Fermion bag solutions to some sign problems in four-fermion field theories

Anyi Li

Institute for Nuclear Theory

In collaboration with

Shailesh Chandrasekharan

Duke University

Outline

- Continuum Gross-Neveu models
- Sign problem in lattice GN models: Examples with staggered fermions
- Fermion bag approach to the sign problem
- Conclusion

Continuum Gross-Neveu models

$$\mathcal{L}_{Z_2} = \bar{\psi}(\not{\partial} + m)\psi - \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

$$\mathcal{L}_{U(1)} = \bar{\psi}(\not{\partial} + m)\psi - \frac{g^2}{2N_f}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2]$$

$$\mathcal{L}_{SU(2)} = \bar{\psi}(\not{\partial} + m)\psi - \frac{g^2}{2N_f}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\boldsymbol{\tau}\psi)^2]$$

- Continuous symmetry broken - Goldstone modes
- QCD-like (NJL model)
- Strongly correlated fermion system
- No trivial fixed point $2<d<4$ for small N_f
- BCS superfluid
- QCD phase transition at finite temperature and density

Lattice GN models: Z_2 and U(1) symmetries
Examples with staggered fermions

i: flavor index
x,y: lattice sites

$$S_{GN} = \sum_{x,y,i} \overline{\chi}_i(x)(D[\bar{\phi}])_{x,y}\chi_i(y) + S_{AF}$$

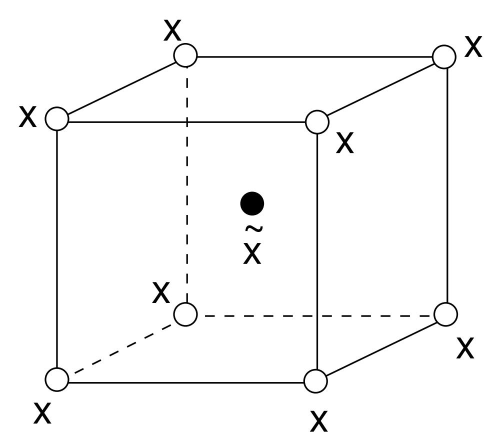
$$\left(D[\bar{\phi}]\right)_{xy} = D_{xy} + \delta_{xy}\,\bar{\phi}(x), \quad D_{x,y} = m\delta_{x,y} + \sum_{\alpha=1,2,3} \frac{\eta_{x,\alpha}}{2}\left[\delta_{x+\alpha,y} - \delta_{x,y+\alpha}\right].$$

Free stagger fermion

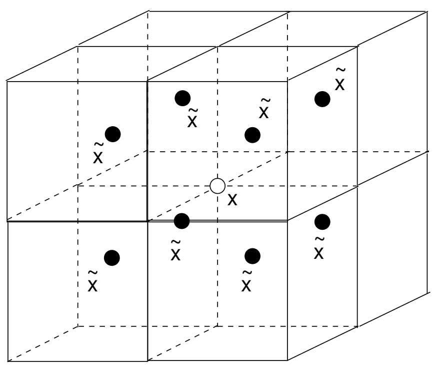
Z_2

$$S_{AF}[\sigma] = \frac{N}{2g^2}\sum_{\tilde{x}} \sigma^2(\tilde{x}),$$

$$\bar{\phi}(x) = \frac{1}{8}\sum_{\langle \tilde{x},x\rangle} \sigma(\tilde{x})$$



symmetry



$$\chi_i(x) \to \varepsilon(x)\chi_i(x), \;\; \overline{\chi}_i(x) \to -\overline{\chi}_i(x)\varepsilon(x)$$
$$\sigma(\tilde{x}) \to -\sigma(\tilde{x})$$

U(1) $$S_{GN} = \sum_{x,y,i} \overline{\chi}_i(x)(D[\bar{\phi}])_{x,y}\chi_i(y) + S_{AF}$$

$$S_{AF}[\sigma,\pi] = \frac{N}{4g^2}\sum_{\tilde{x}}\left(\sigma^2(\tilde{x}) + \pi^2(\tilde{x})\right),$$

$$\bar{\phi}(x) = \frac{1}{8}\sum_{\langle\tilde{x},x\rangle}\Big(\sigma(\tilde{x}) + i\varepsilon(x)\pi(\tilde{x})\Big),$$

symmetry

$$\chi_i(x) \to \mathrm{e}^{i\varepsilon(x)\theta/2}\chi_i(x),\ \ \overline{\chi}_i(x) \to \overline{\chi}_i(x)\mathrm{e}^{i\varepsilon(x)\theta/2}$$

$$\sigma(\tilde{x}) \to \sigma(\tilde{x})\cos\theta + \pi(\tilde{x})\sin\theta,$$
$$\pi(\tilde{x}) \to \pi(\tilde{x})\cos\theta - \sigma(\tilde{x})\sin\theta$$

Partition functions (Conventional approach)

$$\left(D[\bar{\phi}]\right)_{xy} = D_{xy} + \delta_{xy}\,\bar{\phi}(x),$$

V x V matrix. Real, no positivity

$$Z_{Z_2} = \int [\mathcal{D}\sigma]\, e^{-S_{AF}[\sigma]} \left\{ \mathrm{Det} D([\bar{\phi}]) \right\}^N,$$

Odd N

$$Z_{U(1)} = \int [\mathcal{D}\sigma\mathcal{D}\pi]\, e^{-S_{AF}[\sigma,\pi]} \left\{ \mathrm{Det} D([\bar{\phi}]) \right\}^N$$

Even & Odd N

$\Rightarrow$ **Sign problem!**

V x V matrix. Complex

Fermion bag approach

Define a k-point correlator for flavor i:

Free stagger fermion

$$C_i(x_{i_1},...,x_{i_{k_i}}) = \int [d\overline{\chi}_i d\chi_i] \mathrm{e}^{-\sum_{x,y}\ \overline{\chi}_i(x)\ D_{xy}\ \chi_i(y)}$$
$$\overline{\chi}_i(x_{i_1})\chi_i(x_{i_1})\ ...\ \overline{\chi}_i(x_{i_{k_i}})\chi_i(x_{i_{k_i}})$$

$$C_i(x_{i_1},..,x_{i_{k_i}}) = \mathrm{Det}(D)\ \mathrm{Det}(G[\{x\}_i]) = \mathrm{Det}(W[\{x\}_i])$$

$\boxed{>= 0}$

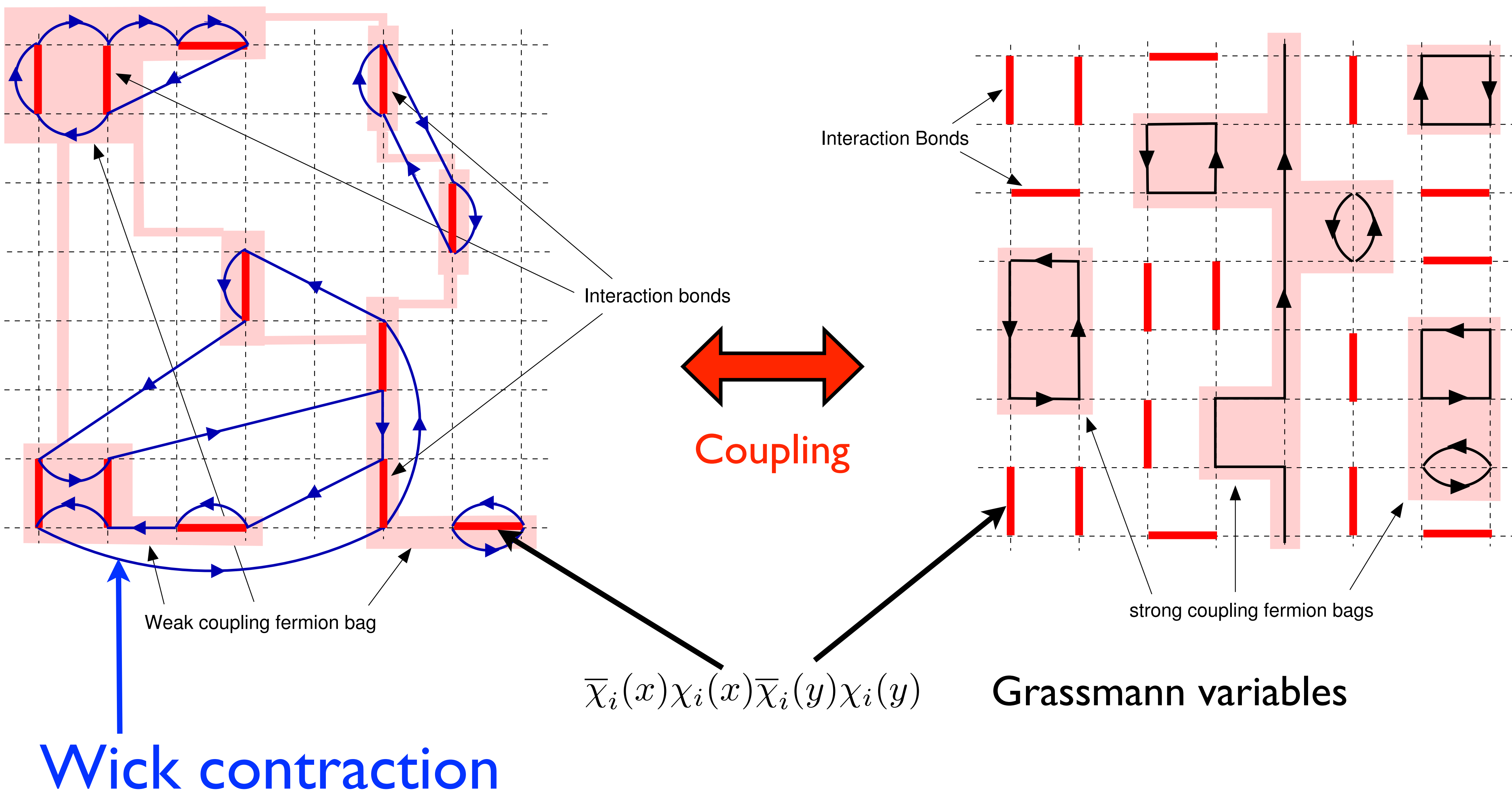
Similar structure as free stagger matrix, but all $\{x\}_i$ sites have been dropped, can be broken into smaller matrixes

Free stagger matrix (V x V matrix)

propagator matrix
(k_i x k_i matrix)

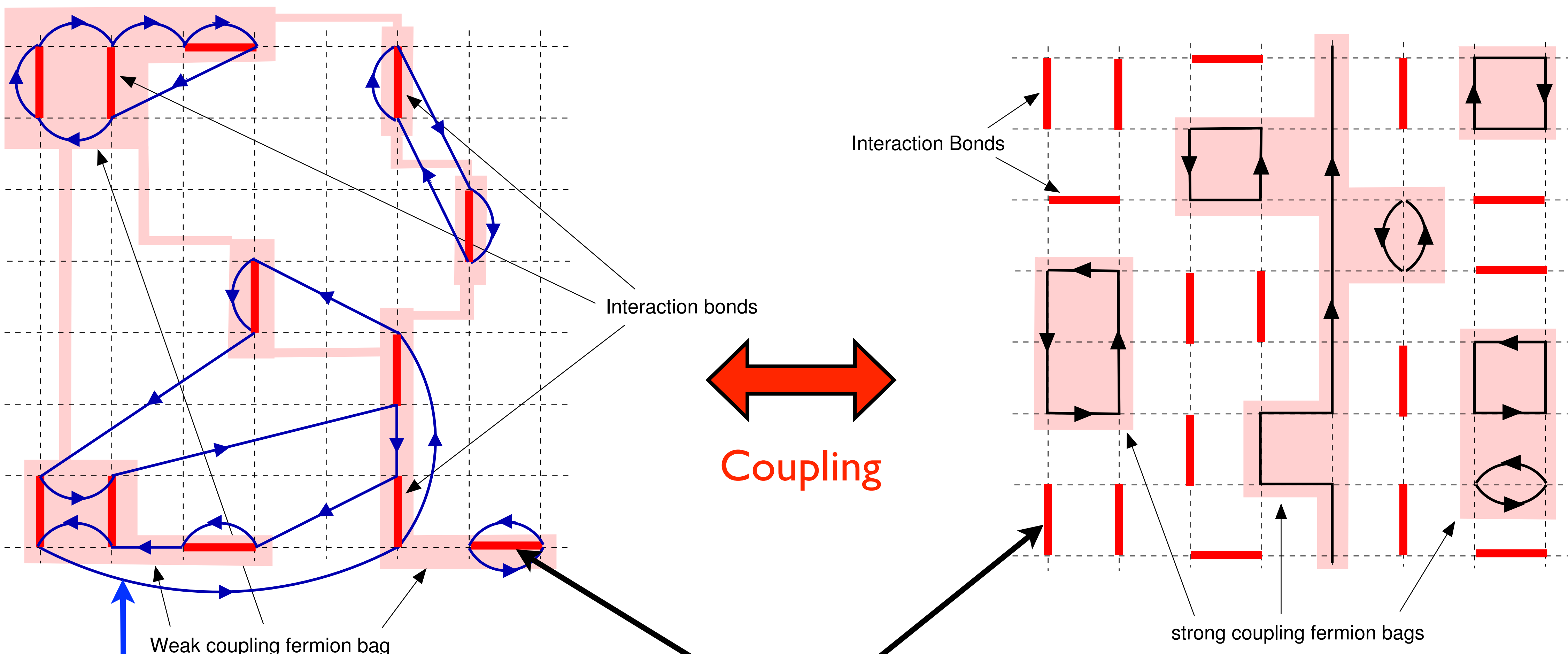
Dual relation

$$\mathrm{Det}(D)\ \mathrm{Det}(G[\{x\}_i]) = \mathrm{Det}(W[\{x\}_i])$$



Dual relation

$$\mathrm{Det}(D)\ \mathrm{Det}(G[\{x\}_i]) = \mathrm{Det}(W[\{x\}_i])$$



$\overline{\chi}_i(x)\chi_i(x)\overline{\chi}_i(y)\chi_i(y)$ Grassmann variables

Wick contraction

Fermion bag : resummation of the "optimized" fermionic degree of freedom

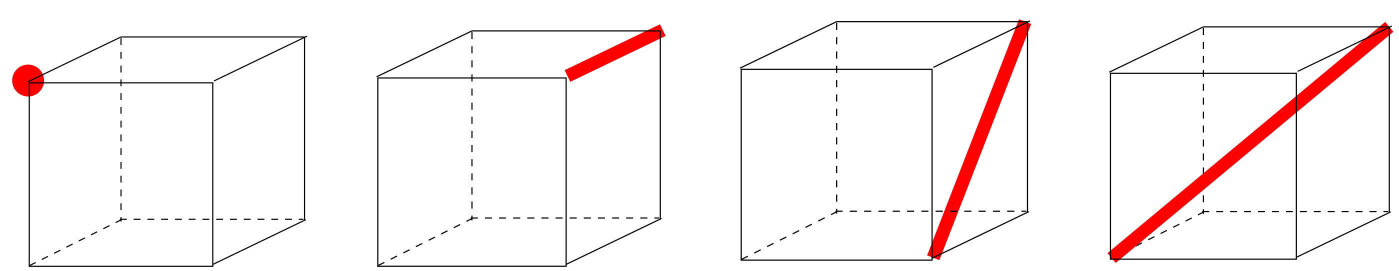
Solution to the sign problem in GN model : Z_2

Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int d\sigma(\tilde{x})\ \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$ $$S_I(\tilde{x}) = -\frac{g^2}{128N}\Big[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\Big]^2$$

$$\mathcal{B}_{\text{bond}} = \sum_{i,j,\langle xy\rangle\in\text{bond}} \overline{\chi}_i(x)\chi_i(x)\overline{\chi}_j(y)\chi_j(y)$$



S L F B

Summation of all $\tilde{x}$

$$\mathcal{B}_{\mathrm{bond}} = \sum_{i,j,\langle xy\rangle \in \mathrm{bond}} \overline{\chi}_i(x)\chi_i(x)\overline{\chi}_j(y)\chi_j(y)$$

$$S_{Z_2,\mathrm{int}} = \sum_{\tilde{x}} S_I(\tilde{x}) = U_S\mathcal{B}_S + U_L\mathcal{B}_L + U_F\mathcal{B}_F + U_B\mathcal{B}_B$$

Couplings for different types of bond

$$U_S/4 = U_L/4 = U_F/2 = U_B = g^2/(64N)$$

Z_2 model partition function:

$$Z_{Z_2} = \int \prod_i [d\overline{\chi}_i d\chi_i]\ \mathrm{e}^{-S_0 - S_{Z_2,int}}$$

$$= \sum_{[b]} U_S^{n_S} U_L^{n_L} U_F^{n_F} U_B^{n_B} \int \prod_i [d\overline{\chi}_i d\chi_i]\ \mathrm{e}^{-S_0}$$

$$\times \prod_i \overline{\chi}_i(x_{i_1})\chi_i(x_{i_2})...\overline{\chi}_i(x_{i_{k_i}})\chi_i(x_{i_{k_i}})$$

$$= \sum_{[b]} U_S^{n_S} U_L^{n_L} U_F^{n_F} U_B^{n_B} \left\{ \prod_i C_i(x_{i_1}, .., x_{i_{k_i}}) \right\}$$

$$= \sum_{[b]} U_S^{n_S} U_L^{n_L} U_F^{n_F} U_B^{n_B} \prod_i \mathrm{Det}(W[\{x\}_i])$$

$$\exp[\overline{\chi}(x_p)_i\chi_i(x_p)\overline{\chi}_j(x_q)\chi_j(x_q)]$$

$$= 1 + \overline{\chi}(x_p)_i\chi_i(x_p)\overline{\chi}_j(x_q)\chi_j(x_q)$$

$$b_{ij}(x_p, x_q) = 0, 1$$

bond variable

Summation of all $\tilde{x}$

$$\mathcal{B}_{\rm bond} = \sum_{i,j,\langle xy\rangle \in {\rm bond}} \overline{\chi}_i(x)\chi_i(x)\overline{\chi}_j(y)\chi_j(y)$$

$$S_{Z_2,{\rm int}} = \sum_{\tilde{x}} S_I(\tilde{x}) = U_S\mathcal{B}_S + U_L\mathcal{B}_L + U_F\mathcal{B}_F + U_B\mathcal{B}_B$$

Couplings for different types of bond

$$U_S/4 = U_L/4 = U_F/2 = U_B = g^2/(64N)$$

Z_2 model partition function:

$$Z_{Z_2} = \int \prod_i [d\overline{\chi}_i d\chi_i]\ {\rm e}^{-S_0 - S_{Z_2,int}}$$

$$= \sum_{[b]} U_S^{n_S} U_L^{n_L} U_F^{n_F} U_B^{n_B} \int \prod_i [d\overline{\chi}_i d\chi_i]\ {\rm e}^{-S_0}$$

$$\times \prod_i \overline{\chi}_i(x_{i_1})\chi_i(x_{i_2})...\overline{\chi}_i(x_{i_{k_i}})\chi_i(x_{i_{k_i}})$$

$$= \sum_{[b]} U_S^{n_S} U_L^{n_L} U_F^{n_F} U_B^{n_B} \left\{ \prod_i C_i(x_{i_1},..,x_{i_{k_i}}) \right\}$$

$$= \sum_{[b]} U_S^{n_S} U_L^{n_L} U_F^{n_F} U_B^{n_B} \prod_i {\rm Det}(W[\{x\}_i])$$

$$\exp[\overline{\chi}(x_p)_i\chi_i(x_p)\overline{\chi}_j(x_q)\chi_j(x_q)]$$

$$= 1 + \overline{\chi}(x_p)_i\chi_i(x_p)\overline{\chi}_j(x_q)\chi_j(x_q)$$

$$\Downarrow$$

$$b_{ij}(x_p, x_q) = 0, 1$$

bond variable

NO Sign problem!

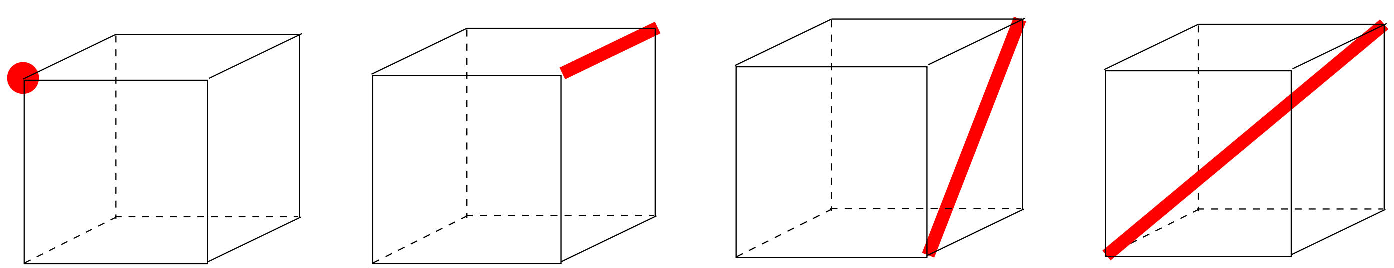
Solution to the sign problem in GN model : U(1)

Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int [d\sigma(\tilde{x}) d\pi(\tilde{x})]\, \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \overline{\chi}_i(x)\chi_i(x)\right)} \times \mathrm{e}^{-i\frac{\pi(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$

$$S_I(\tilde{x}) = \frac{g^2}{64N}\left\{\Big[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\Big]^2 - \Big[\sum_{i,[x,\tilde{x}]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\Big]^2\right\}$$



S L F B

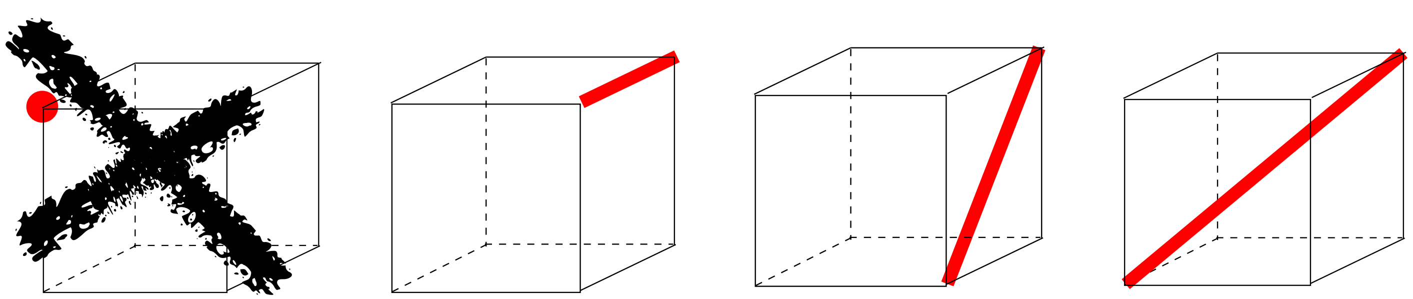
Solution to the sign problem in GN model : U(1)

Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int [d\sigma(\tilde{x}) d\pi(\tilde{x})]\, \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \overline{\chi}_i(x)\chi_i(x)\right)} \times \mathrm{e}^{-i\frac{\pi(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$

$$S_I(\tilde{x}) = \frac{g^2}{64N}\left\{\Big[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\Big]^2 - \Big[\sum_{i,[x,\tilde{x}]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\Big]^2\right\}$$



S L F B

Solution to the sign problem in GN model : U(1)

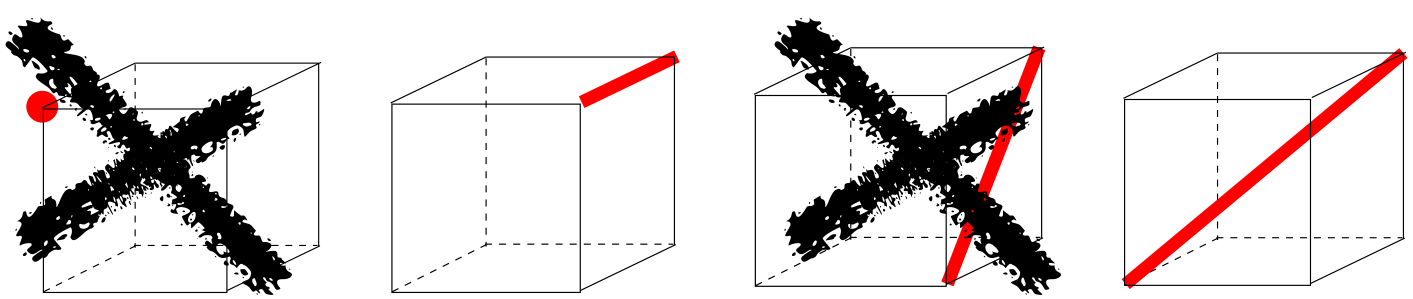
Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int [d\sigma(\tilde{x}) d\pi(\tilde{x})] \, \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \overline{\chi}_i(x)\chi_i(x)\right)}$$

$$\times \, \mathrm{e}^{-i\frac{\pi(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$

$$S_I(\tilde{x}) = \frac{g^2}{64N}\left\{\Big[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\Big]^2 - \Big[\sum_{i,[x,\tilde{x}]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\Big]^2\right\}$$



S L F B

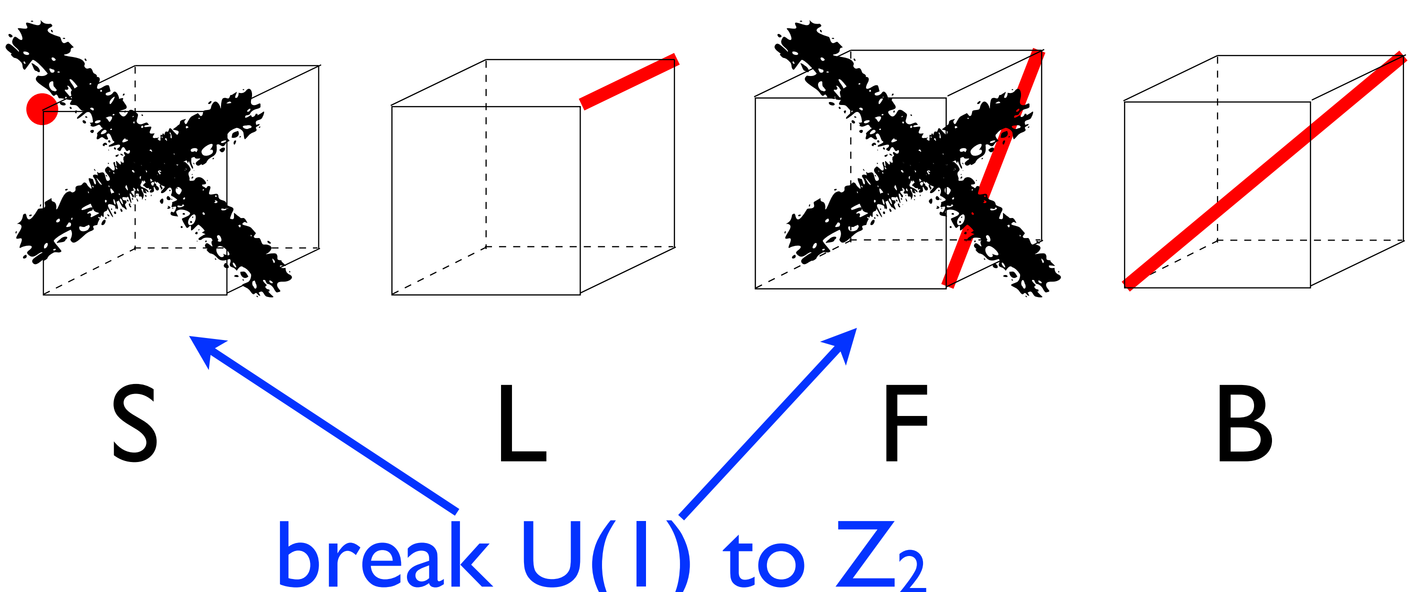
Solution to the sign problem in GN model : U(1)

Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int [d\sigma(\tilde{x}) d\pi(\tilde{x})]\, \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \overline{\chi}_i(x)\chi_i(x)\right)} \times \mathrm{e}^{-i\frac{\pi(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$

$$S_I(\tilde{x}) = \frac{g^2}{64N}\left\{\Big[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\Big]^2 - \Big[\sum_{i,[x,\tilde{x}]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\Big]^2\right\}$$



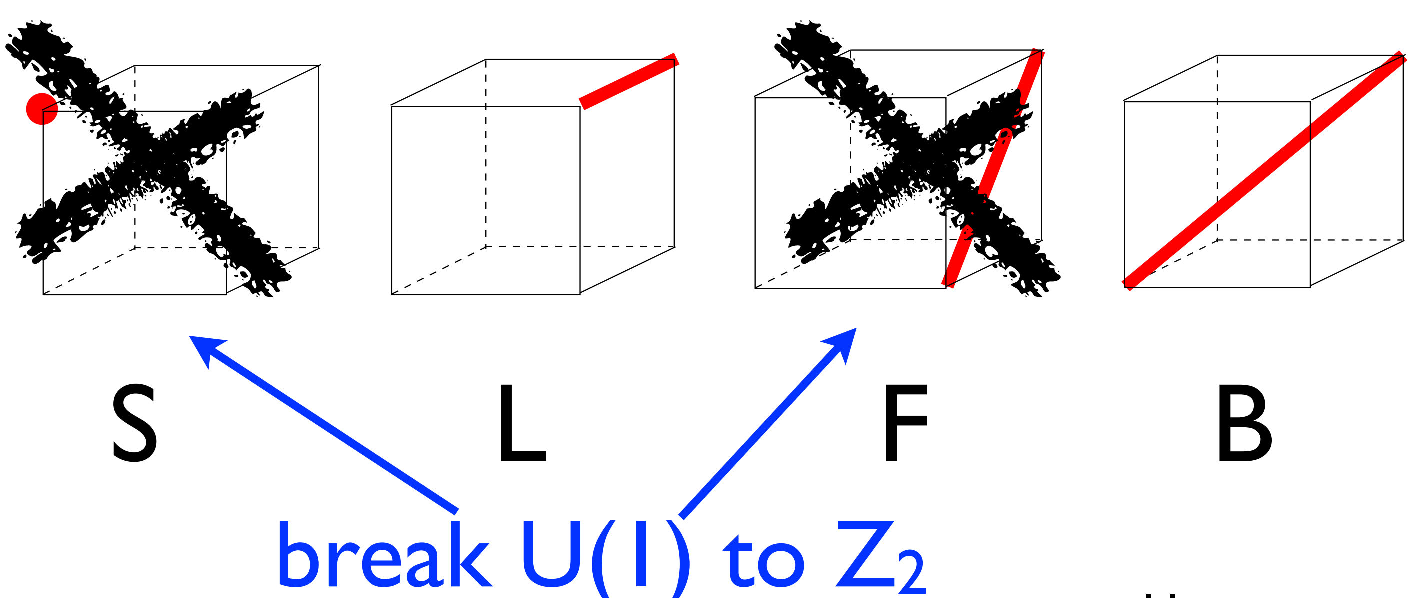
Solution to the sign problem in GN model : U(1)

Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int [d\sigma(\tilde{x}) d\pi(\tilde{x})]\, \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \overline{\chi}_i(x)\chi_i(x)\right)} \times \mathrm{e}^{-i\frac{\pi(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$

$$S_I(\tilde{x}) = \frac{g^2}{64N}\left\{\Big[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\Big]^2 - \Big[\sum_{i,[x,\tilde{x}]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\Big]^2\right\}$$



$$S_{U(1)} = S_0 + U_L\mathcal{B}_L + U_B\mathcal{B}_B$$

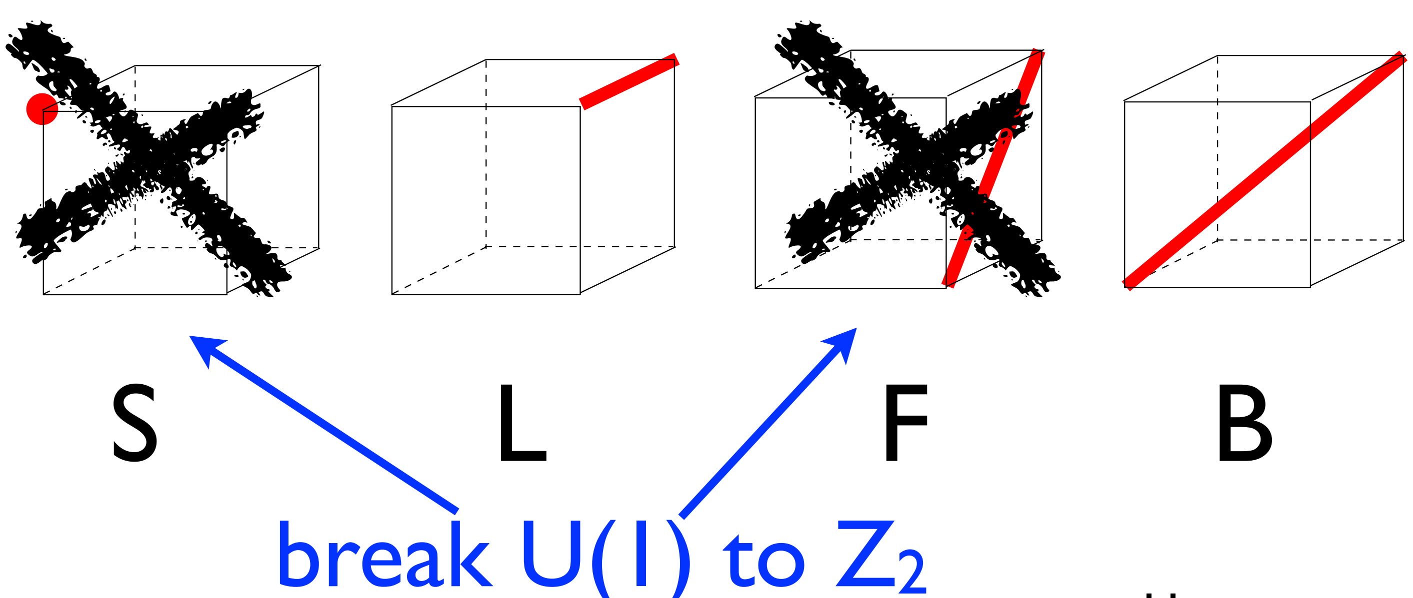
Solution to the sign problem in GN model : U(1)

Integrate out auxiliary field first, needs to compute

$$I_{\tilde{x}} = \int [d\sigma(\tilde{x})d\pi(\tilde{x})]\, \mathrm{e}^{-S_{AF} - \frac{\sigma(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \overline{\chi}_i(x)\chi_i(x)\right)} \times \mathrm{e}^{-i\frac{\pi(\tilde{x})}{8}\left(\sum_{i,[\tilde{x},x]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right)} = \mathcal{N}\mathrm{e}^{-S_I(\tilde{x})}$$

for each $\tilde{x}$

$$S_I(\tilde{x}) = \frac{g^2}{64N}\left\{\left[\sum_{i,[x,\tilde{x}]} \overline{\chi}_i(x)\chi_i(x)\right]^2 - \left[\sum_{i,[x,\tilde{x}]} \varepsilon(x)\overline{\chi}_i(x)\chi_i(x)\right]^2\right\}$$



$$S_{U(1)} = S_0 + U_L\mathcal{B}_L + U_B\mathcal{B}_B$$

NO Sign problem!

New results: unpublished

Solution to the sign problem in SU(2) Yukawa model (nucleon-pion interaction)

$$\mathcal{L}_{SU(2)} = \bar{\psi}(\not\partial + m)\psi + ig\bar{\psi}\boldsymbol{\tau}\boldsymbol{\pi}\gamma_5\psi + \mathcal{L}_0(\boldsymbol{\pi})$$

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix} \qquad \boldsymbol{\pi} = (\pi_1 \quad \pi_2 \quad \pi_3) \qquad \boldsymbol{\tau} = (\tau_1 \quad \tau_2 \quad \tau_3)$$

global SU(2) rotation $\psi \to U\psi, \boldsymbol{\tau}\boldsymbol{\pi} \to U\boldsymbol{\tau}\boldsymbol{\pi}U^\dagger \quad U = \exp(-i\boldsymbol{\alpha}\boldsymbol{\tau}/2)$

On lattice with Wilson fermion

$$\begin{pmatrix} \not\partial + m + ig\pi_3\gamma_5 & (ig\pi_1 + g\pi_2)\gamma_5 \\ (ig\pi_1 - g\pi_2)\gamma_5 & \not\partial + m - ig\pi_3\gamma_5 \end{pmatrix} \longrightarrow \gamma_5 \begin{pmatrix} \gamma_5(\not\partial + m) + ig\pi_3 & ig\pi_1 + g\pi_2 \\ ig\pi_1 - g\pi_2 & \gamma_5(\not\partial + m) - ig\pi_3 \end{pmatrix}$$

$$\gamma_5 \begin{pmatrix} D[\pi_3] & \phi[\pi_1, \pi_2] \\ -\phi^\dagger[\pi_1, \pi_2] & D^\dagger[\pi_3] \end{pmatrix}$$

Conventional: real, no positivity sign problem

Fermion bag approach: NO sign problem

S. Ch. hep-lat/arXiv:1205.0084, (accepted for PRD(R))

Conclusions

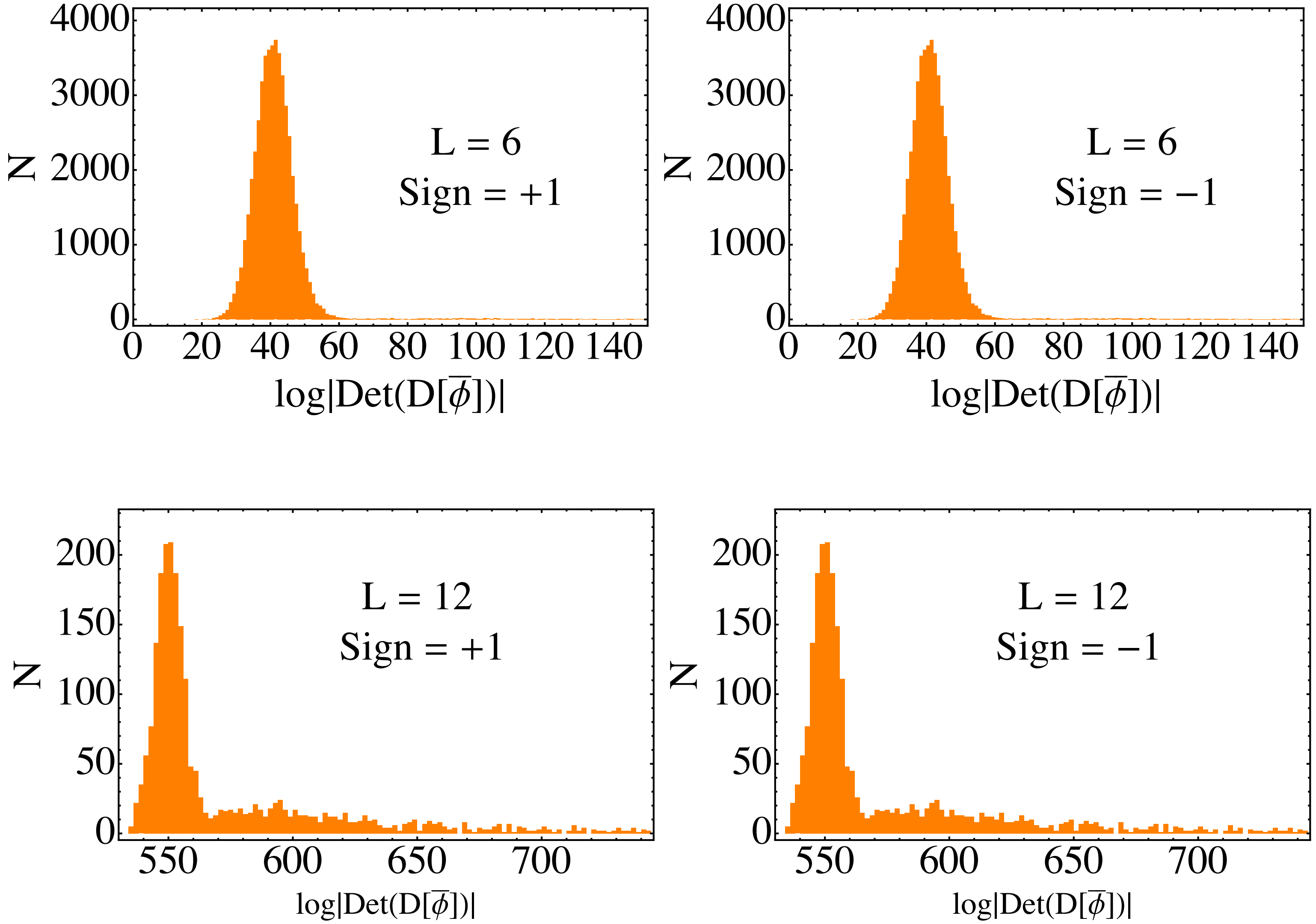
- Fermion bag approach: A new method for resummation of fermionic degrees of freedom
- Solutions to previously unsolved sign problems in GN and Yukawa models
- Implementation through efficient algorithms (large volume, chiral limit, no critical slowing down...)
- Applications to many interesting physics (nuclear effective field theory, unitary fermi gas, graphene physics...)

References:

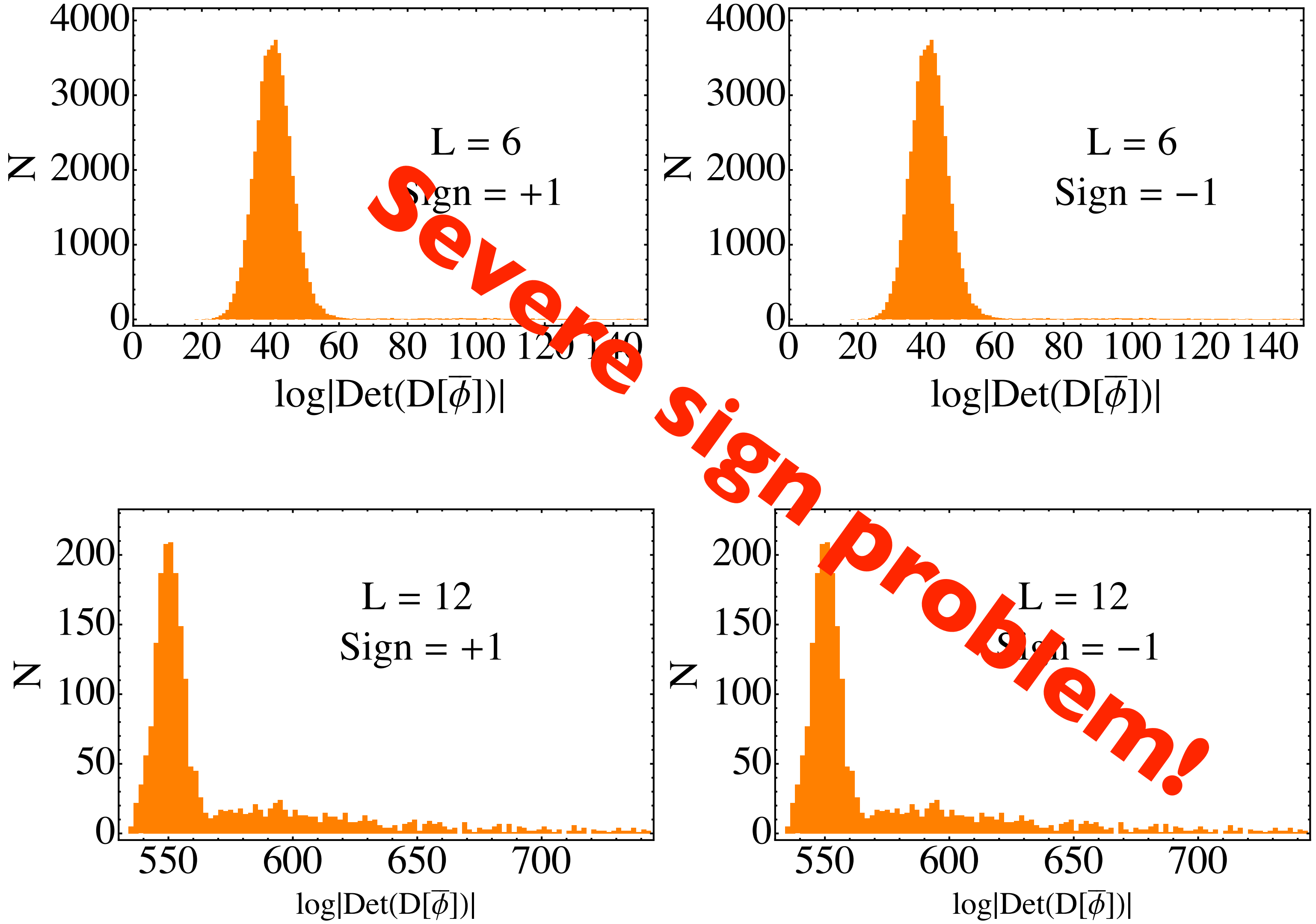
S. Ch. & Anyi Li hep-lat/arXiv:1111.5276, PoS Lattice (2011)
S. Ch. & Anyi Li hep-lat/arXiv:1111.7204, Phys. Rev. Lett. 108, 140404 (2012)
S. Ch. & Anyi Li hep-lat/arXiv:1202.6572, Phys.Rev. D85 (2012) 091502 (R)
S. Ch. hep-lat/arXiv:1205.0084, (accepted for PRD(R))

Backup

Severity of the sign problem in GN model

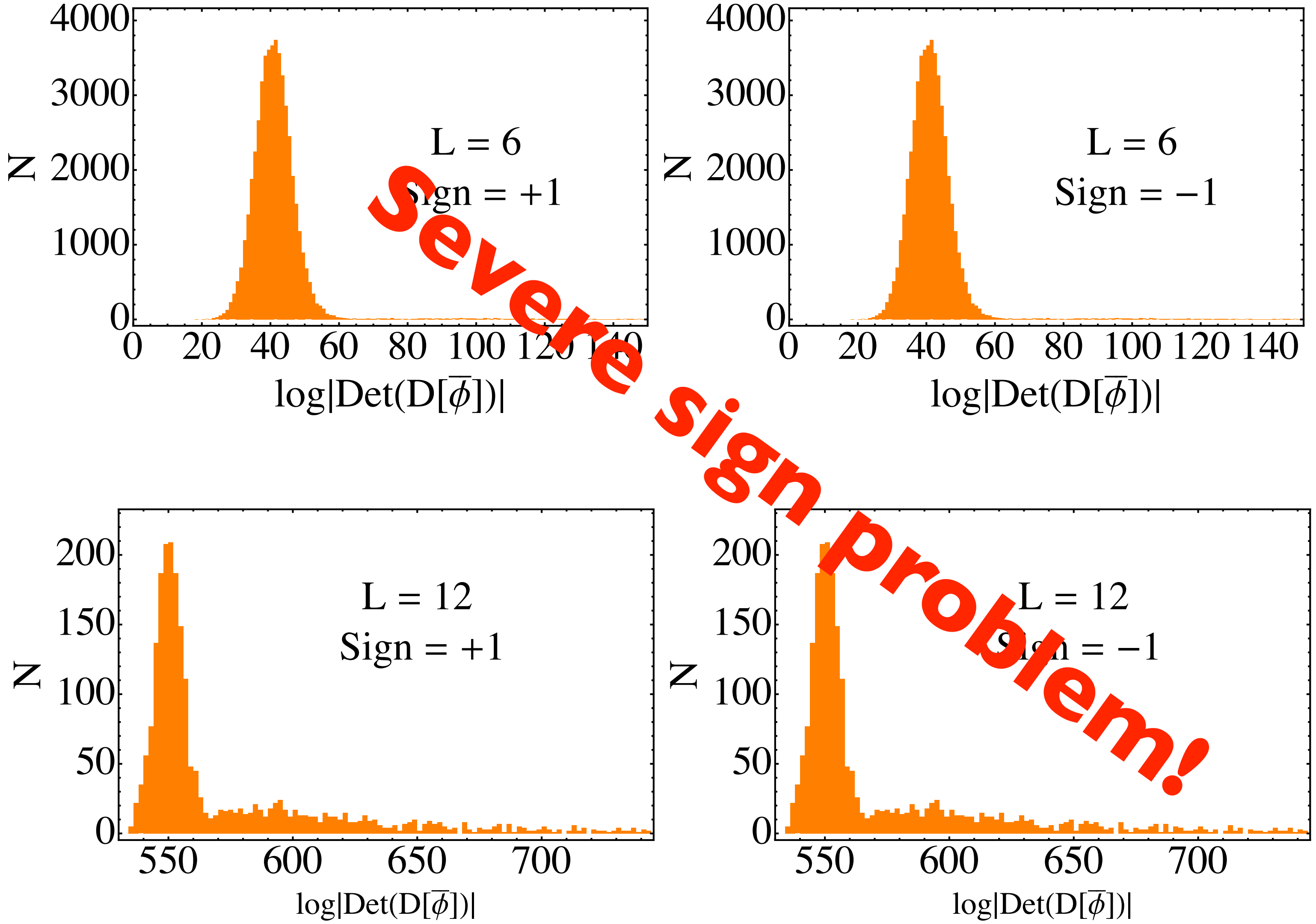


Severity of the sign problem in GN model



Tuesday, June 26, 12

Severity of the sign problem in GN model

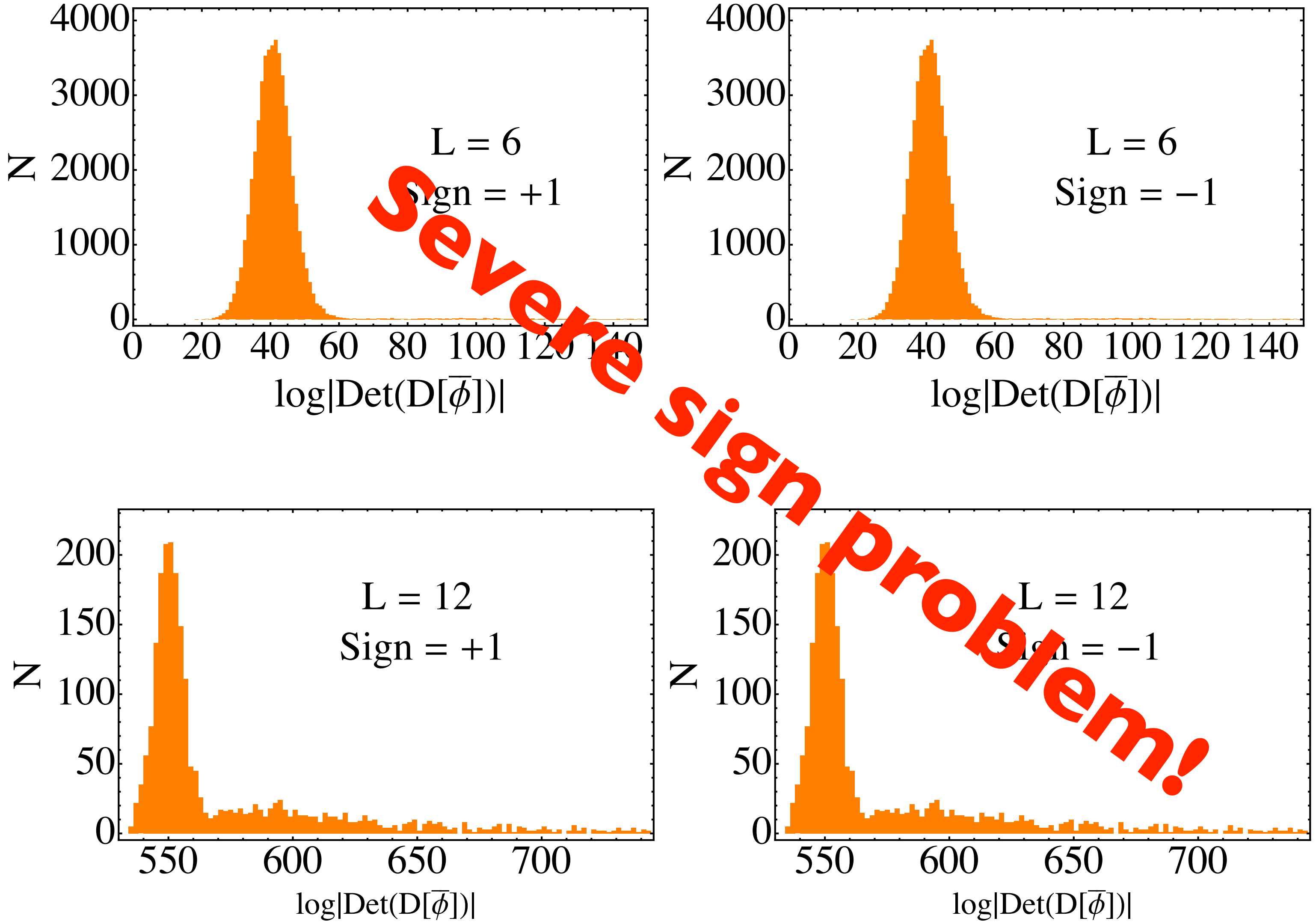


Solutions?

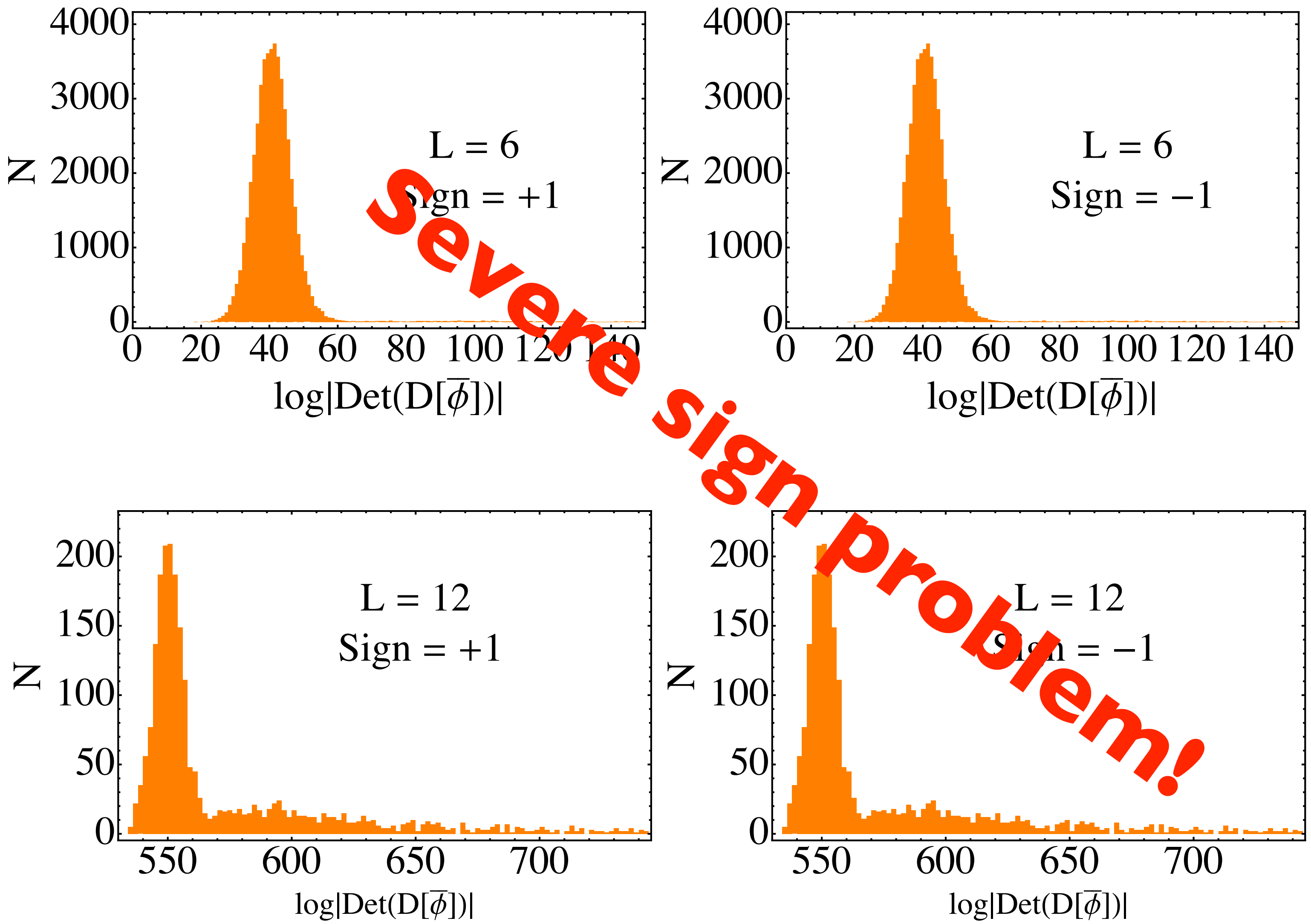
Z_2: even flavors

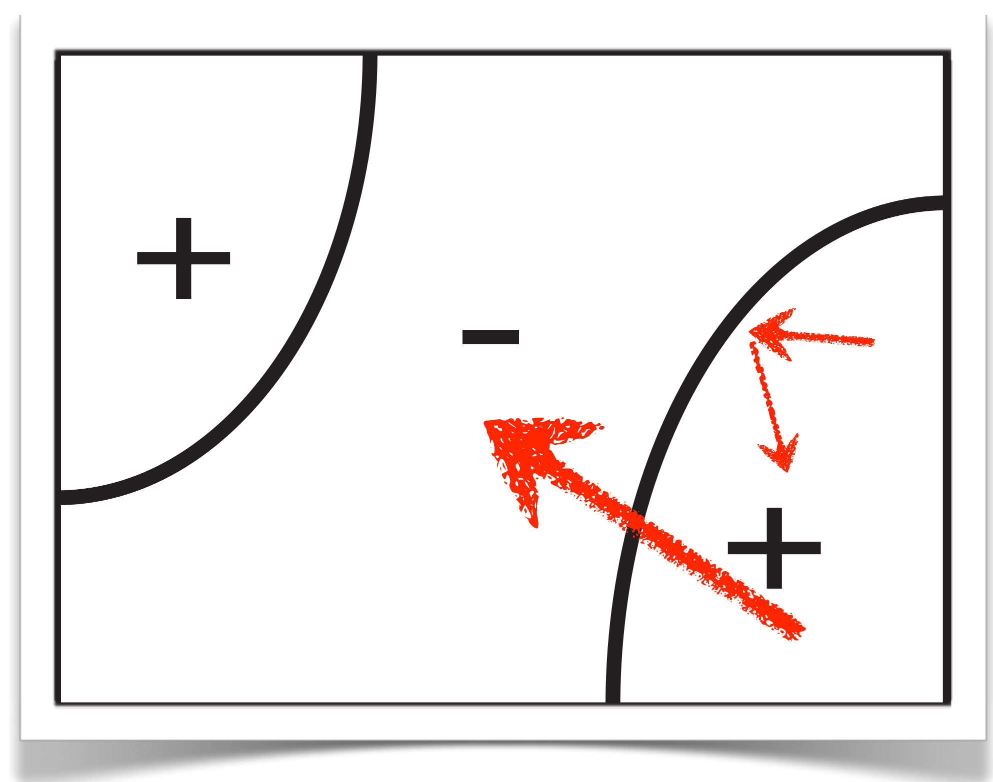
U(1): conjugate field

Severity of the sign problem in GN model

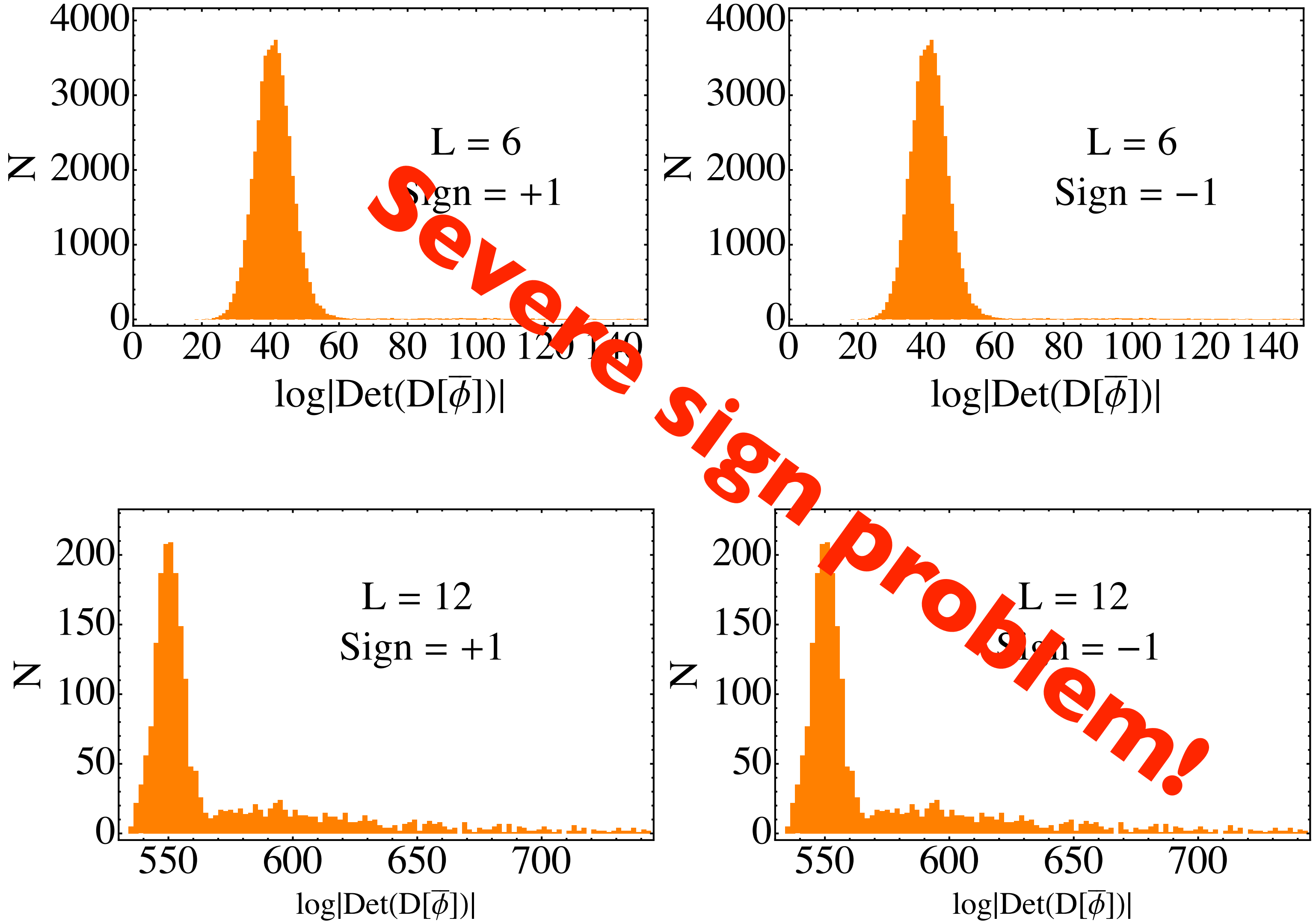


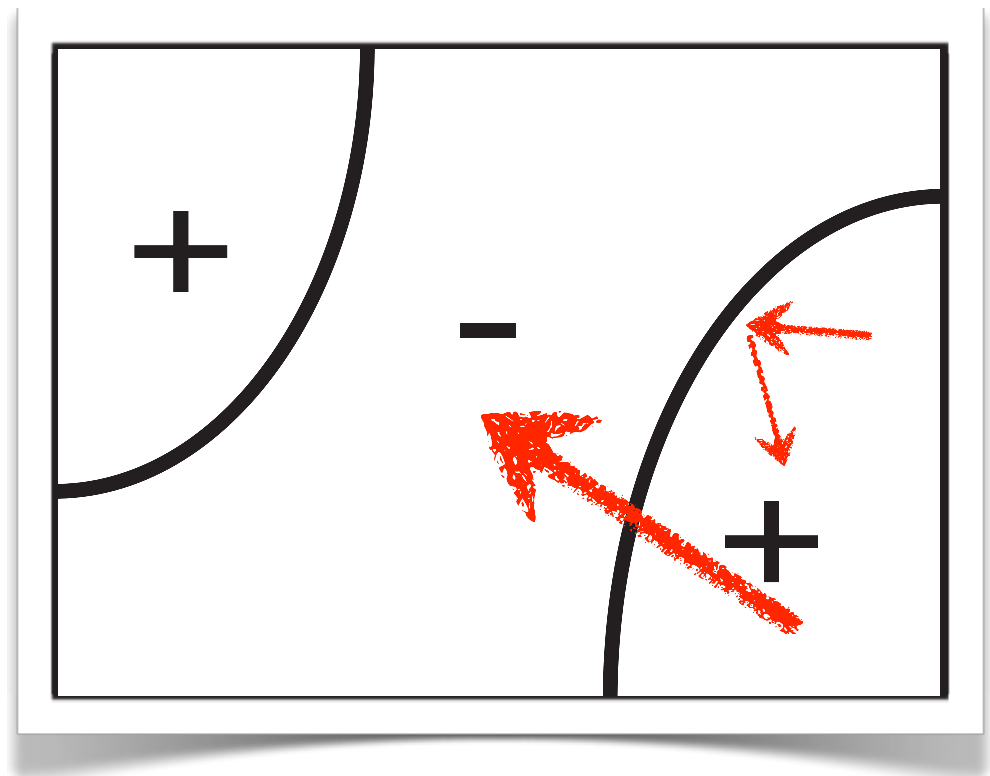
Severity of the sign problem in GN model





Severity of the sign problem in GN model





Ergodicity ?

Long autocorrelation time