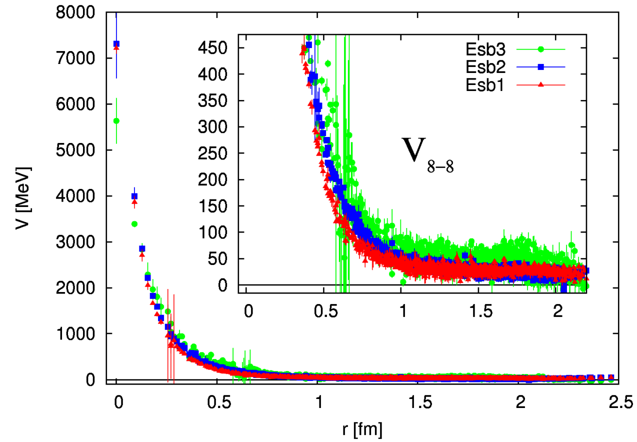
Off-diagonal part of the potential matrix in the SU(3) IR basis
would be an effectual measure of the SU(3) breaking effect.

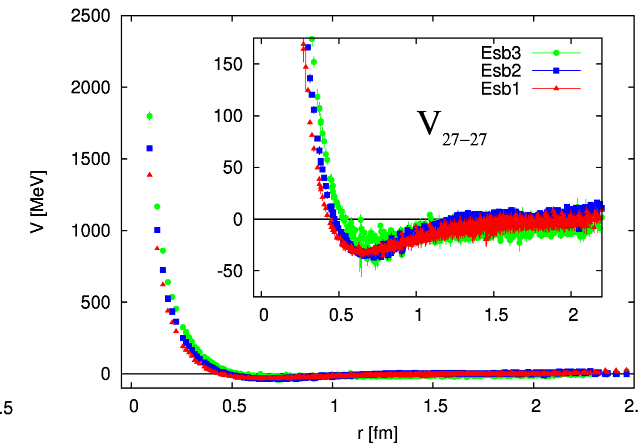
We will see how the SU(3) symmetry of potential will be broken
by changing the u,d quark masses lighter.

ΛΛ, NΞ, ΣΣ (I=0) $^1S_0$ channel

Esb1
Esb2
Esb3

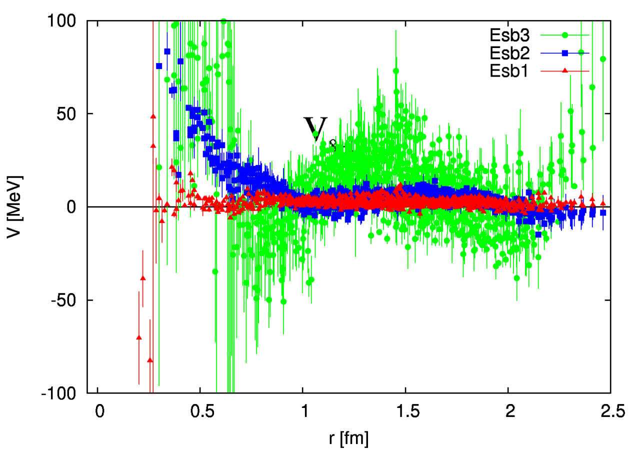


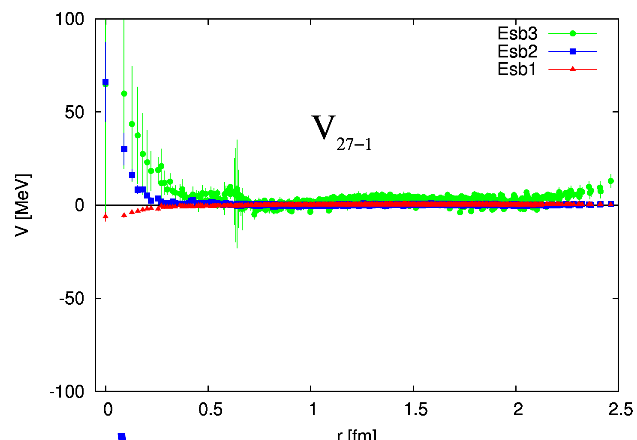


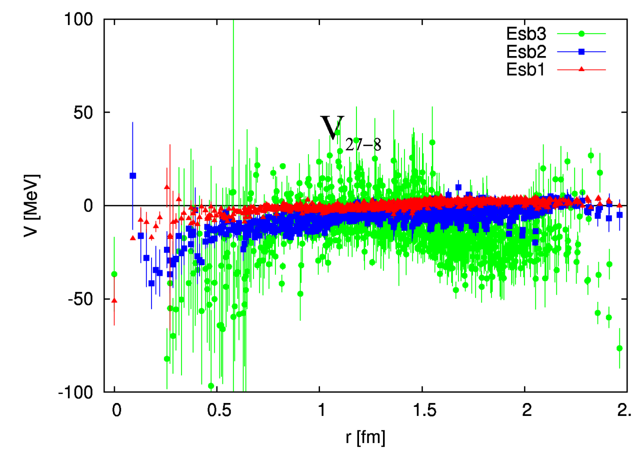


Strongly attractive
H-dibaryon channel

Pauli blocking effect





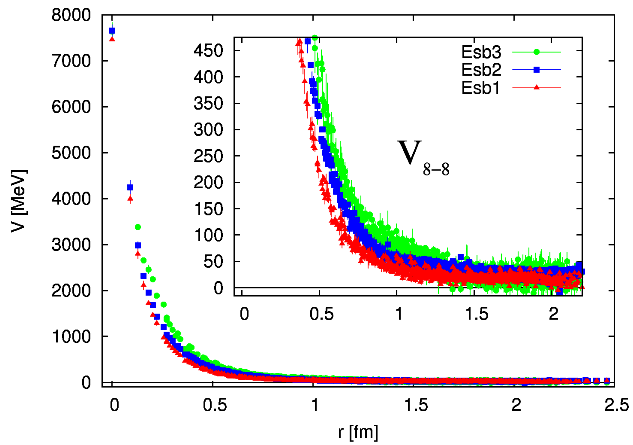


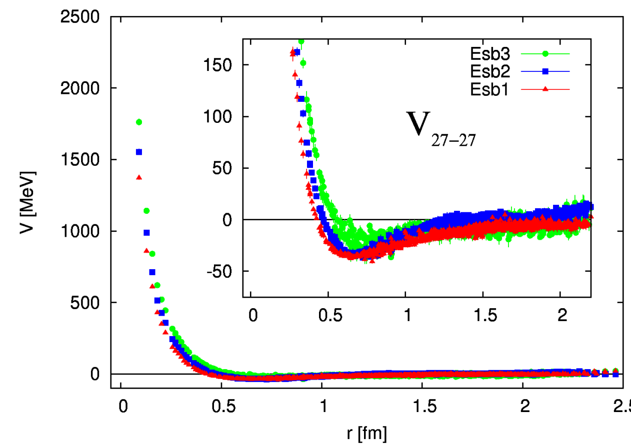
Mixture of singlet and octet
Is relatively larger than the others

27 plet does not mix so much to the other representations

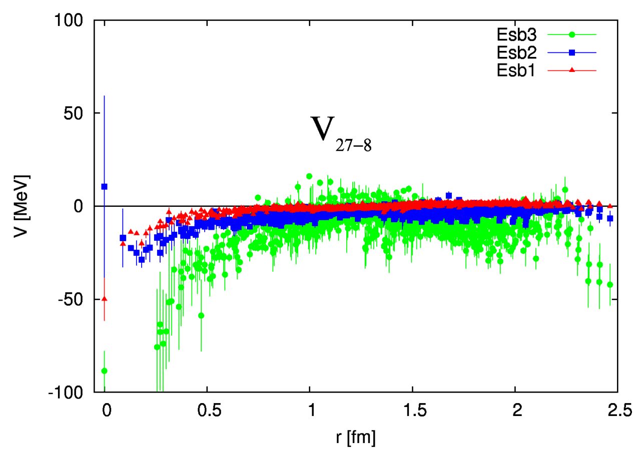
$N\Xi, \Lambda\Sigma$ (I=1) $^1S_0$ channel

Esb1
Esb2
Esb3





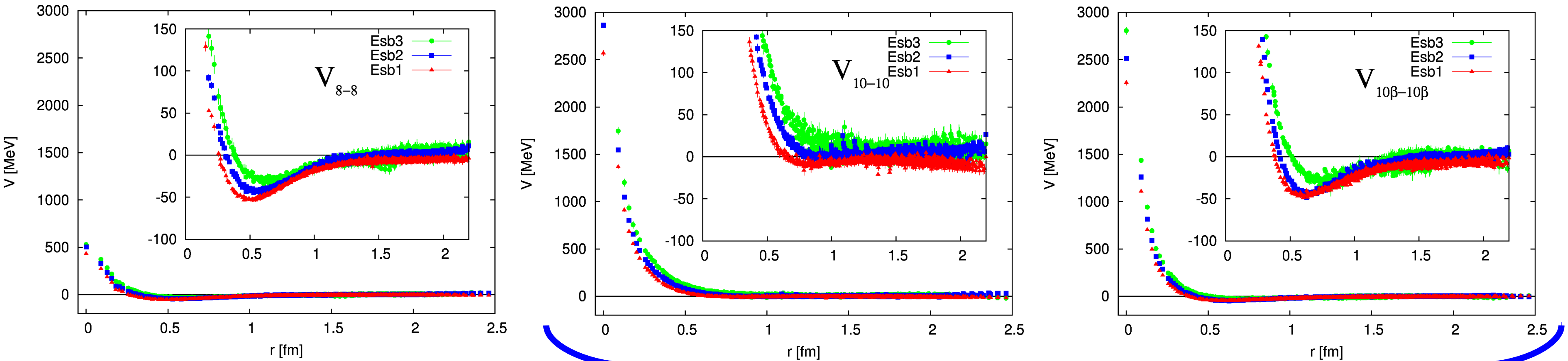
Strongly repulsive



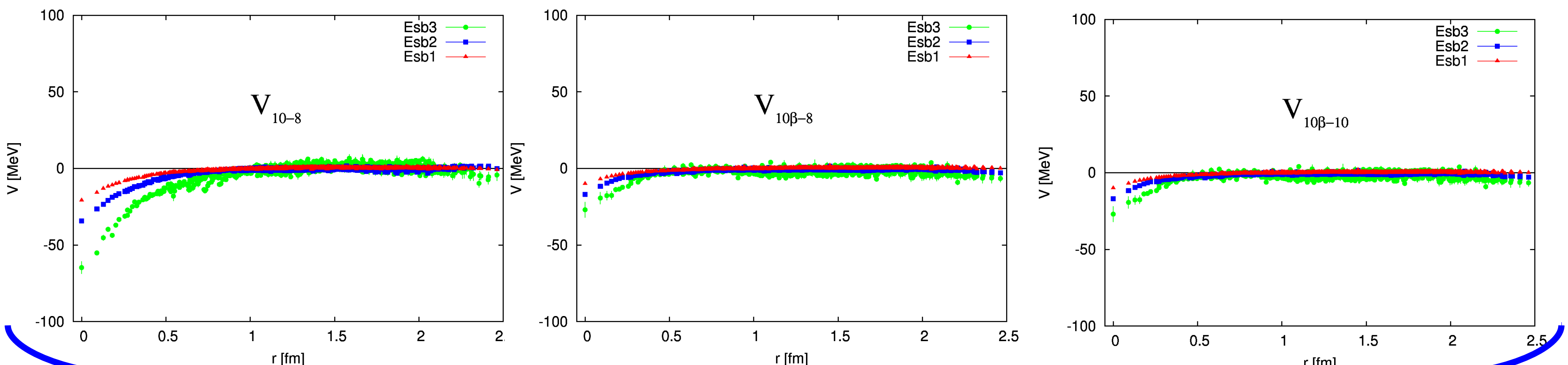
8_s and 27plet mixing is similar strength to the I=1 $^1S_0$ case

$N\Xi$, $\Sigma\Sigma$, $\Lambda\Sigma$ (I=1) $^3S_1$ channel

Esb1
Esb2
Esb3



Repulsive core grow as decreasing quark mass



Mixture of these three IRs get larger and larger as decreasing quark mass.

Summaries and outlooks

- We have investigated the S=-2 BB interactions from lattice QCD.
- In order to deal with a variety of interactions, we extend our method to the coupled channel formalism.
- Potentials are derived from NBS wave functions calculated with PACS-CS configurations
- Quark mass dependence of potentials can be seen not in long range region but in shot range region as an enhancement of repulsive core.
- Small mixture between different SU(3) IRs can be seen as the flavor SU(3) breaking effect.
- SU(3) breaking effects are still small even in $m\pi/mK$=0.65 situation but it would be change drastically at physical situation $m\pi/mK$=0.28.

Backup slides

Energy independent potential in coupled channel S.E.

Inside the interaction range, we can define the interaction kernel

$$\begin{pmatrix} p^2+\nabla & q^2+\nabla \\ p^2+\nabla & q^2+\nabla \end{pmatrix} \begin{pmatrix} \psi_a^a(\vec{x},E) & \psi_a^b(\vec{x},E) \\ \psi_a^b(\vec{x},E) & \psi_a^b(\vec{x},E) \end{pmatrix} = \begin{pmatrix} K_a^a(\vec{x},E) & K_b^a(\vec{x},E) \\ K_b^a(\vec{x},E) & K_b^a(\vec{x},E) \end{pmatrix}$$

Factorization of the interaction kernel

$$\begin{pmatrix} K_a^a(\vec{x},E) & K_b^a(\vec{x},E) \\ K_b^a(\vec{x},E) & K_b^a(\vec{x},E) \end{pmatrix} = \int dy \begin{pmatrix} U_a^a(\vec{x},\vec{y}) & U_a^b(\vec{x},\vec{y}) \\ U_b^a(\vec{x},\vec{y}) & U_b^b(\vec{x},\vec{y}) \end{pmatrix} \begin{pmatrix} \psi_a^a(\vec{y},E) & \psi_a^b(\vec{y},E) \\ \psi_a^b(\vec{y},E) & \psi_a^b(\vec{y},E) \end{pmatrix}$$

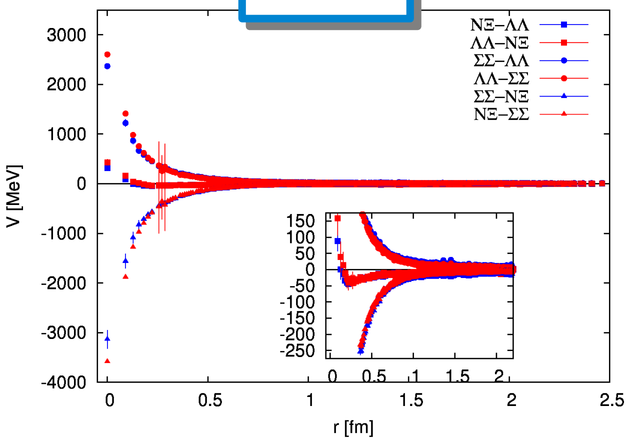
$$\int dx \begin{pmatrix} \tilde{\psi}_a^a(\vec{x},E') & \tilde{\psi}_a^b(\vec{x},E') \\ \tilde{\psi}_b^a(\vec{x},E') & \tilde{\psi}_b^b(\vec{x},E') \end{pmatrix} \begin{pmatrix} \psi_a^a(\vec{x},E) & \psi_a^b(\vec{x},E) \\ \psi_b^a(\vec{x},E) & \psi_b^b(\vec{x},E) \end{pmatrix} = 2\pi\delta(E-E')$$

$$\begin{pmatrix} U_a^a(\vec{x},\vec{y}) & U_b^a(\vec{x},\vec{y}) \\ U_a^b(\vec{x},\vec{y}) & U_b^b(\vec{x},\vec{y}) \end{pmatrix} = \int \frac{dE}{2\pi} \begin{pmatrix} K_a^a(\vec{x},E) & K_a^b(\vec{x},E) \\ K_a^b(\vec{x},E) & K_b^b(\vec{x},E) \end{pmatrix} \begin{pmatrix} \tilde{\psi}_a^a(\vec{y},E) & \tilde{\psi}_b^a(\vec{y},E) \\ \tilde{\psi}_a^b(\vec{y},E) & \tilde{\psi}_b^b(\vec{y},E) \end{pmatrix}$$

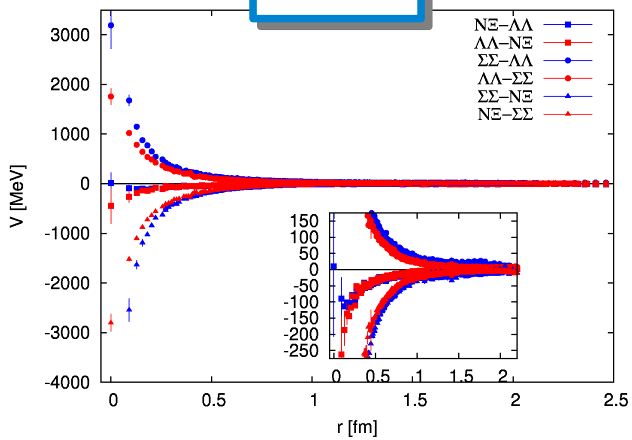
Energy independent potential in Schrödinger equation.

Harmiticity check for $^{1}S_{0}$ I=0

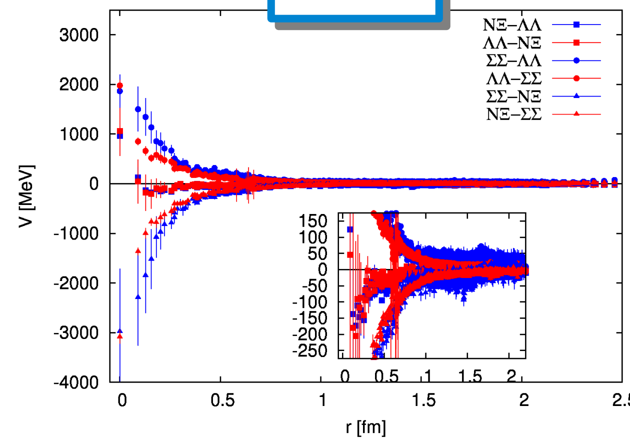
Esb 1



Esb 2

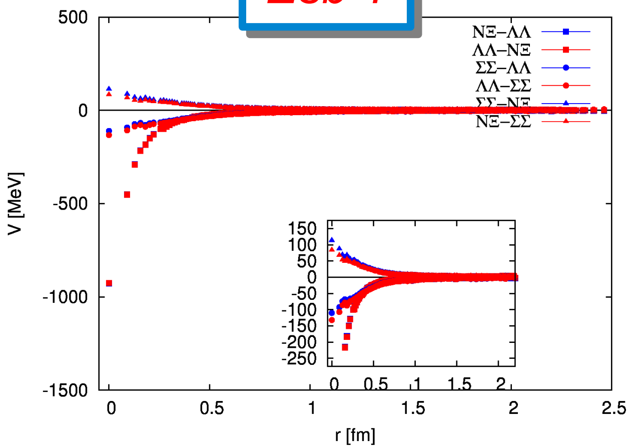


Esb 3

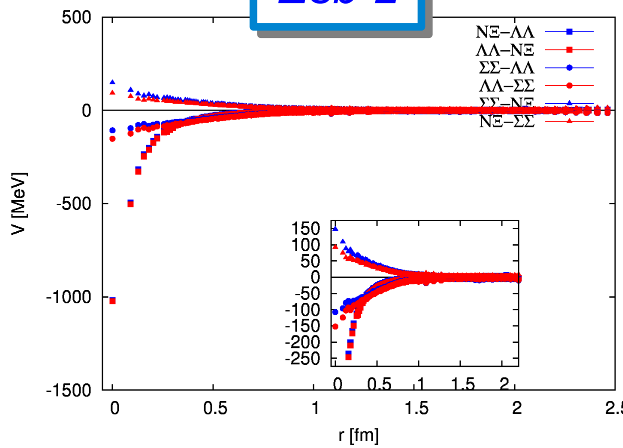


Harmiticity check for $^{3}S_{1}$ I=1

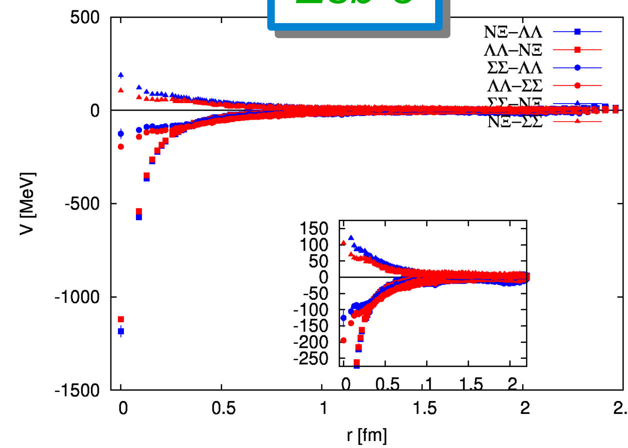
Esb 1



Esb 2



Esb 3

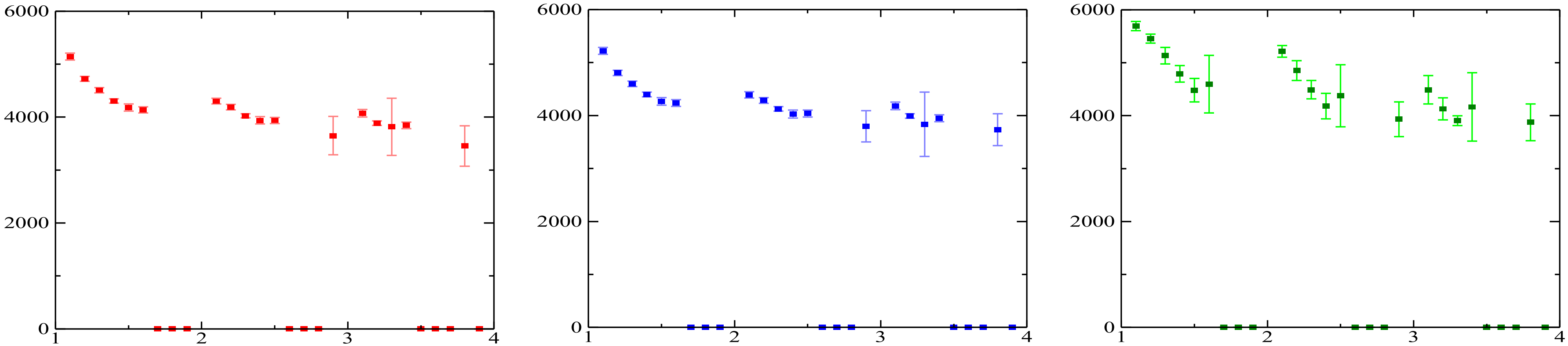


Hermiticity is roughly fine, but we need more statistics

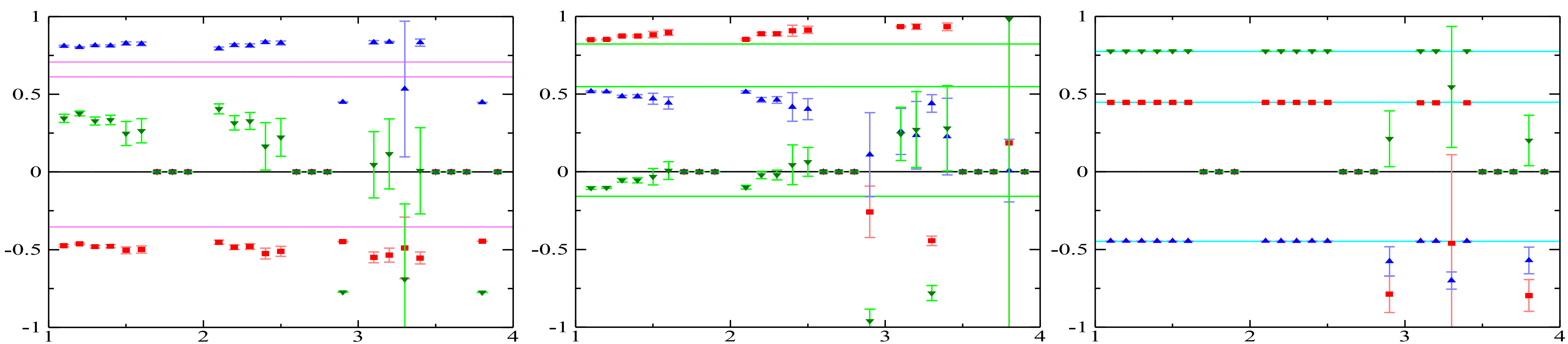
Effective masses and eigen vectors

Set 1 : m_π= 875

Eigen values (effective mass) of diagonalized correlator matrix



Eigen vectors of diagonalized correlator matrix

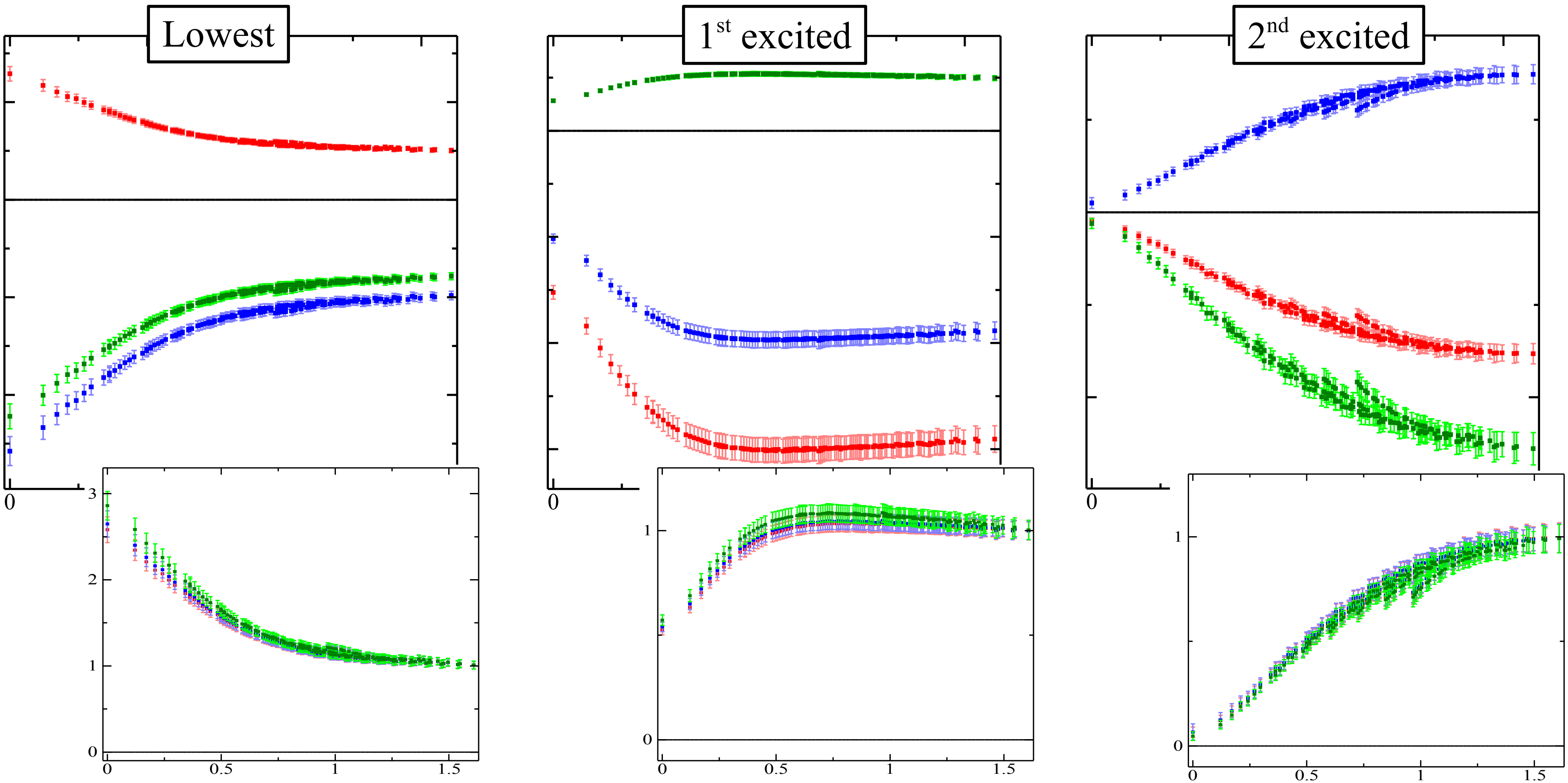


NBS wave functions for (S,I)=(-2,0) channel.

Set 1 : m_π= 875

Wave functions with diagonalized souces

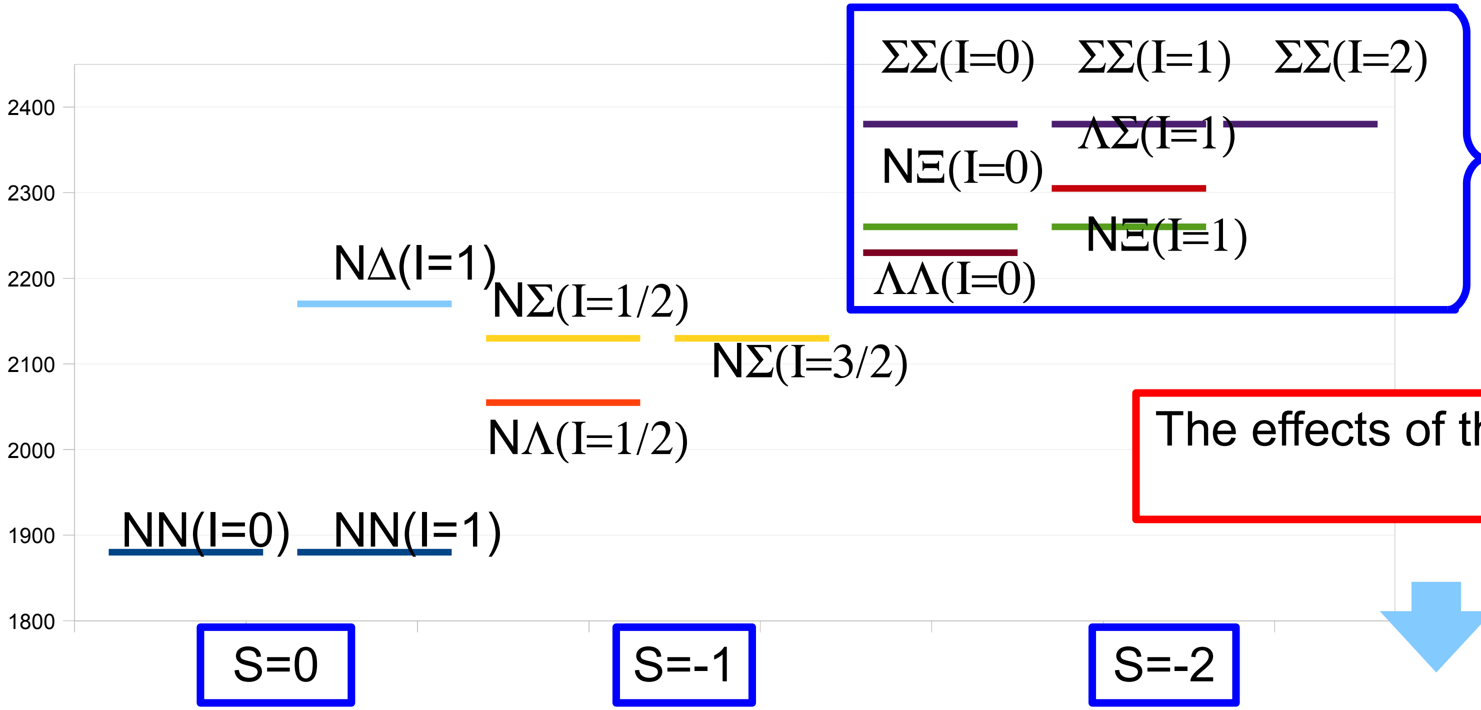
ΛΛ, NΞ, ΣΣ



The qualitative behavior of wave functions are same as the result of $SU(3)_f$ limit.
We can see a dependence on the sink operator due to the $SU(3)_f$ breaking effect.

Channel coupling

Energy levels of baryon-baryon system in the real world



small energy difference

The effects of the neighboring states can not be ignored.

We have to extend our method to the coupled channel formalism.

NBS wave function

$$W(t-t_0,\vec{r}) = \sum_{\vec{x}} \sum_{i} \langle 0 | B_\alpha(\vec{x}+\vec{r}) B_\beta(\vec{x}) | m_i \rangle e^{-E_i(t-t_0)} \langle m_i | \bar{B}_\gamma \bar{B}_\delta | 0 \rangle$$

Source operator can generate some states (m_1, m_2, …) with the same quantum number.

Small energy difference becomes the origin of mixture of energy eigen states

Optimization of the source operator.

Variational method

Linear combination of several independent operators

$$I(T) = \sum_\alpha v_\alpha J^\alpha(T)$$

Forming the effective mass of this operator

$$m(T_D) = -\frac{1}{T_D} \ln\left[\frac{\langle I(T+T_D) I^\dagger(T)\rangle}{\langle I(T) I^\dagger(T)\rangle}\right] = -\frac{1}{T_D} \ln\left[\frac{\sum_{\alpha\beta} v_\alpha v_\beta C_{\alpha\beta}(T+T_D)}{\sum_{\alpha\beta} v_\alpha v_\beta C_{\alpha\beta}(T)}\right]$$

with the correlation matrix defined as

$$C_{\alpha\beta}(T) \equiv \langle J_\alpha(T) J_\beta^\dagger(0)\rangle$$

Flat wall sink was considered to symmetrize the correlation matrix

Search the stationary point of m against *v*

$$\frac{\partial m(T_D)}{\partial v_\gamma} = 0$$

Diagonalization

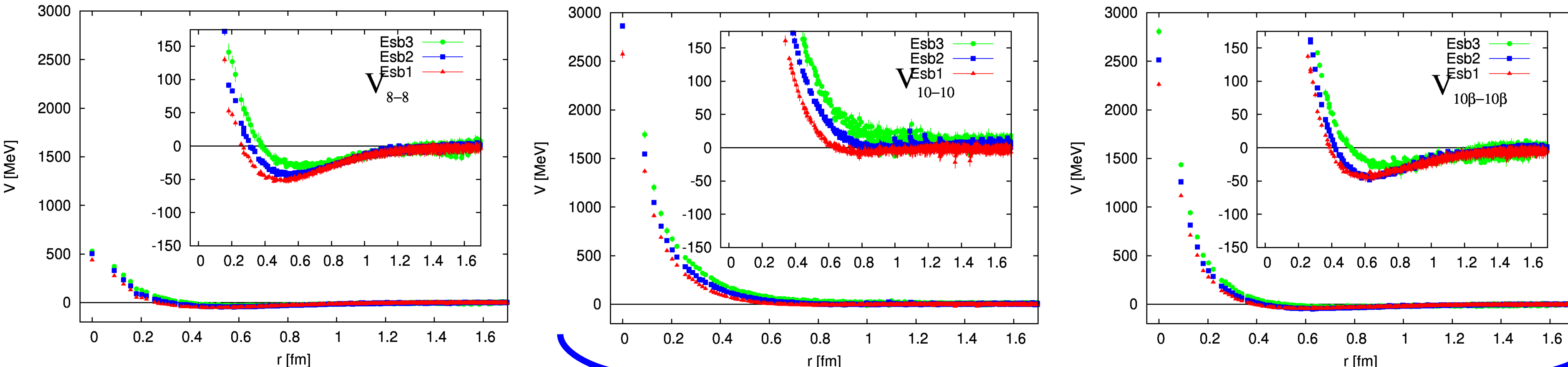
$$C(T+T_D)\vec{v} = e^{-m(T_D)T_D} C(T)\vec{v}$$

We can obtain the source operators I_0, I_1, …

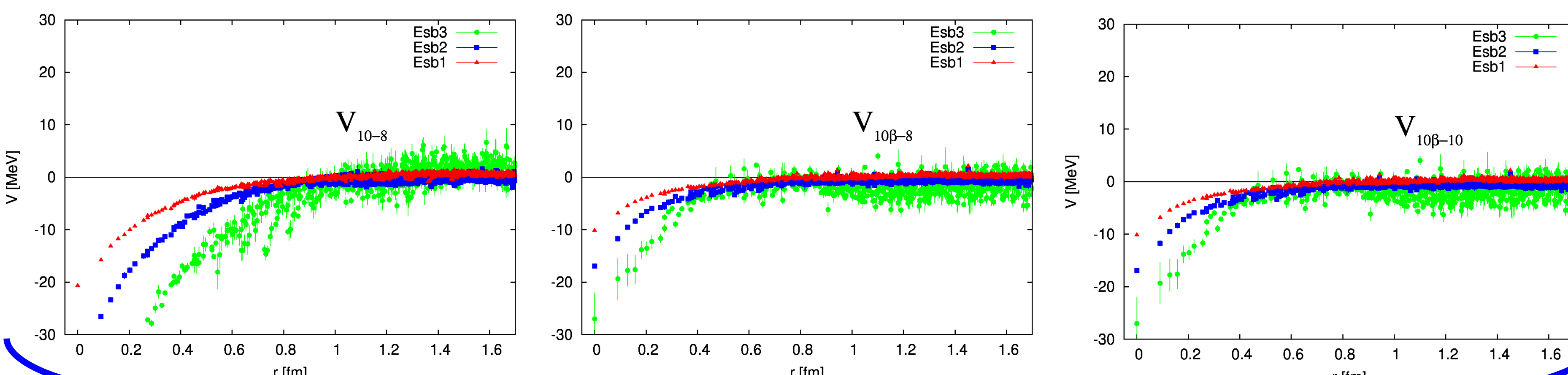
which strongly couples to the ground state, 1^{st} excited state, … .

$N\Xi$, $\Sigma\Sigma$, $\Lambda\Sigma$ (I=1) $^3S_1$ channel

Esb1
Esb2
Esb3



Repulsive core grow as decreasing quark mass



Mixture of these three IRs get larger and larger as decreasing quark mass.

Baryon operators

$$p_\alpha = \epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, u(\xi_1) d(\xi_2) u(\xi_3)$$
$$n_\alpha = \epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, u(\xi_1) d(\xi_2) d(\xi_3)$$

$$\Sigma^+_\alpha = -\epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, u(\xi_1) s(\xi_2) u(\xi_3)$$
$$\Sigma^0_\alpha = -\epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, \sqrt{\frac{1}{2}} \left[d(\xi_1) s(\xi_2) u(\xi_3) + u(\xi_1) s(\xi_2) d(\xi_3) \right]$$
$$\Sigma^-_\alpha = -\epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, d(\xi_1) s(\xi_2) d(\xi_3)$$

$$\Lambda_\alpha = -\epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, \sqrt{\frac{1}{6}} \left[d(\xi_1) s(\xi_2) u(\xi_3) + s(\xi_1) u(\xi_2) d(\xi_3) - 2u(\xi_1) d(\xi_2) s(\xi_3) \right]$$

$$\Xi^0_\alpha = \epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, s(\xi_1) u(\xi_2) s(\xi_3)$$
$$\Xi^-_\alpha = \epsilon_{c_1 c_2 c_3} (C\gamma_5)_{d_1 d_2} \delta_{d_3 \alpha}\, s(\xi_1) d(\xi_2) s(\xi_3)$$

- With corrected phase $\bar{1} = -\epsilon^{123} = -(ds - sd) = sd - ds$

Isospin combinations of BB operator

$\Lambda\Lambda,\ p\Xi^-,\ n\Xi^0,\ \Xi^-p,\ \Xi^0 n,\ \Sigma^+\Sigma^-,\ \Sigma^0\Sigma^0,\ \Sigma^-\Sigma^+,\ \Lambda\Sigma^0,\ \Sigma^0\Lambda$

I=0 operstors

$$\begin{cases} \Lambda\Lambda \\ N\Xi = +\sqrt{\frac{1}{2}}\,p\Xi^- - \sqrt{\frac{1}{2}}\,n\Xi^0 \\ \Sigma\Sigma = +\sqrt{\frac{1}{3}}\,\Sigma^+\Sigma^- - \sqrt{\frac{1}{3}}\,\Sigma^0\Sigma^0 + \sqrt{\frac{1}{3}}\,\Sigma^-\Sigma^+ \end{cases}$$

Flavor symmetric

I=1 operstors

$$\begin{cases} N\Xi = +\sqrt{\frac{1}{2}}\,p\Xi^- + \sqrt{\frac{1}{2}}\,n\Xi^0 \\ \Sigma\Sigma = +\sqrt{\frac{1}{2}}\,\Sigma^+\Sigma^- - \sqrt{\frac{1}{2}}\,\Sigma^-\Sigma^+ \\ \Lambda\Sigma \end{cases}$$

Flavor anti-symmetric

I=2 operstors

$$\Sigma\Sigma = +\sqrt{\frac{1}{6}}\,\Sigma^+\Sigma^- + \sqrt{\frac{4}{6}}\,\Sigma^0\Sigma^0 + \sqrt{\frac{1}{6}}\,\Sigma^-\Sigma^+$$

Irreducible BB source operator

$$\overline{BB^{(27)}} = +\sqrt{\frac{27}{40}}\,\overline{\Lambda}\,\overline{\Lambda} - \sqrt{\frac{1}{40}}\,\overline{\Sigma}\,\overline{\Sigma} + \sqrt{\frac{12}{40}}\,\overline{N}\,\overline{\Xi}$$

$$\overline{BB^{(8s)}} = -\sqrt{\frac{1}{5}}\,\overline{\Lambda}\,\overline{\Lambda} - \sqrt{\frac{3}{5}}\,\overline{\Sigma}\,\overline{\Sigma} + \sqrt{\frac{1}{5}}\,\overline{N}\,\overline{\Xi}$$

$$\overline{BB^{(1)}} = -\sqrt{\frac{1}{8}}\,\overline{\Lambda}\,\overline{\Lambda} + \sqrt{\frac{3}{8}}\,\overline{\Sigma}\,\overline{\Sigma} + \sqrt{\frac{4}{8}}\,\overline{N}\,\overline{\Xi}$$

$$\overline{\Sigma}\,\overline{\Sigma} = +\sqrt{\frac{1}{3}}\,\overline{\Sigma^{+}}\,\overline{\Sigma^{-}} - \sqrt{\frac{1}{3}}\,\overline{\Sigma^{0}}\,\overline{\Sigma^{0}} + \sqrt{\frac{1}{3}}\,\overline{\Sigma^{-}}\,\overline{\Sigma^{+}}$$

$$\overline{N}\,\overline{\Xi} = +\sqrt{\frac{1}{4}}\,\overline{p}\,\overline{\Xi^{-}} + \sqrt{\frac{1}{4}}\,\overline{\Xi^{-}}\,\overline{p} - \sqrt{\frac{1}{4}}\,\overline{n}\,\overline{\Xi^{0}} - \sqrt{\frac{1}{4}}\,\overline{\Xi^{0}}\,\overline{n}$$

$$\overline{BB^{(10*)}} = +\sqrt{\frac{1}{2}}\,\overline{p}\,\overline{n} - \sqrt{\frac{1}{2}}\,\overline{n}\,\overline{p}$$

$$\overline{BB^{(10)}} = +\sqrt{\frac{1}{2}}\,\overline{p}\,\overline{\Sigma^{+}} - \sqrt{\frac{1}{2}}\,\overline{\Sigma^{+}}\,\overline{p}$$

$$\overline{BB^{(8a)}} = +\sqrt{\frac{1}{4}}\,\overline{p}\,\overline{\Xi^{-}} - \sqrt{\frac{1}{4}}\,\overline{\Xi^{-}}\,\overline{p} - \sqrt{\frac{1}{4}}\,\overline{n}\,\overline{\Xi^{0}} + \sqrt{\frac{1}{4}}\,\overline{\Xi^{0}}\,\overline{n}$$

Determination of the energy part

Using four-point correlator *W* with an optimized source such as,

$$W_\alpha(\vec{x}, E) = A\Psi_\alpha(\vec{x}, E)\,\mathrm{e}^{-Et}$$

The coupled channel Schrödinger equation can be rewritten as

$$\left(\frac{p_\alpha^2}{2\mu_\alpha} - H_0^\alpha\right) W_\alpha(\vec{x}, E) = V_{\alpha\alpha}(\vec{x}) W_\alpha(\vec{x}, E) + V_{\alpha\beta}(\vec{x}) W_\beta(\vec{x}, E) + V_{\alpha\gamma}(\vec{x}) W_\gamma(\vec{x}, E)$$

Define

$$R_\alpha(\vec{x}, E) \equiv \frac{W_\alpha(\vec{x}, E)}{C_\alpha(t)} \propto \exp\left(-(E - M_\alpha)t\right) \simeq \exp\left(-\frac{p_\alpha^2}{2\mu_\alpha} t\right)$$

Product of single baryon correlators

Taking time derivative of R,

$$\partial_t R_\alpha(\vec{x}, E) = -\frac{p_\alpha^2}{2\mu_\alpha} R_\alpha(\vec{x}, E)$$

Source optimization is not necessary !

Thus the potential matrix can be obtained as

$$\begin{pmatrix} V^{\Lambda\Lambda}_{\Lambda\Lambda}(\vec{x}) \\ V^{\Lambda\Lambda}_{N\Xi}(\vec{x}) \\ V^{\Lambda\Lambda}_{\Sigma\Sigma}(\vec{x}) \end{pmatrix} = \begin{pmatrix} W_{\Lambda\Lambda}(\vec{x}, E_0) & W_{N\Xi}(\vec{x}, E_0) & W_{\Sigma\Sigma}(\vec{x}, E_0) \\ W_{\Lambda\Lambda}(\vec{x}, E_1) & W_{N\Xi}(\vec{x}, E_1) & W_{\Sigma\Sigma}(\vec{x}, E_1) \\ W_{\Lambda\Lambda}(\vec{x}, E_2) & W_{N\Xi}(\vec{x}, E_2) & W_{\Sigma\Sigma}(\vec{x}, E_2) \end{pmatrix}^{-1} \begin{pmatrix} -C_{\Lambda\Lambda}\partial R_{\Lambda\Lambda}(\vec{x}, E_0) - H_0^\alpha W_{\Lambda\Lambda}(\vec{x}, E_0) \\ -C_{\Lambda\Lambda}\partial R_{\Lambda\Lambda}(\vec{x}, E_1) - H_0^\alpha W_{\Lambda\Lambda}(\vec{x}, E_1) \\ -C_{\Lambda\Lambda}\partial R_{\Lambda\Lambda}(\vec{x}, E_2) - H_0^\alpha W_{\Lambda\Lambda}(\vec{x}, E_2) \end{pmatrix}$$

The asymptotic momentum (energy shift) is determined through the time derivative of R-correlator.