Non-trivial θ-vacuum effects in the 2-d $O(3)$ model

Michael Bögli

Albert Einstein Center for Fundamental Physics, University of Bern

June 27, 2012

M. Bögli, F. Niedermayer, M. Pepe and U.-J. Wiese,
JHEP **1204** (2012) 117. [arXiv:1112.1873 [hep-lat]]

Open questions in the 2d $O(3)$ model

Vacuum angle θ in the continuum:

$$\begin{aligned} S[\vec{e}] &= \frac{1}{2g^2}\int d^2x\, \partial_\mu \vec{e}\cdot\partial_\mu \vec{e} + i\theta Q[\vec{e}], \quad \vec{e}(x)\in S^2 \\ Q[\vec{e}] &= \frac{1}{8\pi}\int d^2x\, \varepsilon_{\mu\nu}\vec{e}\cdot(\partial_\mu \vec{e}\times\partial_\nu \vec{e}) \in \Pi_2(S^2) \end{aligned}$$

- Asymptotically free
- Instantons, θ-vacua
- Sign problem, $e^{i\pi Q} = (-1)^Q$

Open questions in the 2d $O(3)$ model

Vacuum angle θ in the continuum:

$$
\begin{aligned}
S[\vec{e}] &= \frac{1}{2g^2}\int d^2x\, \partial_\mu \vec{e}\cdot\partial_\mu \vec{e} + i\theta Q[\vec{e}], \quad \vec{e}(x)\in S^2 \\
Q[\vec{e}] &= \frac{1}{8\pi}\int d^2x\, \varepsilon_{\mu\nu}\vec{e}\cdot(\partial_\mu \vec{e}\times\partial_\nu \vec{e}) \in \Pi_2(S^2)
\end{aligned}
$$

- Asymptotically free
- Instantons, θ-vacua
- Sign problem, $e^{i\pi Q} = (-1)^Q$
- The conjectured exact S-matrix implies

  A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP **63** (1986) 913.

  - $\theta = 0$: non-perturbatively generated mass gap $m = \frac{8}{e}\Lambda_{\overline{MS}}$
  - $\theta = \pi$: WZNW model as low-energy effective field theory, $m = 0$

Open questions in the 2d $O(3)$ model

Vacuum angle θ in the continuum:

$$
\begin{aligned}
S[\vec{e}] &= \frac{1}{2g^2}\int d^2x\, \partial_\mu \vec{e}\cdot\partial_\mu \vec{e} + i\theta Q[\vec{e}], \quad \vec{e}(x)\in S^2 \\
Q[\vec{e}] &= \frac{1}{8\pi}\int d^2x\, \varepsilon_{\mu\nu}\vec{e}\cdot(\partial_\mu\vec{e}\times\partial_\nu\vec{e}) \in \Pi_2(S^2)
\end{aligned}
$$

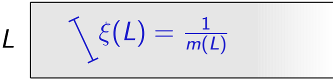
- Asymptotically free
- Instantons, θ-vacua
- Sign problem, $e^{i\pi Q} = (-1)^Q$
- The conjectured exact S-matrix implies

  A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. JETP **63** (1986) 913.

  - $\theta = 0$: non-perturbatively generated mass gap $m = \frac{8}{e}\Lambda_{\overline{MS}}$
  - $\theta = \pi$: WZNW model as low-energy effective field theory, $m = 0$
- Is the exact S-matrix theory correct from lattice first principles?
- Is θ renormalized non-pertubatively?

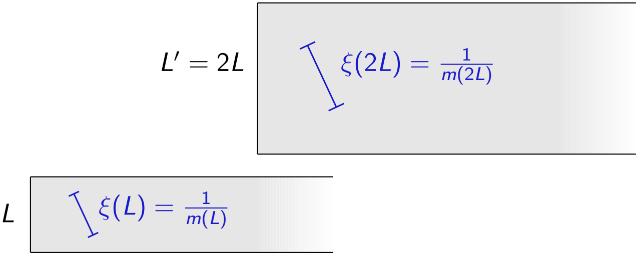
Finite volume approach

- Exact S-Matrix solved in finite volume
- Determine the step scaling function:



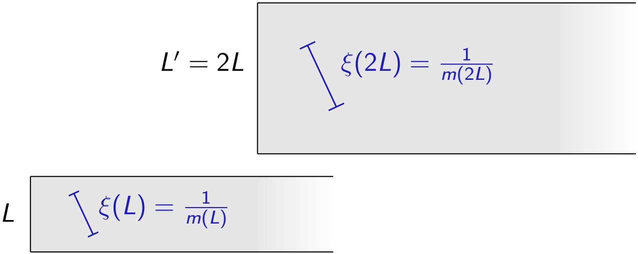
Finite volume approach

- Exact S-Matrix solved in finite volume
- Determine the step scaling function: $\sigma(2, m(L) \cdot L) = m(2L) \cdot 2L$



Finite volume approach

- Exact S-Matrix solved in finite volume
- Determine the step scaling function: $\sigma(2, m(L) \cdot L) = m(2L) \cdot 2L$



$$\theta = 0: \ \sigma(2, m(L) \cdot L = 1.0595) = 1.2612103$$

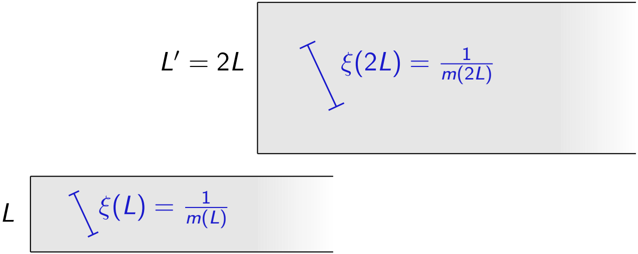
J. Balog and A. Hegedus, J. Phys. A **37** (2004) 1881.

$$\theta = \pi: \ \sigma(2, m(L) \cdot L = 1.0595) = 1.231064$$

J. Balog, private communication.

Finite volume approach

- Exact S-Matrix solved in finite volume
- Determine the step scaling function: $\sigma(2, m(L) \cdot L) = m(2L) \cdot 2L$



$$\theta = 0: \;\; \sigma(2, m(L) \cdot L = 1.0595) = 1.2612103$$

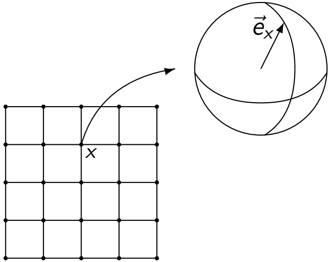
J. Balog and A. Hegedus, J. Phys. A **37** (2004) 1881.

$$\theta = \pi: \;\; \sigma(2, m(L) \cdot L = 1.0595) = 1.231064$$

J. Balog, private communication.

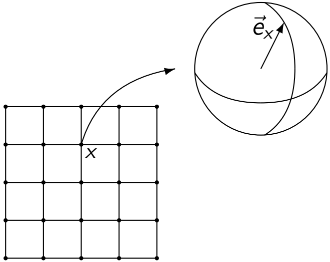
- Simulation on the lattice

The 2d lattice $O(3)$ model



- Each site x has a spin $\vec{e}_x \in S^2$

The 2d lattice $O(3)$ model

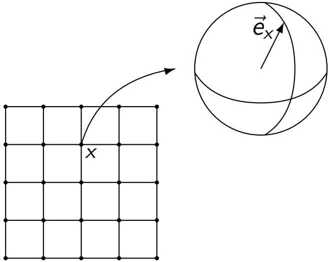


$$Z = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}$$

all field configurations of $\vec{e}_x \in S^2$

- Each site x has a spin $\vec{e}_x \in S^2$
- Action with nearest-neighbor coupling $S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)$

The 2d lattice $O(3)$ model



$$Z(\theta) = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q[\vec{e}]}$$

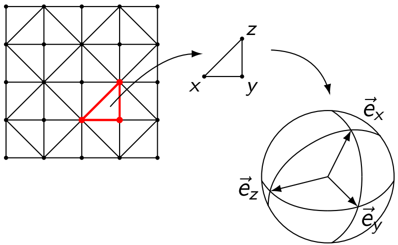
Topological charge

all field configurations of $\vec{e}_x \in S^2$

- Each site x has a spin $\vec{e}_x \in S^2$
- Action with nearest-neighbor coupling $S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)$
- Add θ combined with the topological charge

The topological charge

$$Z(\theta) = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q[\vec{e}]}$$

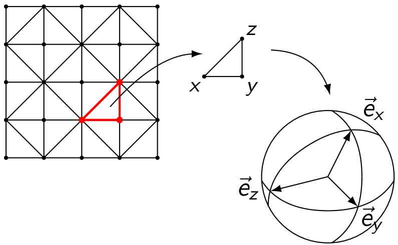


- Triangulate the lattice
- Triangle $\langle xyz \rangle \longmapsto \ \vec{e}_x, \vec{e}_y, \vec{e}_z \sim$ oriented area $A_{\langle xyz \rangle}/4\pi = q_{\langle xyz \rangle}$

B. Berg and M. Lüscher, Nucl. Phys. B **190**, (1981) 412.

The topological charge

$$Z(\theta) = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q[\vec{e}]}$$

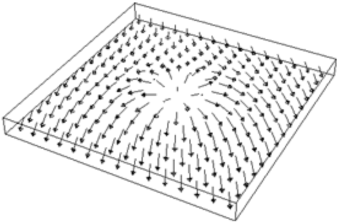


- Triangulate the lattice
- Triangle $\langle xyz \rangle \longmapsto \vec{e}_x, \vec{e}_y, \vec{e}_z \sim$ oriented area $A_{\langle xyz \rangle}/4\pi = q_{\langle xyz \rangle}$

B. Berg and M. Lüscher, Nucl. Phys. B **190**, (1981) 412.

- $Q[\vec{e}] = \sum_{\langle xyz \rangle} q_{\langle xyz \rangle} \in \mathbb{Z}$

Problem: dislocations



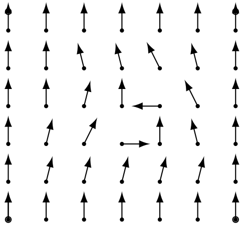
Instantons

Field configurations $[\vec{e}]$ with

- $Q = 1$
- minimal action $S[\vec{e}]$
- $S_{\text{inst}}[\vec{e}] = 4\pi/g^2$

M. Lüscher, Nucl. Phys. B **200** (1982) 61.

Problem: dislocations



Dislocations on the lattice

- $Q = 1$
- minimum of lattice action $S[\vec{e}]$
- for standard action $S_{\text{disloc}}[\vec{e}] = 6.69\ldots/g^2$

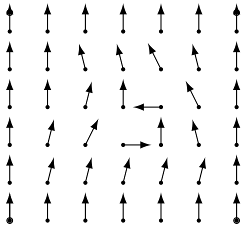
Instantons

Field configurations $[\vec{e}]$ with

- $Q = 1$
- minimal action $S[\vec{e}]$
- $S_{\text{inst}}[\vec{e}] = 4\pi/g^2$

M. Lüscher, Nucl. Phys. B **200** (1982) 61.

Problem: dislocations



Dislocations on the lattice

- $Q = 1$
- minimum of lattice action $S[\vec{e}]$
- for standard action $S_{\text{disloc}}[\vec{e}] = 6.69\ldots/g^2 < 4\pi/g^2$

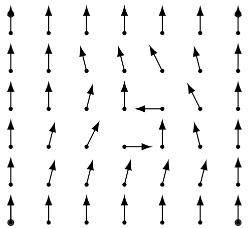
Instantons

Field configurations $[\vec{e}]$ with

- $Q = 1$
- minimal action $S[\vec{e}]$
- $S_{\text{inst}}[\vec{e}] = 4\pi/g^2$

M. Lüscher, Nucl. Phys. B **200** (1982) 61.

Problem: dislocations



Instantons

Field configurations $[\vec{e}]$ with

- $Q = 1$
- minimal action $S[\vec{e}]$
- $S_{\text{inst}}[\vec{e}] = 4\pi/g^2$

M. Lüscher, Nucl. Phys. B **200** (1982) 61.

Dislocations on the lattice

- $Q = 1$
- minimum of lattice action $S[\vec{e}]$
- for standard action $S_{\text{disloc}}[\vec{e}] = 6.69\ldots/g^2 < 4\pi/g^2$

Topological susceptibility χ_t

$$\chi_t = \frac{\langle Q^2 \rangle}{V}$$

is ultra-violet divergent:

- Logarithmic divergence in the continuum
- Depending on the dislocation action, one may expect a power-law divergence on the lattice

Different lattice actions

$$Z(\theta) = \int \mathcal{D}\vec{e}\ e^{-S[\vec{e}]}\ e^{i\theta Q[\vec{e}]}$$

Action $S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)$

Standard action $s(\vec{e}_x, \vec{e}_y) = -\frac{1}{g^2} \vec{e}_x \cdot \vec{e}_y$

Different lattice actions

$$Z(\theta) = \int \mathcal{D}\vec{e}\ e^{-S[\vec{e}]}\ e^{i\theta Q[\vec{e}]}$$

Action $S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)$

Standard action $s(\vec{e}_x, \vec{e}_y) = -\frac{1}{g^2} \vec{e}_x \cdot \vec{e}_y$

Topological action $s(\vec{e}_x, \vec{e}_y) = \begin{cases} 0 & \vec{e}_x \cdot \vec{e}_y > \cos\delta \\ \infty & \text{else} \end{cases}$

δ is the maximally allowed angle between neighboring spins

Different lattice actions

$$Z(\theta) = \int \mathcal{D}\vec{e}\ e^{-S[\vec{e}]}\ e^{i\theta Q[\vec{e}]}$$

Action $S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)$

Standard action $s(\vec{e}_x, \vec{e}_y) = -\frac{1}{g^2}\vec{e}_x \cdot \vec{e}_y$

Topological action $s(\vec{e}_x, \vec{e}_y) = \begin{cases} 0 & \vec{e}_x \cdot \vec{e}_y > \cos\delta \\ \infty & \text{else} \end{cases}$

δ is the maximally allowed angle between neighboring spins

Optimized constraint action $s(\vec{e}_x, \vec{e}_y) = \begin{cases} -\frac{1}{g^2}\vec{e}_x \cdot \vec{e}_y & \vec{e}_x \cdot \vec{e}_y > \cos\delta \\ \infty & \text{else} \end{cases}$

The constraint angle δ needs to be adjusted. Can choose δ such that cut-off effects are minimized.

Different lattice actions

$$Z(\theta) = \int \mathcal{D}\vec{e}\ e^{-S[\vec{e}]}\ e^{i\theta Q[\vec{e}]}$$

Action $S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)$

$\delta \to \pi$: Standard action $s(\vec{e}_x, \vec{e}_y) = -\frac{1}{g^2} \vec{e}_x \cdot \vec{e}_y$

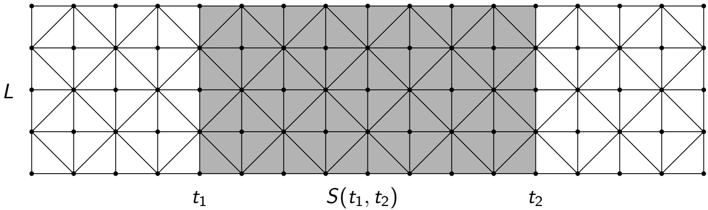
$g^2 \to \infty$: Topological action $s(\vec{e}_x, \vec{e}_y) = \begin{cases} 0 & \vec{e}_x \cdot \vec{e}_y > \cos\delta \\ \infty & \text{else} \end{cases}$

δ is the maximally allowed angle between neighboring spins

Optimized constraint action $s(\vec{e}_x, \vec{e}_y) = \begin{cases} -\frac{1}{g^2} \vec{e}_x \cdot \vec{e}_y & \vec{e}_x \cdot \vec{e}_y > \cos\delta \\ \infty & \text{else} \end{cases}$

The constraint angle δ needs to be adjusted. Can choose δ such that cut-off effects are minimized.

Correlation function



$$C(t_1 - t_2, \theta) = \frac{1}{Z(t_1, t_2, \theta)} \int \mathcal{D}\vec{e}\ \vec{E}(t_1) \cdot \vec{E}(t_2)\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q(t_1, t_2)}$$

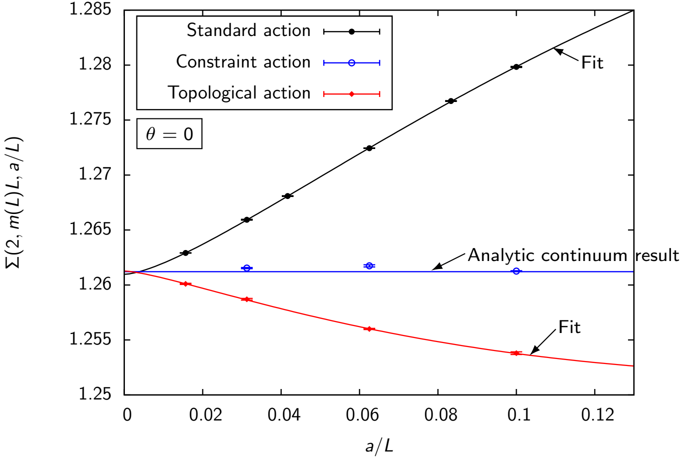
$$\sim \mathrm{e}^{-m(\theta, L)(t_2 - t_1)}$$

$$\vec{E}(t) = \sum_{x_1} \vec{e}_{x_1, t}, \qquad Z(t_1, t_2, \theta) = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q(t_1, t_2)}$$

Cut-off effects

$$\Sigma(2, m(L)L, a/L) = \sigma(2, m(L)L) + \frac{a^2}{L^2}[B \log^3(L/a) + C \log^2(L/a) + \ldots]$$

J. Balog, F. Niedermayer, and P. Weisz, Phys. Lett. **B676** (2009) 188

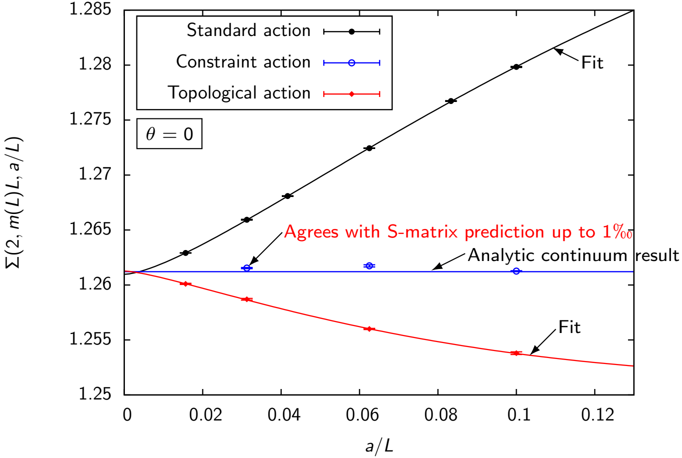


Cut-off effects of the step scaling function $\Sigma(2, m(L)L = 1.0595, a/L)$

Cut-off effects

$$\Sigma(2, m(L)L, a/L) = \sigma(2, m(L)L) + \frac{a^2}{L^2}[B \log^3(L/a) + C \log^2(L/a) + \ldots]$$

J. Balog, F. Niedermayer, and P. Weisz, Phys. Lett. **B676** (2009) 188

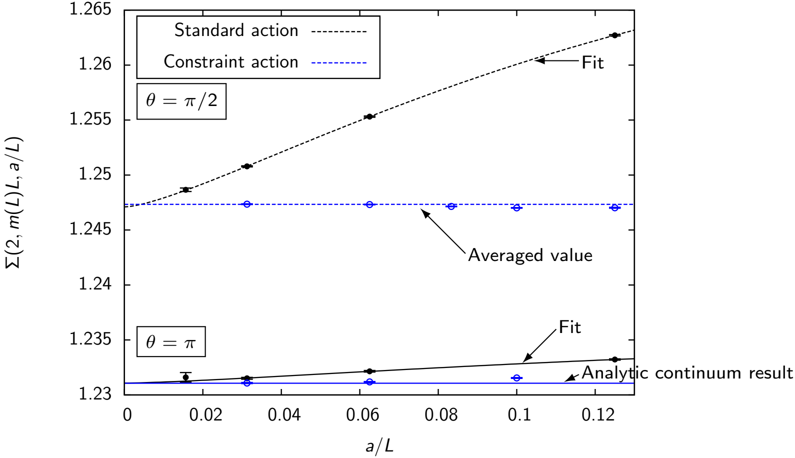


Cut-off effects of the step scaling function $\Sigma(2, m(L)L = 1.0595, a/L)$

Cut-off effects

$$\Sigma(2, m(L)L, a/L) = \sigma(2, m(L)L) + \frac{a^2}{L^2}[B \log^3(L/a) + C \log^2(L/a) + \ldots]$$

J. Balog, F. Niedermayer, and P. Weisz, Phys. Lett. **B676** (2009) 188

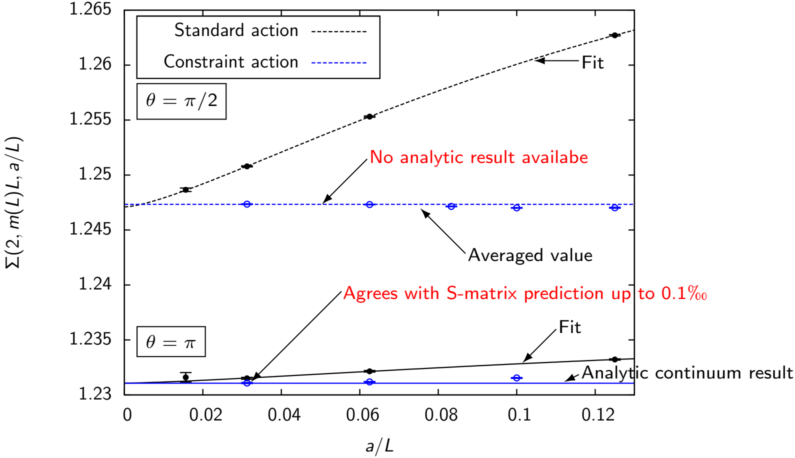


Cut-off effects of the step scaling function $\Sigma(2, m(L)L = 1.0595, a/L)$

Cut-off effects

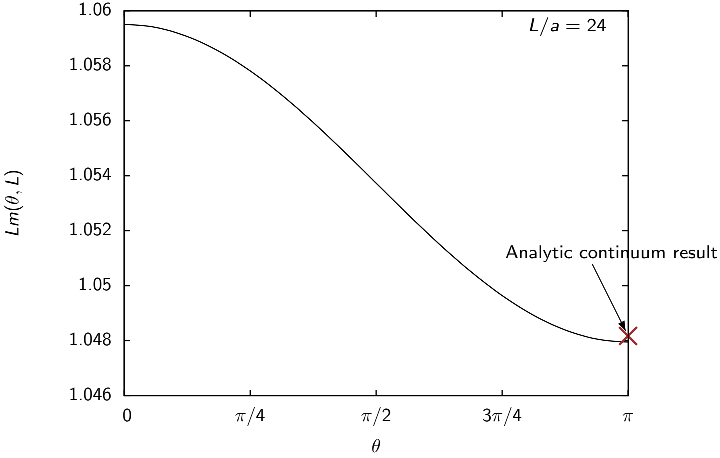
$$\Sigma(2, m(L)L, a/L) = \sigma(2, m(L)L) + \frac{a^2}{L^2}\left[B \log^3(L/a) + C \log^2(L/a) + \ldots\right]$$

J. Balog, F. Niedermayer, and P. Weisz, Phys. Lett. **B676** (2009) 188



Cut-off effects of the step scaling function $\Sigma(2, m(L)L = 1.0595, a/L)$

Continuum theory for $0 \leq \theta \leq \pi$



Mass gap $Lm(\theta, L)$ at $Lm(\theta, L) = 1.0595$ using the constraint action

Conclusion and Outlook

Conclusion

- We saw that θ is a relevant parameter, each value of θ corresponds to a different continuum theory
- Dislocations do not spoil the continuum limit
- We can confirm the exact S-matrix prediction at $\theta = 0, \pi$
- Optimized constraint action has very small cut-off effects
- The modified Hasenbusch improved estimator allowed to reach very high accuracy

Conclusion and Outlook

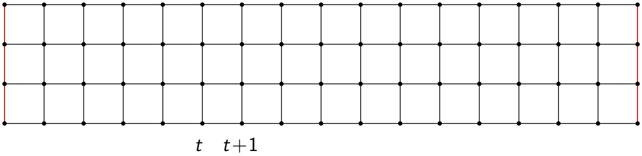
Conclusion

- We saw that θ is a relevant parameter, each value of θ corresponds to a different continuum theory
- Dislocations do not spoil the continuum limit
- We can confirm the exact S-matrix prediction at $\theta = 0, \pi$
- Optimized constraint action has very small cut-off effects
- The modified Hasenbusch improved estimator allowed to reach very high accuracy

Outlook

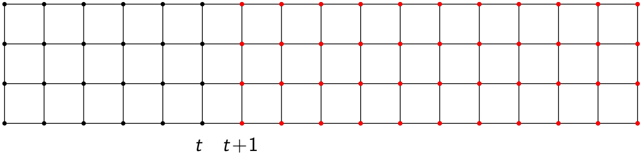
- Simulate gauge theories at $\theta = \pi$
- Implementation of the optimized constraint action for $O(N)$
- Implementation of the optimized constraint action for Yang-Mills theories

Modified Hasenbusch improved estimator



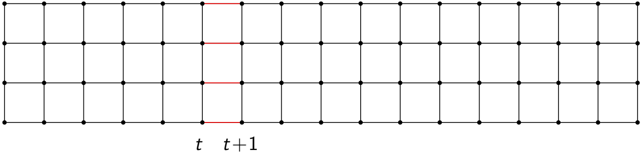
- Use free boundary conditions in t direction

Modified Hasenbusch improved estimator



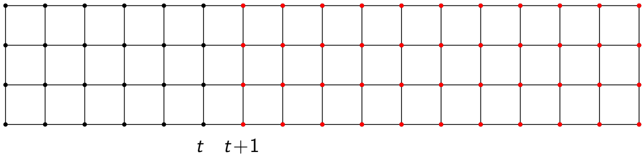
- Use free boundary conditions in t direction
- Rotate all spins after timeslice t

Modified Hasenbusch improved estimator



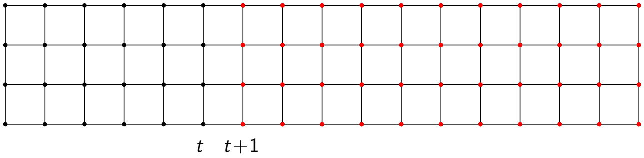
- Use free boundary conditions in t direction
- Rotate all spins after timeslice t
- $\longrightarrow$ Action only changes between timeslice t and $t+1$

Modified Hasenbusch improved estimator



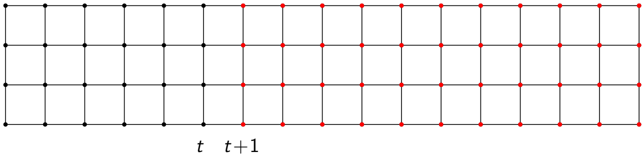
- Use free boundary conditions in t direction
- Rotate all spins after timeslice t
- $\longrightarrow$ Action only changes between timeslice t and $t+1$
- Integrate over all rotations, weighted with the action

Modified Hasenbusch improved estimator



- Use free boundary conditions in t direction
- Rotate all spins after timeslice t
- $\longrightarrow$ Action only changes between timeslice t and $t+1$
- Integrate over all rotations, weighted with the action
- Integration can not be done analytically
- Modification1: Only average over 4 rotation matrices

Modified Hasenbusch improved estimator



- Use free boundary conditions in t direction
- Rotate all spins after timeslice t
- $\longrightarrow$ Action only changes between timeslice t and $t+1$
- Integrate over all rotations, weighted with the action
- Integration can not be done analytically
- Modification1: Only average over 4 rotation matrices
- Modification2: Incorporate θ-term
- Allows to extract very high precision data

Correlation function

- Correlation between $\vec{E}(t_1)$ and $\vec{E}(t_2)$.
  $\vec{E}(t)$ is the average spin in timeslice t.

Correlation function

- Correlation between $\vec{E}(t_1)$ and $\vec{E}(t_2)$.
  $\vec{E}(t)$ is the average spin in timeslice t.
- Partition function of stripe $Z(t_1, t_2, \theta) = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q(t_1,t_2)}$

$$\begin{aligned} C(t_1 - t_2, \theta) &= \left\langle \vec{E}(t_1) \cdot \vec{E}(t_2) \right\rangle_\theta \\ &= \frac{1}{Z(t_1, t_2, \theta)} \int \mathcal{D}\vec{e}\ \vec{E}(t_1) \cdot \vec{E}(t_2)\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q(t_1,t_2)} \end{aligned}$$

Correlation function

- Correlation between $\vec{E}(t_1)$ and $\vec{E}(t_2)$.
  $\vec{E}(t)$ is the average spin in timeslice t.
- Partition function of stripe $Z(t_1, t_2, \theta) = \int \mathcal{D}\vec{e}\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q(t_1,t_2)}$

$$\begin{aligned} C(t_1 - t_2, \theta) &= \left\langle \vec{E}(t_1) \cdot \vec{E}(t_2) \right\rangle_\theta \\ &= \frac{1}{Z(t_1, t_2, \theta)} \int \mathcal{D}\vec{e}\ \vec{E}(t_1) \cdot \vec{E}(t_2)\ \mathrm{e}^{-S[\vec{e}]}\ \mathrm{e}^{i\theta Q(t_1,t_2)} \end{aligned}$$

Numerically, we calculate

$$C(t_1 - t_2, \theta) = \frac{C(t_1-t_2,\theta)Z(t_1,t_2,\theta)/Z(0)}{Z(t_1,t_2,\theta)/Z(0)} = \frac{\left\langle \vec{E}(t_1)\cdot\vec{E}(t_2)\ \mathrm{e}^{i\theta Q(t_1,t_2)} \right\rangle_{\theta=0}}{\left\langle \mathrm{e}^{i\theta Q(t_1,t_2)} \right\rangle_{\theta=0}}$$

Define $Q(t_1, t_2) = \sum_{\langle xyz \rangle \in S(t_1,t_2)} q_{\langle xyz \rangle}$, non integer

The step scaling function Σ

- Consider the correlation function at fixed $\theta = \theta_0$: $C(t, \theta_0)$
- Calculate exponential mass: $C(t, \theta_0) \sim e^{-m(\theta_0, L)t}$

The step scaling function Σ

- Consider the correlation function at fixed $\theta = \theta_0$: $C(t, \theta_0)$
- Calculate exponential mass: $C(t, \theta_0) \sim e^{-m(\theta_0, L)t}$
- Define quantity: $u_0 = m(\theta_0, L) \cdot L$
- Adjust until for example: $u_0 \approx 1.0595$

The step scaling function Σ

- Consider the correlation function at fixed $\theta = \theta_0$: $C(t, \theta_0)$
- Calculate exponential mass: $C(t, \theta_0) \sim \mathrm{e}^{-m(\theta_0, L)t}$
- Define quantity: $u_0 = m(\theta_0, L) \cdot L$
- Adjust until for example: $u_0 \approx 1.0595$
- Measure on the double size lattice: $u_1 = m(\theta, 2L) \cdot 2L$

The step scaling function Σ

- Consider the correlation function at fixed $\theta = \theta_0$: $C(t, \theta_0)$
- Calculate exponential mass: $C(t, \theta_0) \sim \mathrm{e}^{-m(\theta_0, L)t}$
- Define quantity: $u_0 = m(\theta_0, L) \cdot L$
- Adjust until for example: $u_0 \approx 1.0595$
- Measure on the double size lattice: $u_1 = m(\theta, 2L) \cdot 2L$
- Given L, u_0: $\Sigma(2, u_0, a/L) = u_0$

The step scaling function Σ

- Consider the correlation function at fixed $\theta = \theta_0$: $C(t, \theta_0)$
- Calculate exponential mass: $C(t, \theta_0) \sim \mathrm{e}^{-m(\theta_0, L)t}$
- Define quantity: $u_0 = m(\theta_0, L) \cdot L$
- Adjust until for example: $u_0 \approx 1.0595$
- Measure on the double size lattice: $u_1 = m(\theta, 2L) \cdot 2L$
- Given L, u_0: $\Sigma(2, u_0, a/L) = u_0$
- Continuum limit: $\sigma(2, u_0) = \lim\limits_{a/L \to 0} \Sigma(2, u_0, a/L)$