HQET Flavor Currents Using Automated Lattice Perturbation Theory

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Thanks to: P. Fritzsch, N. Garron, G.M. von Hippel, M. Kurth, H. Simma, S. Takeda, U. Wolff

LATTICE 2012, Cairns

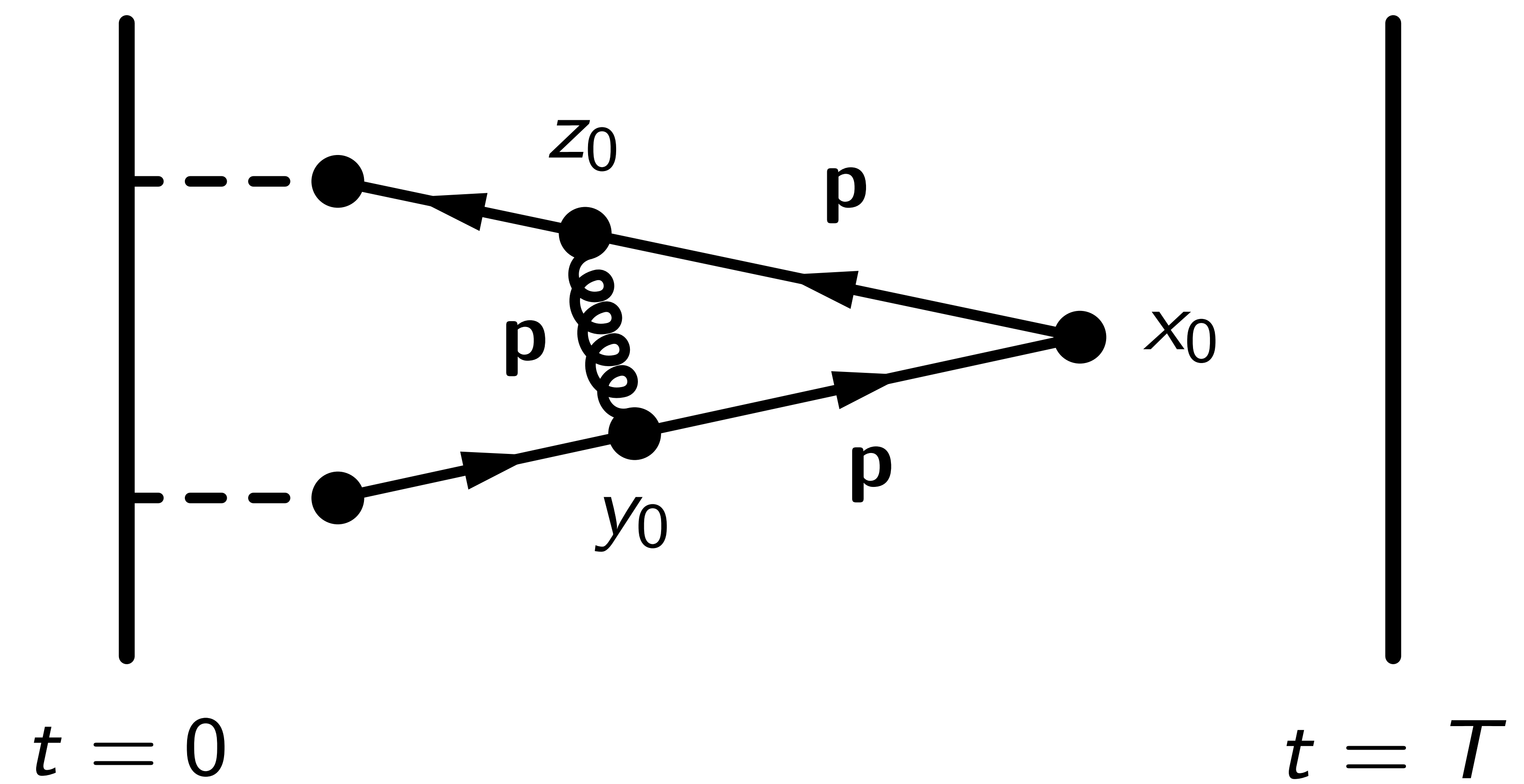
How to/why automate lattice PT?

Automated LPT and HQET matching.

Automation?!

We want to evaluate expressions like this one:

$$\sum_{\mathbf{p},y_0,z_0} \mathrm{tr} \left\{ \right.$$



$$\left. \right\}$$

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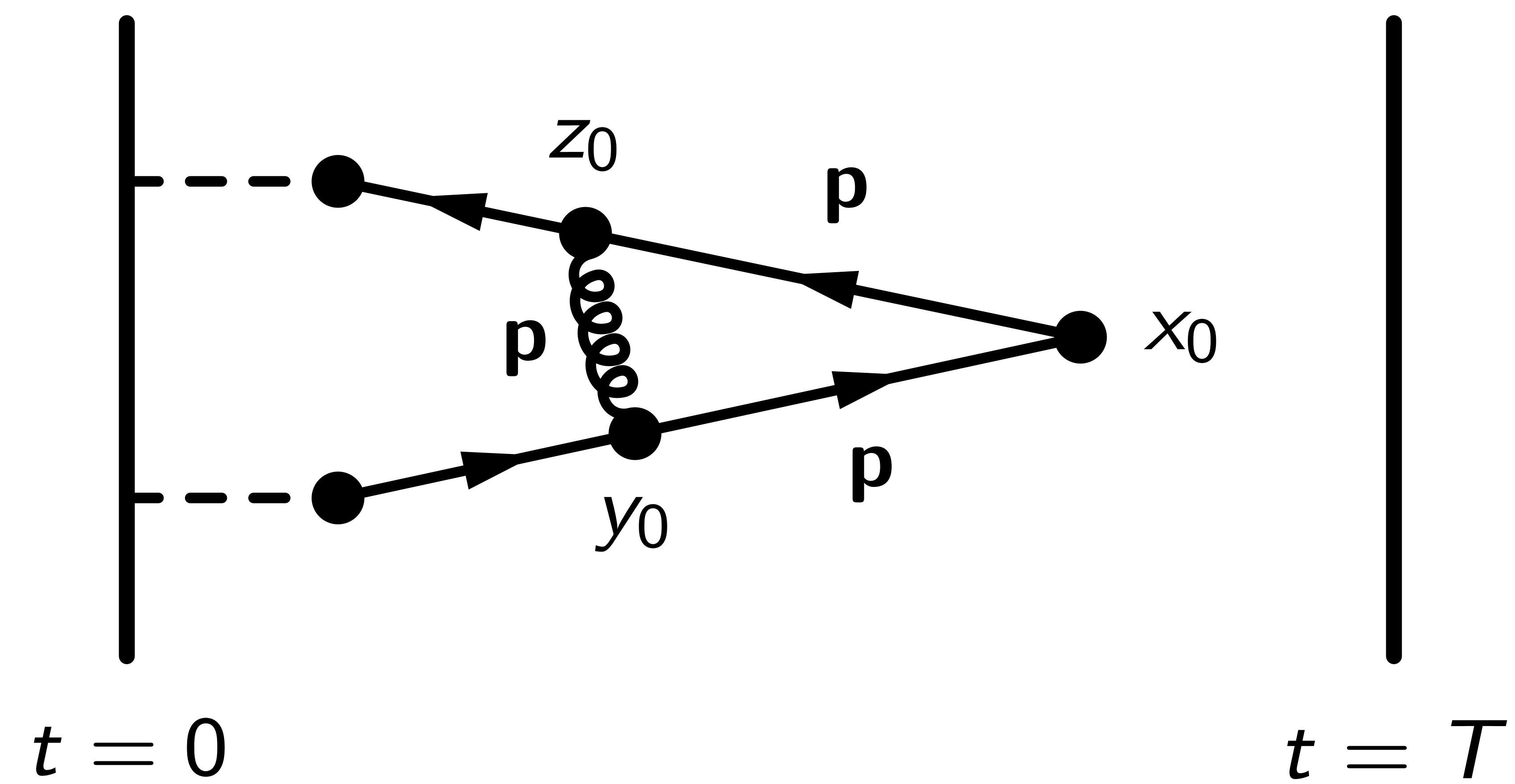

With *automation* I mean automatic generation of

- Feynman rules
- Feynman diagrams
- Code for evaluation

The Schrödinger Functional

	Fermions	**Gluons**
Dirichlet boundary conditions in **time**.	$\zeta(\mathbf{x}) = \sum_y \tilde{K}(\mathbf{x}, y)\psi(y)$ $\overline{\zeta}(\mathbf{x}) = \sum_y \overline{\psi}(y)K(y, \mathbf{x})$ $P_+\psi(x)\|_{x_0=0} = \overline{\psi}(x)P_-\|_{x_0=0} = 0$	$U_k(x)\|_{x_0=0} = e^{C(\mathbf{x})}$ $U_k(x)\|_{x_0=T} = e^{C'(\mathbf{x})}$
Periodic boundary conditions in **space**.	$\psi(x + \hat{k}L) = e^{i\theta_k}\psi(x)$	$U_\mu(x + \hat{k}L) = U_\mu(x)$

M. Lüscher, P. Weisz, R. Narayanan, U. Wolff, 1992. S. Sint, 1996, M. Lüscher 2006.

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The Schrödinger Functional

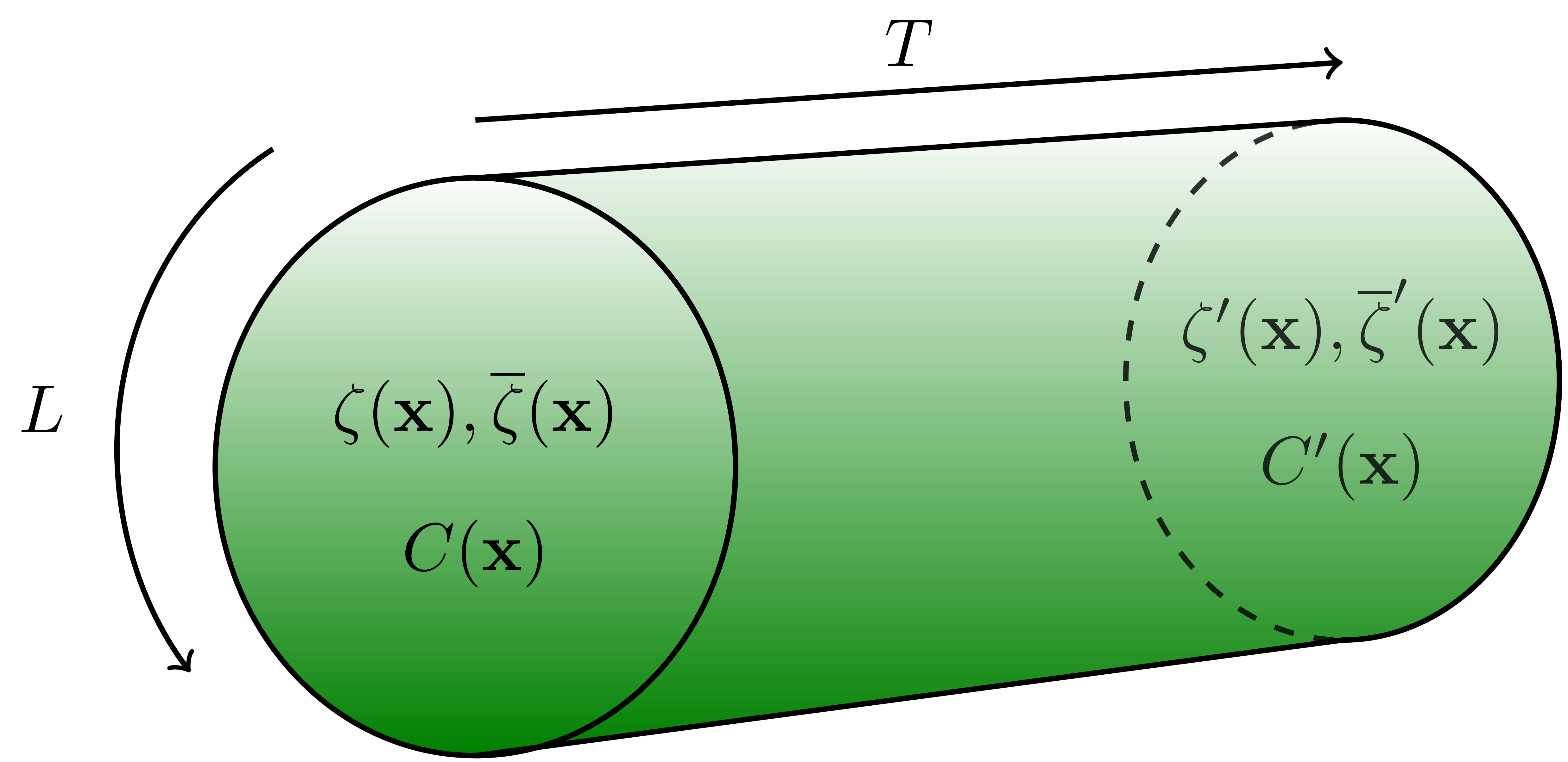
<table>
<tr><th></th><th>Fermions</th><th>Gluons</th></tr>
<tr><td>Dirichlet boundary conditions in time.</td><td colspan="2" rowspan="2">
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Diagram Generation

For example, take $\mathcal{O} = \overline{\psi}_1(x)\,\gamma_0\gamma_5\,\psi_2(x)\,\overline{\zeta}_2(\mathbf{y})\,\gamma_5\,\zeta_1(\mathbf{z})$.

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First perform fermion Wick contractions,

$$\langle\mathcal{O}\rangle = \langle\, \overline{\psi}_1(x)\,\gamma_5\,\psi_2(x)\,\overline{\zeta}_2(\mathbf{y})\,\gamma_5\,\zeta_1(\mathbf{z})\,\rangle_G$$

$$= \langle\, [\zeta_1(\mathbf{z})\overline{\psi}_1(x)]_F\,\gamma_5\,[\psi_2(x)\overline{\zeta}_2(\mathbf{y})]_F\,\gamma_5\,\rangle_G.$$

Then the gauge average,

$$\langle f\rangle_G = \frac{1}{Z}\int \mathcal{D}[U]e^{-S_G[U]}\mathcal{Z}_F[U]f[U],$$

$$\mathcal{Z}_F[U] = \int \mathcal{D}[\overline{\psi},\psi]\,e^{-S_F[\overline{\psi},\psi,U]}.$$

Fermion Wick Contractions

$[\mathcal{O}]_F = [\mathcal{O}]_F^{(0)} + g_0[\mathcal{O}]_F^{(1)} + \ldots$ contains only

$$S(x,y) = S^{(0)}(x,y) + g_0 S^{(1)}(x,y) + \ldots$$

$K(x,\mathbf{y}) = K^{(0)}(x,\mathbf{y}) + g_0 K^{(1)}(x,\mathbf{y}) + \ldots$ etc.

Which can be constructed **automatically**.

$S = (D+m)^{-1}$ (for a given gauge field)

M. Lüscher, P. Weisz, 1996.

M. Lüscher, P. Weisz, 1986. A. Hart, G.M. von Hippel, et. al., 2009. S. Takeda, U. Wolff, 2007.

ALPT - The Comic

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Your favorite observable

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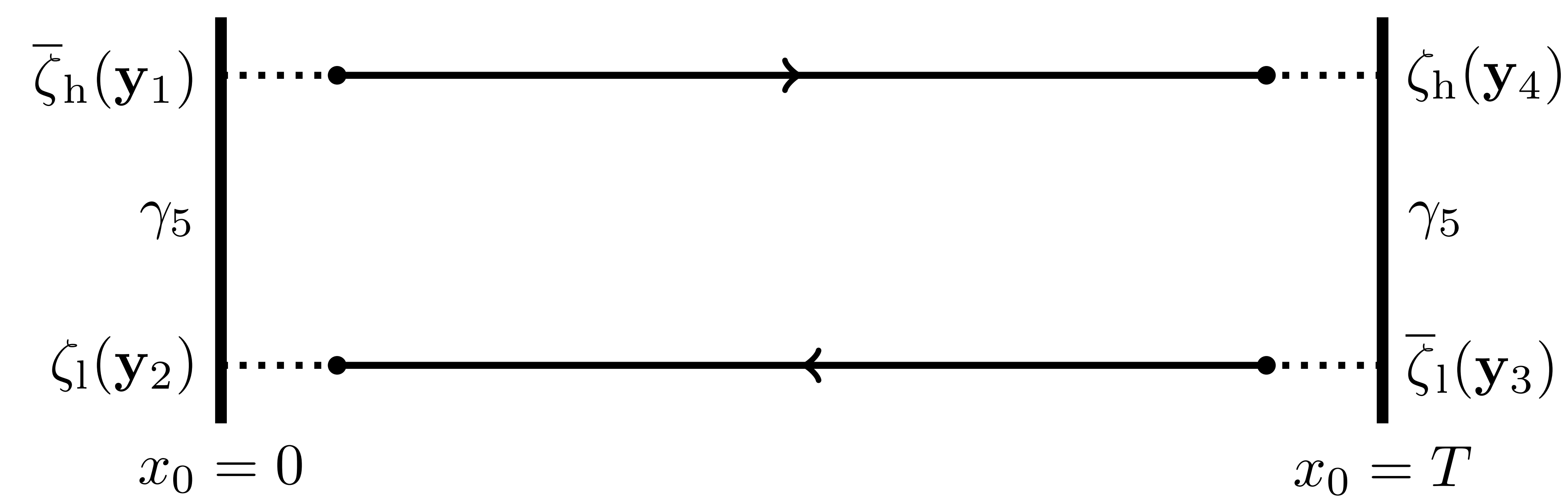
Your favorite observable

Fermion Wick contractions



ALPT - The Comic

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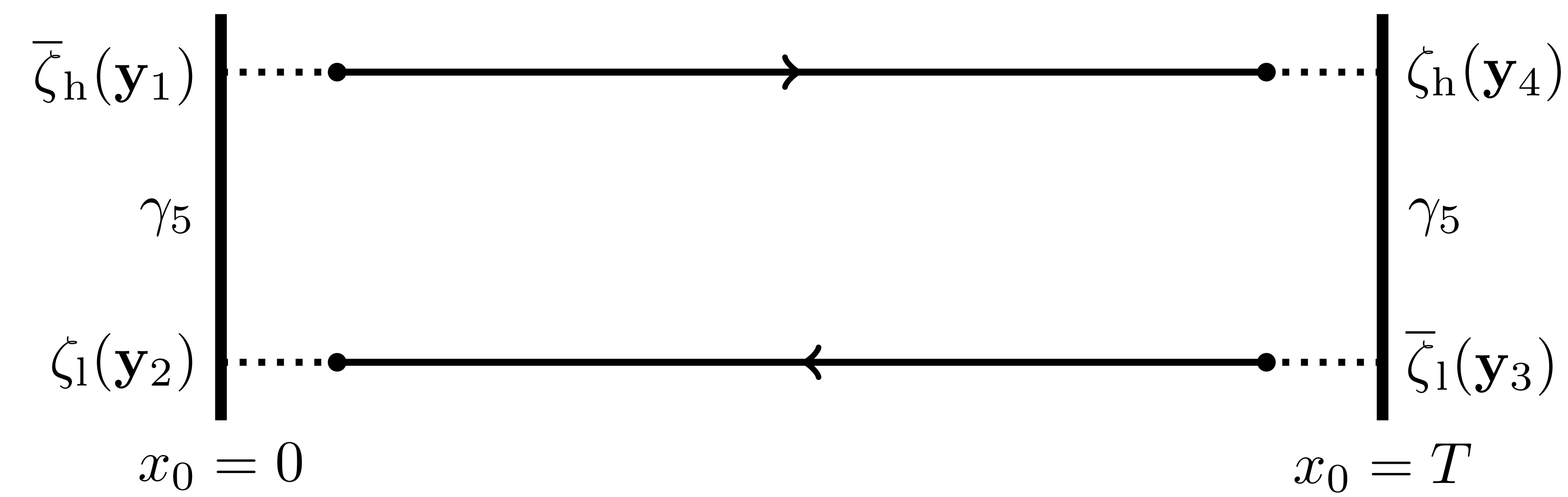
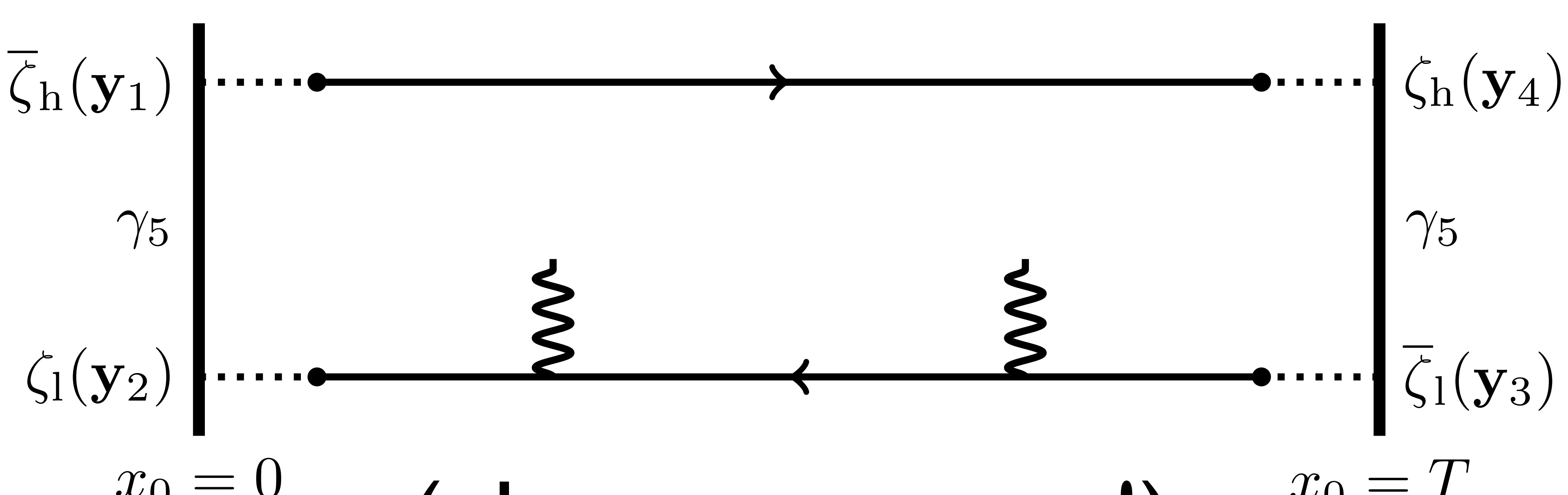


Expand in powers of g_0



(plus many more!)

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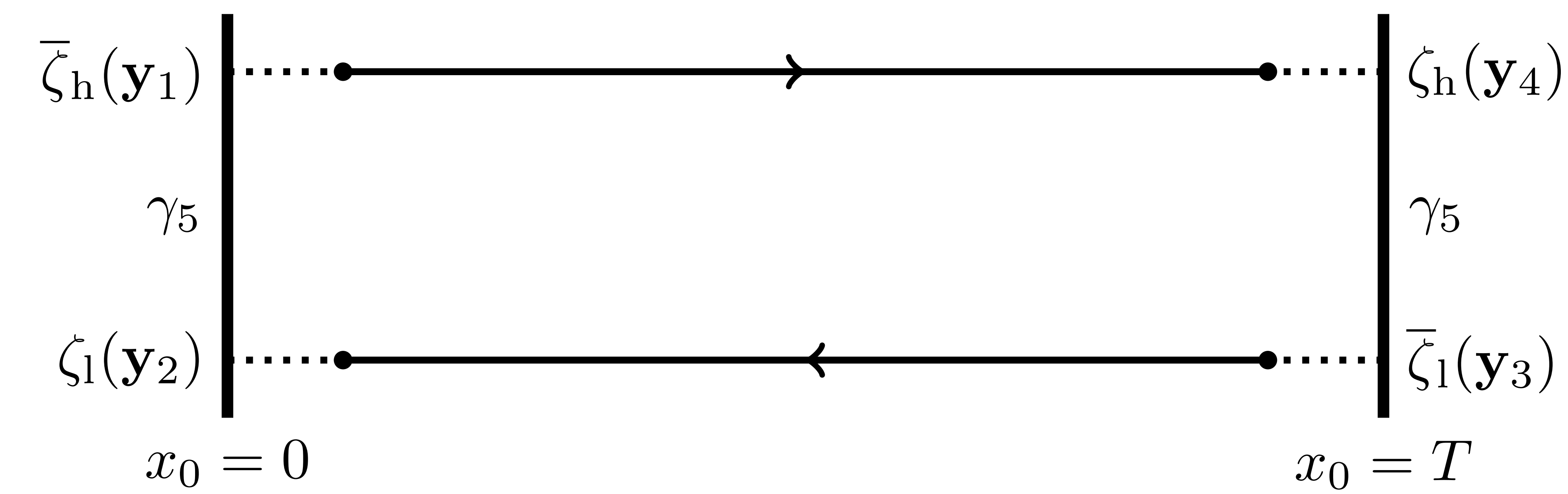
Fermion Wick contractions



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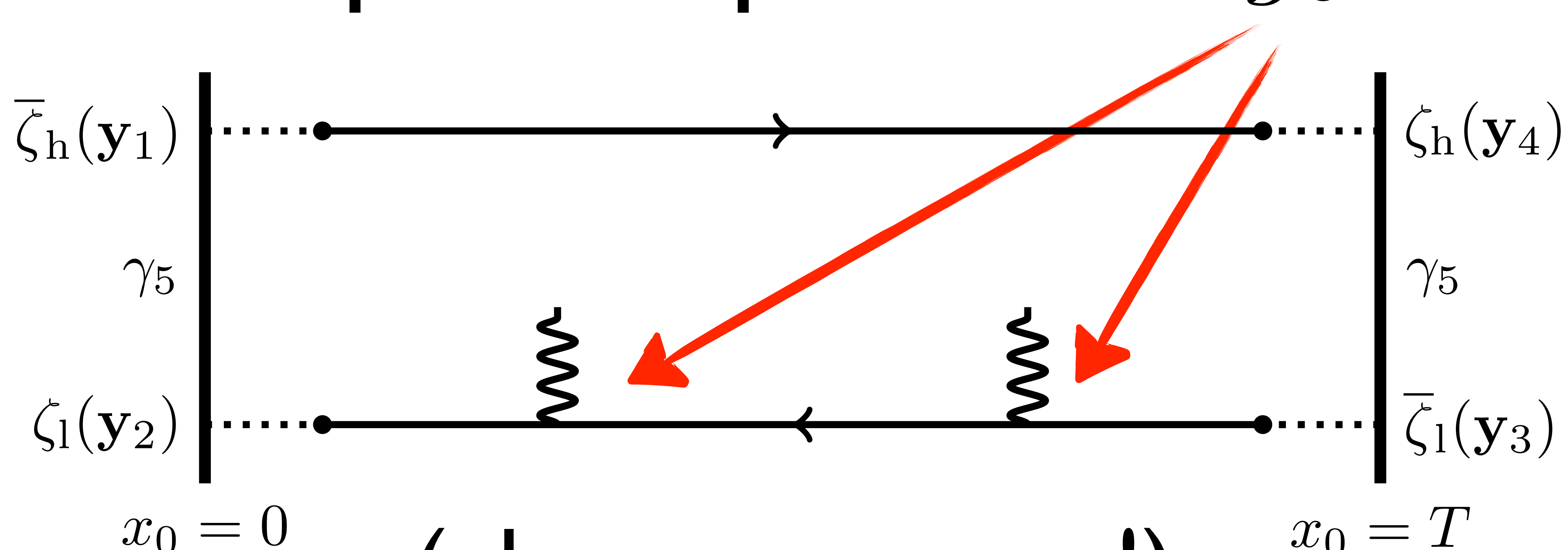

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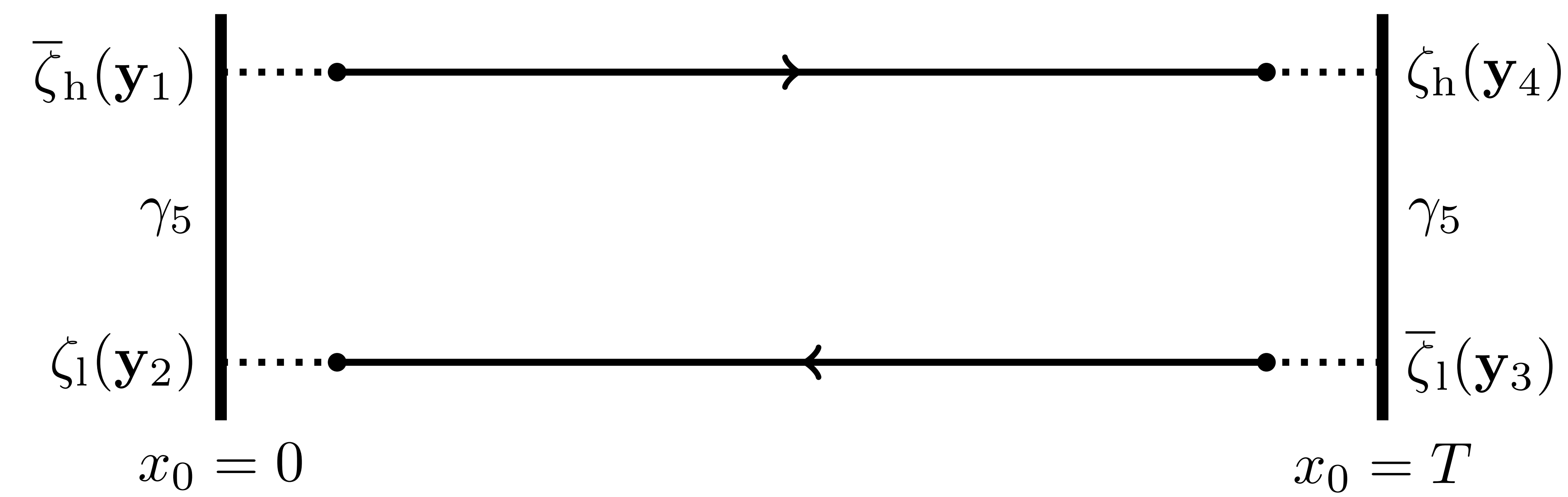

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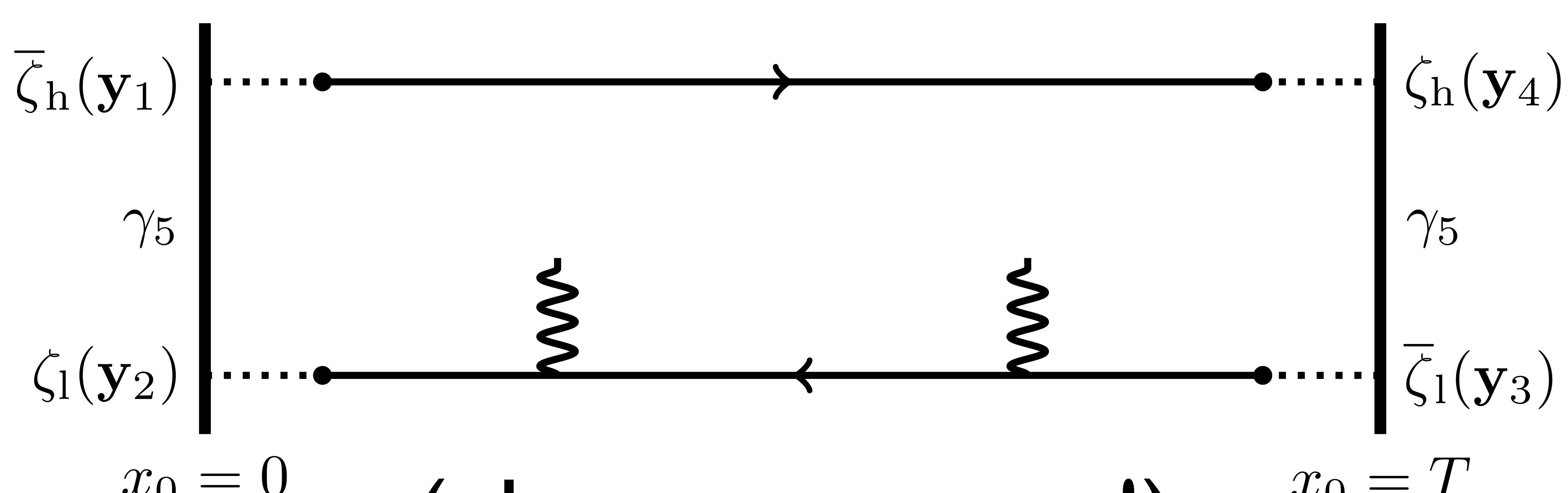
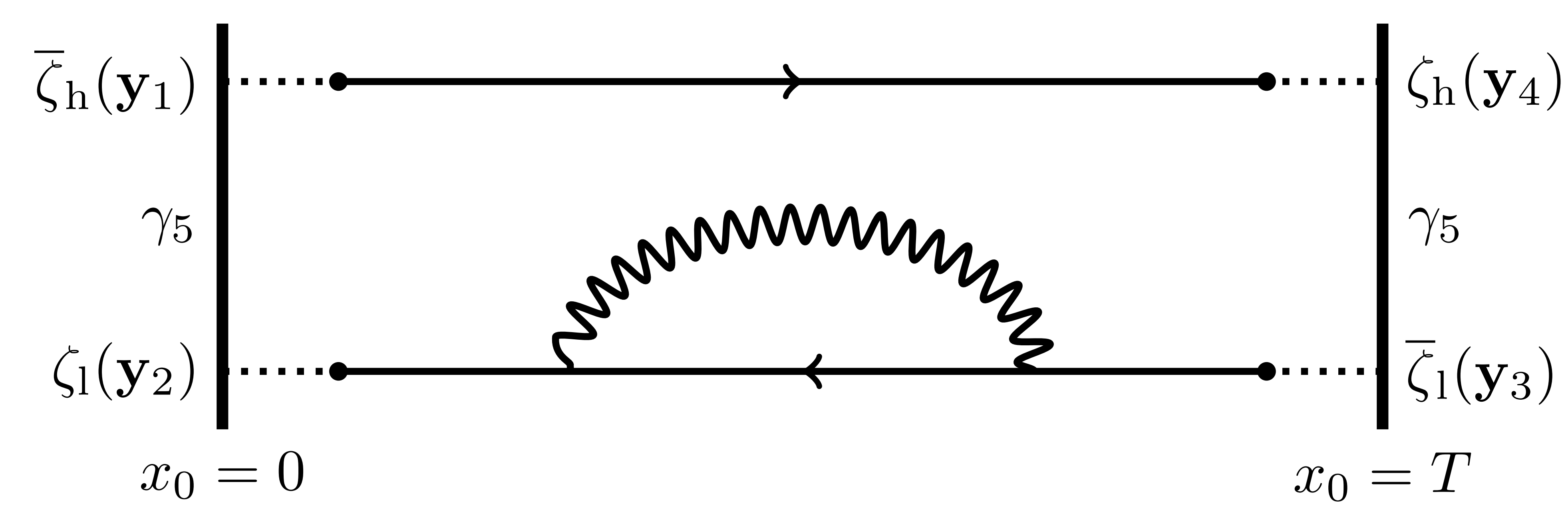


(plus many more!)

Gluon Wick contractions



pastor

User input:

- Fermion and gluon action in symbolic form.
- Observables in terms of propagators and boundary kernels.

Output:

All contributions up to $O(g_0^2)$.

Three **basic steps:**

1) Create a XML input file.
2) Parse the file to generate C++ programs.
3) Run the programs.

Using pastor as an Aid for the Matching of HQET and QCD

HQET Flavor Currents

$$\left(V_{\mathrm{R}}^{\mathrm{HQET}}\right)_0(x) = Z_{\mathrm{V}}^{\mathrm{HQET}}\left[V_0^{\mathrm{stat}} + \sum_{i=1}^{2} c_{\mathrm{V}}^{(i)} V_0^{(i)}(x)\right],$$

$$V_0^{\mathrm{stat}}(x) = \overline{\psi}_{\mathrm{l}}(x)\gamma_0\psi_{\mathrm{h}}(x),$$

$$V_0^{(1)}(x) = \overline{\psi}_{\mathrm{l}}(x)\frac{1}{2}\gamma_i\left(\nabla_i^S - \overleftarrow{\nabla}_i^S\right)\psi_{\mathrm{h}}(x),$$

$$V_0^{(2)}(x) = \overline{\psi}_{\mathrm{l}}(x)\frac{1}{2}\gamma_i\left(\nabla_i^S + \overleftarrow{\nabla}_i^S\right)\psi_{\mathrm{h}}(x)$$

$$\left(V_{\mathrm{R}}^{\mathrm{HQET}}\right)_k(x) = Z_{\mathbf{V}}^{\mathrm{HQET}}\left[V_k^{\mathrm{stat}} + \sum_{i=3}^{6} c_{\mathrm{V}}^{(i)} V_k^{(i)}(x)\right],$$

$$V_k^{(3)}(x) = \overline{\psi}_{\mathrm{l}}(x)\frac{1}{2}\gamma_k\gamma_i\left(\nabla_i^S - \overleftarrow{\nabla}_i^S\right)\psi_{\mathrm{h}}(x),$$

$$V_k^{(4)}(x) = \overline{\psi}_{\mathrm{l}}(x)\frac{1}{2}\left(\nabla_k^S - \overleftarrow{\nabla}_k^S\right)\psi_{\mathrm{h}}(x),$$

$$V_k^{(5)}(x) = \overline{\psi}_{\mathrm{l}}(x)\frac{1}{2}\gamma_k\gamma_i\left(\nabla_i^S + \overleftarrow{\nabla}_i^S\right)\psi_{\mathrm{h}}(x),$$

$$V_k^{(6)}(x) = \overline{\psi}_{\mathrm{l}}(x)\frac{1}{2}\left(\nabla_k^S + \overleftarrow{\nabla}_k^S\right)\psi_{\mathrm{h}}(x)$$

Heitger and Sommer, 2004

... and for the axial vector current ...

$$Z_{\mathbf{A}}^{\mathrm{HQET}},\ Z_{\mathrm{A}}^{\mathrm{HQET}},\ c_{\mathrm{A}}^{(i)}$$

In total16 matching coefficients for the currents (plus 3 in the action)!

Matching the Vector Current

Consider: $$\Phi^{\mathrm{V}_0}(\theta, z) = Z_{\mathrm{V}} \frac{F_{\mathrm{V}_0}(T/2; \theta, z)}{\sqrt{f_1(\theta, z) f_1(\theta, 0)}}$$

Matching the Vector Current

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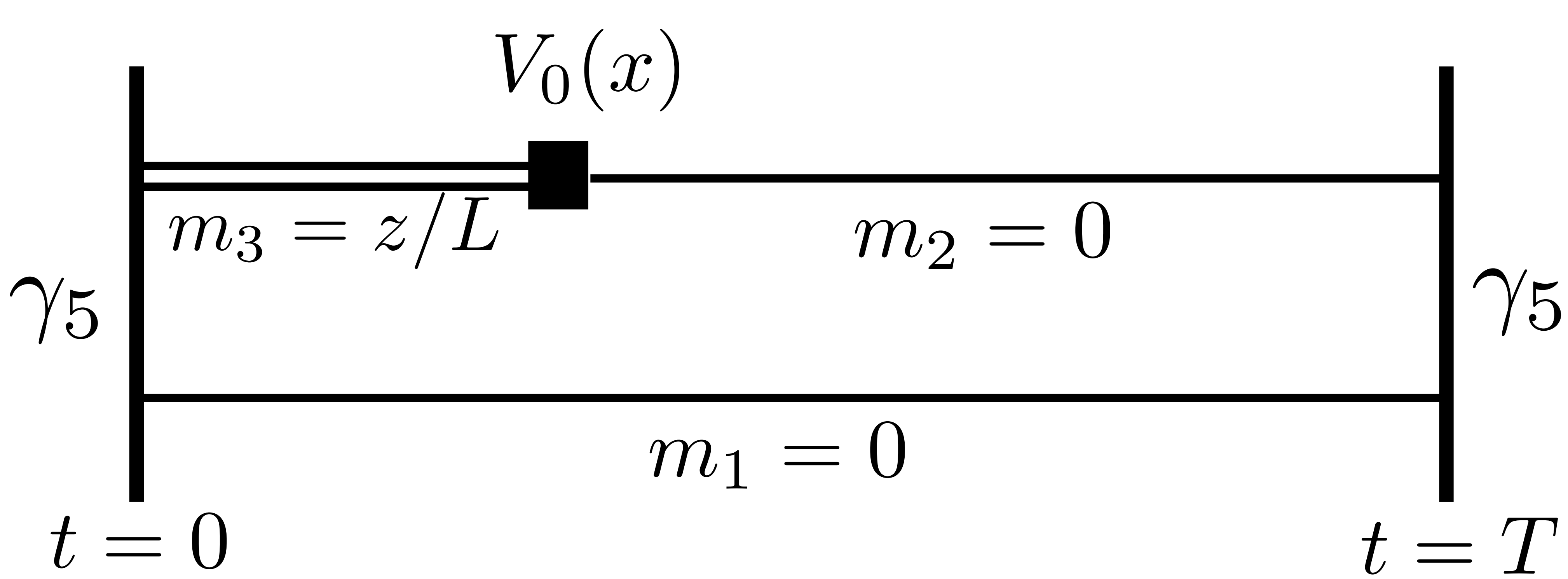
$$F_{\mathrm{V}_0}(x_0; \theta, z = Lm_3) = -\frac{1}{2} \sum_{\mathbf{x}} \langle \overline{\zeta}_1 \, \gamma_5 \, \zeta_3 \, V_0(x) \, \overline{\zeta}'_2 \, \gamma_5 \, \zeta'_1 \rangle,$$



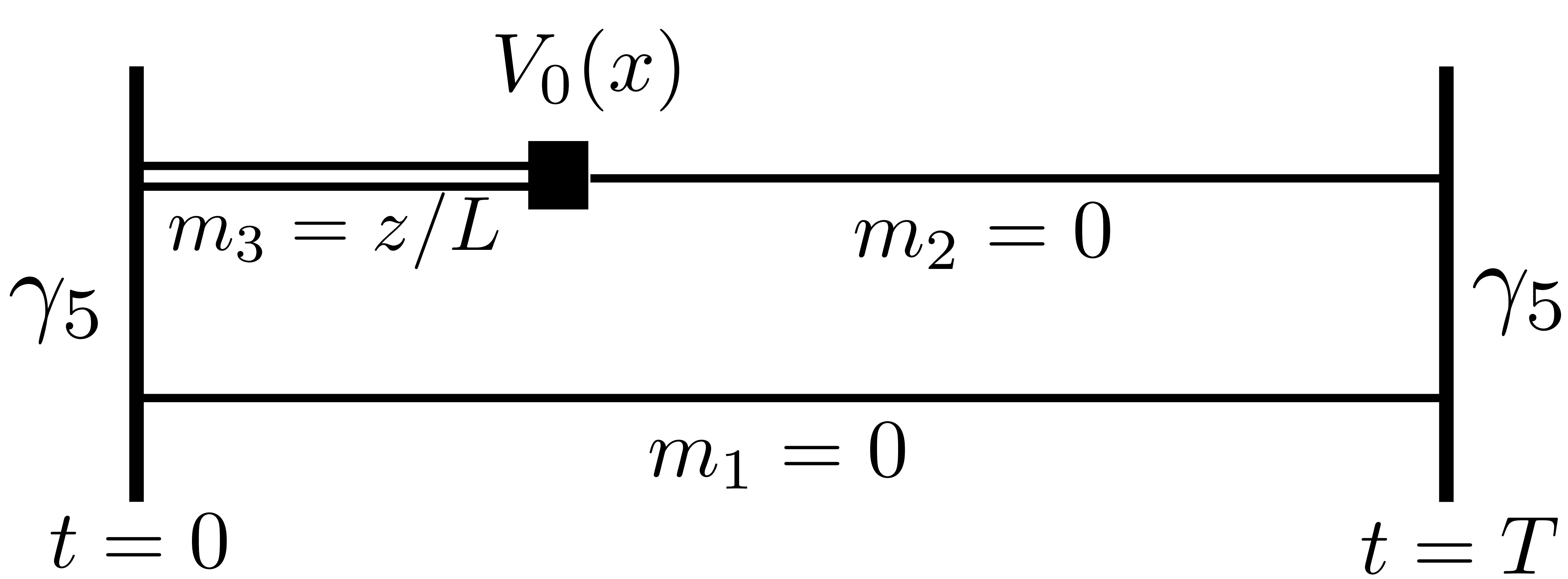
Matching the Vector Current

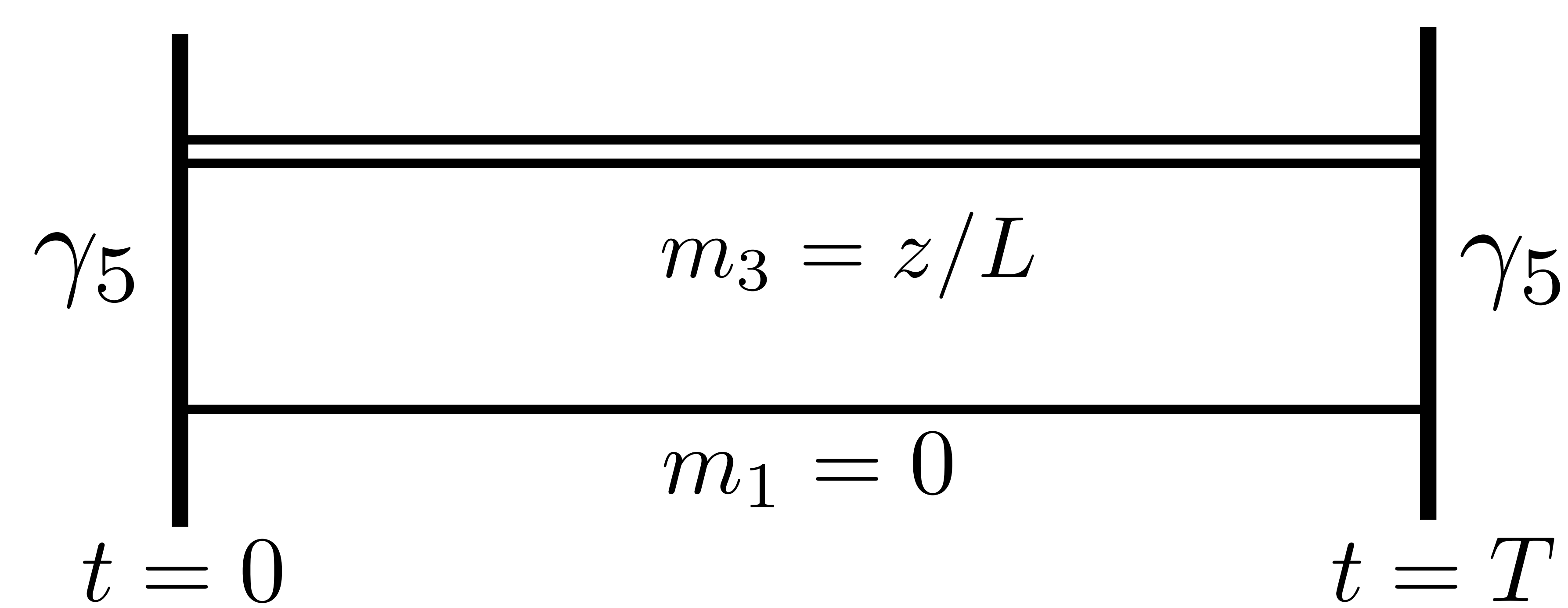
Consider: $$\Phi^{V_0}(\theta, z) = Z_V \frac{F_{V_0}(T/2; \theta, z)}{\sqrt{f_1(\theta, z) f_1(\theta, 0)}}$$

$$F_{V_0}(x_0; \theta, z = Lm_3) = -\frac{1}{2}\sum_{\mathbf{x}} \langle \bar{\zeta}_1\, \gamma_5\, \zeta_3\, V_0(x)\, \bar{\zeta}'_2\, \gamma_5\, \zeta'_1 \rangle,$$

$$f_1(\theta, z) = -\langle \bar{\zeta}'_2\, \gamma_5\, \zeta'_3\, \bar{\zeta}_3\, \gamma_5\, \zeta_2 \rangle$$

Lüscher, Sint, Sommer, and Weisz, 1996.





The Static Approximation

$$X_{\rm V}(\mu) = Z^{\rm stat}_{\rm A,lat}(\mu)\, Z^{\rm stat}_{\rm V/A} X^{\rm bare}_{\rm V}, \quad X^{\rm bare}_{\rm V} = \frac{F^{\rm stat}_{\rm V_0}(T/2;\theta,z)}{\sqrt{f^{\rm stat}_1(\theta,z) f_1(\theta,0)}},$$

$$Z^{\rm stat}_{\rm A,lat}(\mu) = 1 - \gamma_0 \log(a\mu)\, g_0^2 + O(g_0^4).$$

We expect up to **one loop**: $\Phi^{\rm V_0}(z) = (1 + B^{\rm stat}_{\rm A}\, \overline{g}_0^2) X_{\rm V}(z/L) + O(1/z).$

M. Kurth, R. Sommer, 2001.

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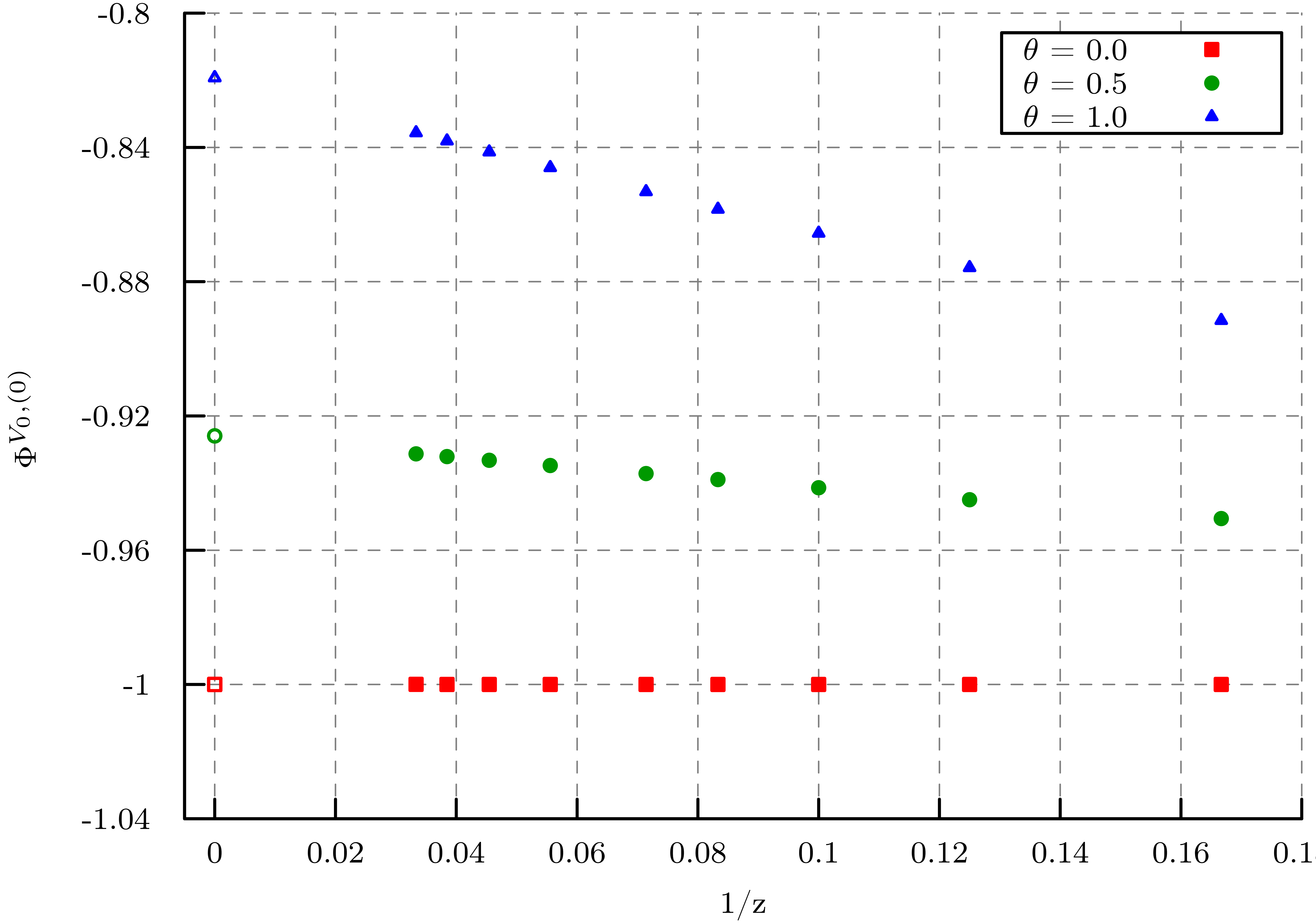
Tree Level



$\theta = 0.0$
$\theta = 0.5$
$\theta = 1.0$
-0.8
-0.84
-0.88
-0.92
-0.96
-1
-1.04
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
$\Phi^{V_0,(0)}$
$1/z$

The Static Approximation

$$X_{\rm V}(\mu) = Z^{\rm stat}_{\rm A,lat}(\mu)\, Z^{\rm stat}_{\rm V/A} X^{\rm bare}_{\rm V}, \qquad X^{\rm bare}_{\rm V} = \frac{F^{\rm stat}_{\rm V_0}(T/2;\theta,z)}{\sqrt{f^{\rm stat}_1(\theta,z) f_1(\theta,0)}},$$

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M. Kurth, R. Sommer, 2001.

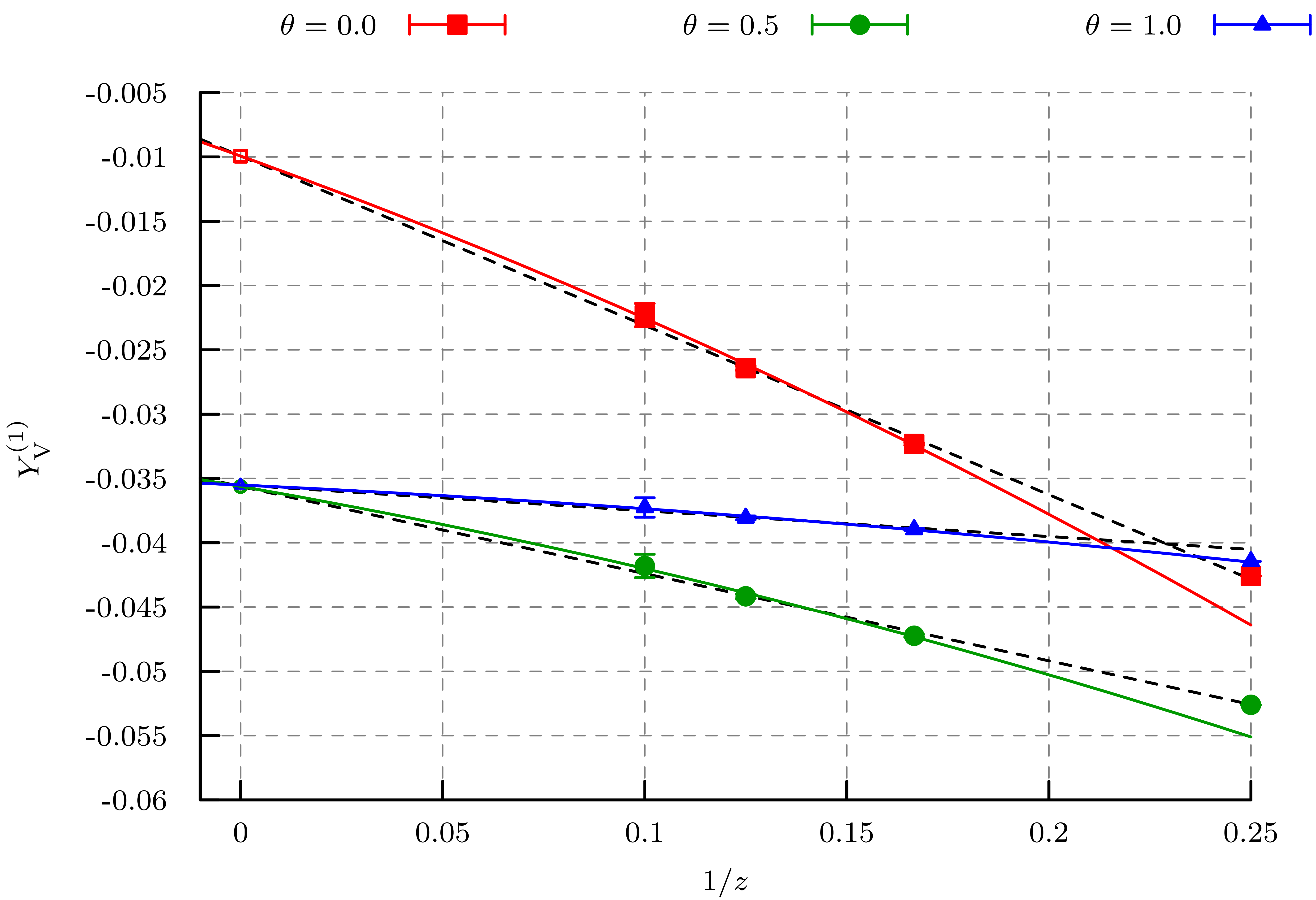
Defining

$$Y^{(1)}_{\rm V} = \Phi^{{\rm V_0},(1)}(z) - \left[B^{\rm stat}_{\rm A} - \gamma_0 \log(z) + \left(Z^{\rm stat}_{\rm V/A} \right)^{(1)} \right] X^{(0)}_{\rm V},$$

we get

$$Y^{(1)}_{\rm V}(z) \xrightarrow{1/z \to 0} X^{(1)}_{\rm V,lat}(1/L) = X^{{\rm bare},(1)}_{\rm V} - \gamma_0 \log(a/L) X^{{\rm bare},(0)}_{\rm V}.$$

One Loop - Results



The fit functions have the form

$$Y_V^{(1)}(z) = X_{V,\mathrm{lat}}^{(1)} + c_1/z + c_2/z^2.$$

Very small $1/z, 1/z^2$ corrections (note the scale)!

Conclusions/Outlook

- Improve pastor.
  - Better support for Abelian background.
  - Smearing.
  - Staggered quarks.
  - Chirally twisted boundary conditions.
  - Two loops?!
- More applications.
  - One loop matching of all components of the currents (in progress with P. Korcyl).

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Appendix

Performance

Diagram count

	QCD	static
$f_1^{\mathrm{V}_0}$	30	29
f_1	22	21

Three values for θ
L/a = 4,...,40
Four values for z

Total time (incl. idle) on DESY PC farm in Zeuthen: Two weeks.

Some Definitions

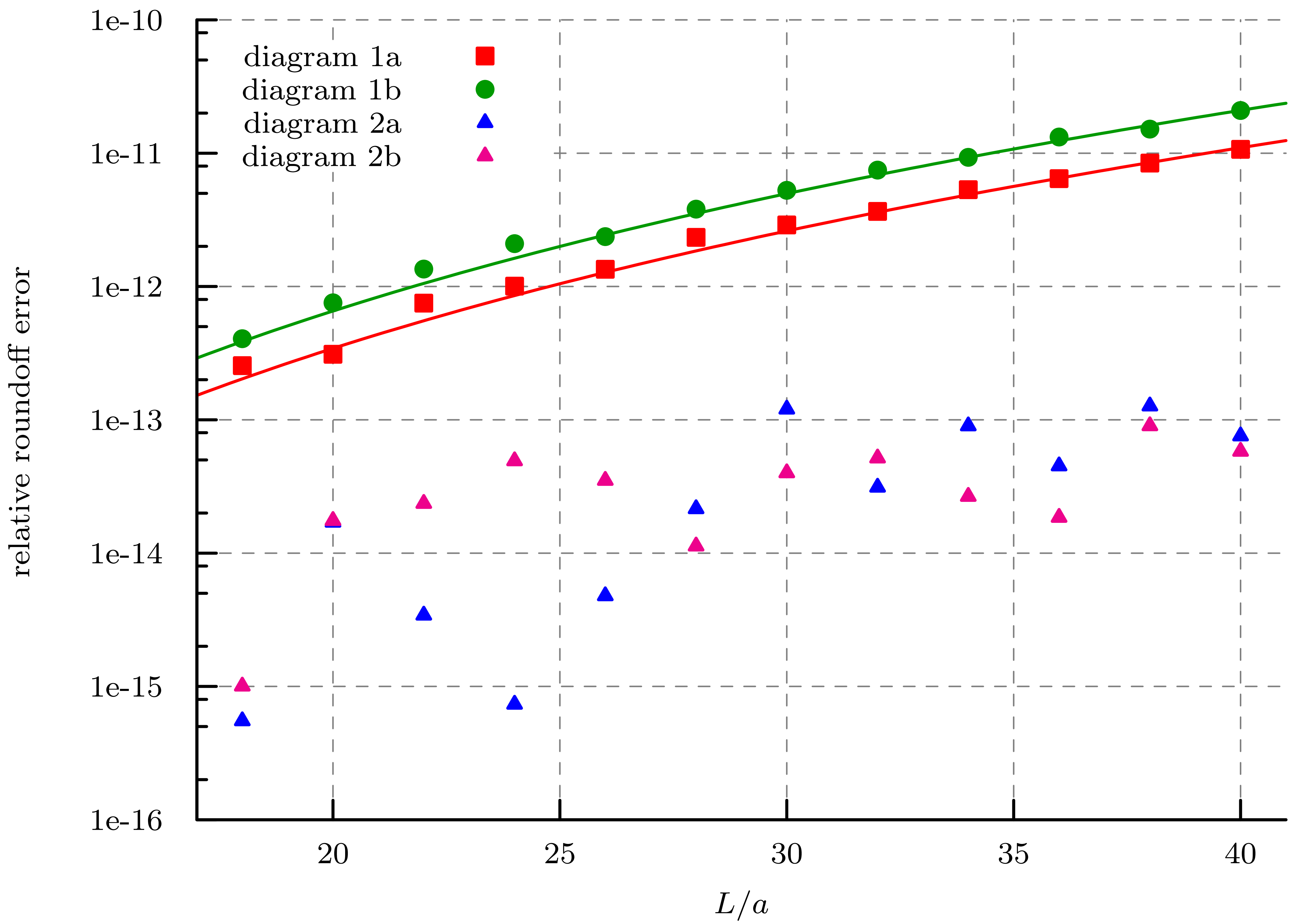
$$\mathcal{O}_{\rm kin}(x) = \overline{\psi}_{\rm h}(x)\nabla_k^*\nabla_k\psi_{\rm h}(x),$$

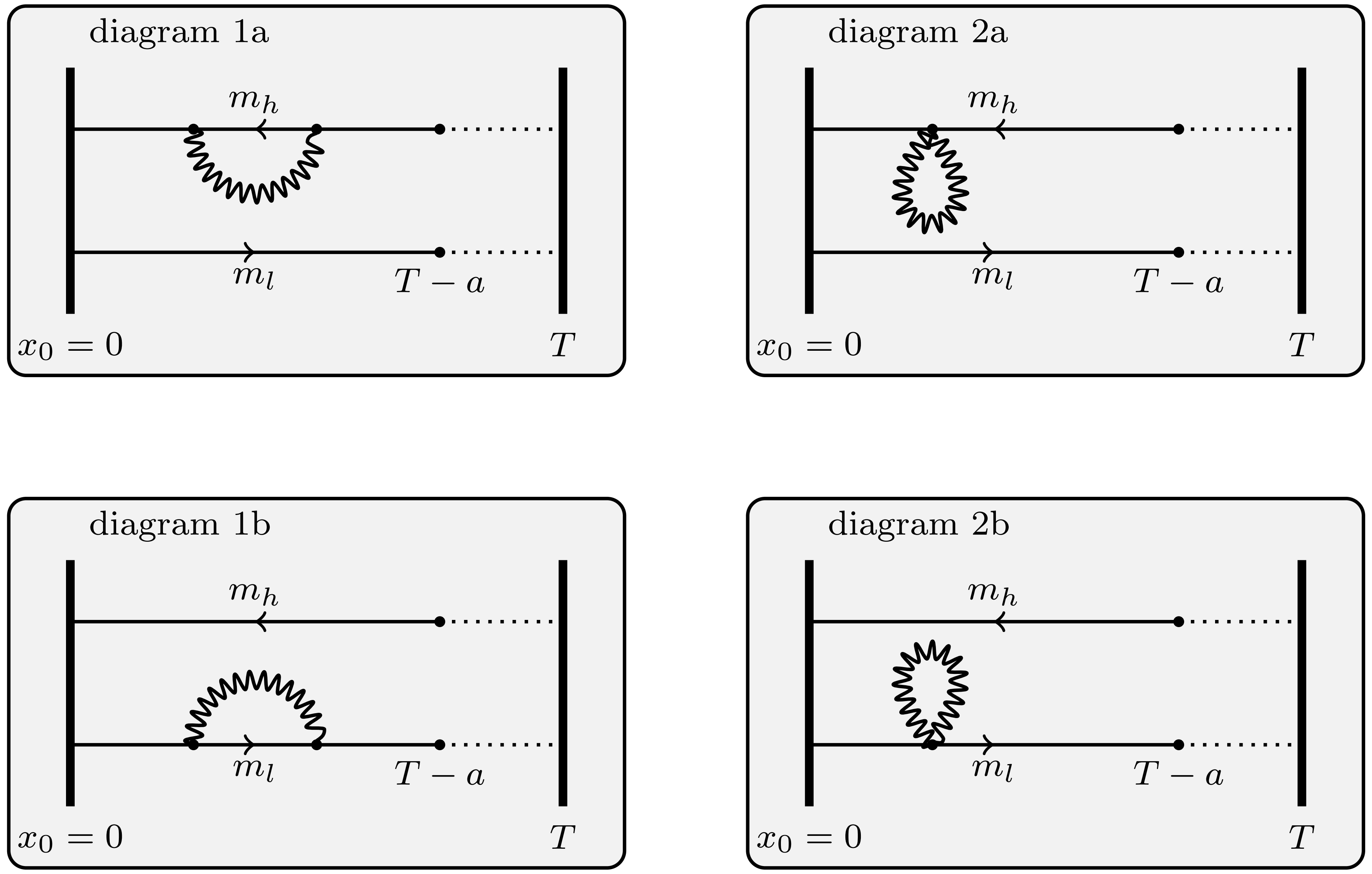
$$\mathcal{O}_{\rm spin}(x) = \overline{\psi}_{\rm h}(x)\,\sigma\cdot\mathbf{B}(x)\psi_{\rm h}(x),$$

$$\sigma_k = \frac{1}{2}\epsilon_{ijk}\sigma_{ij}\,,$$

$$B_k(x) = \frac{i}{2}\epsilon_{ijk}\mathcal{F}_{ij}(x).$$

Round-off





Per-diagram comparison with data by Kurth and Sommer.

MC Cross Checks

We use pastor to calculate

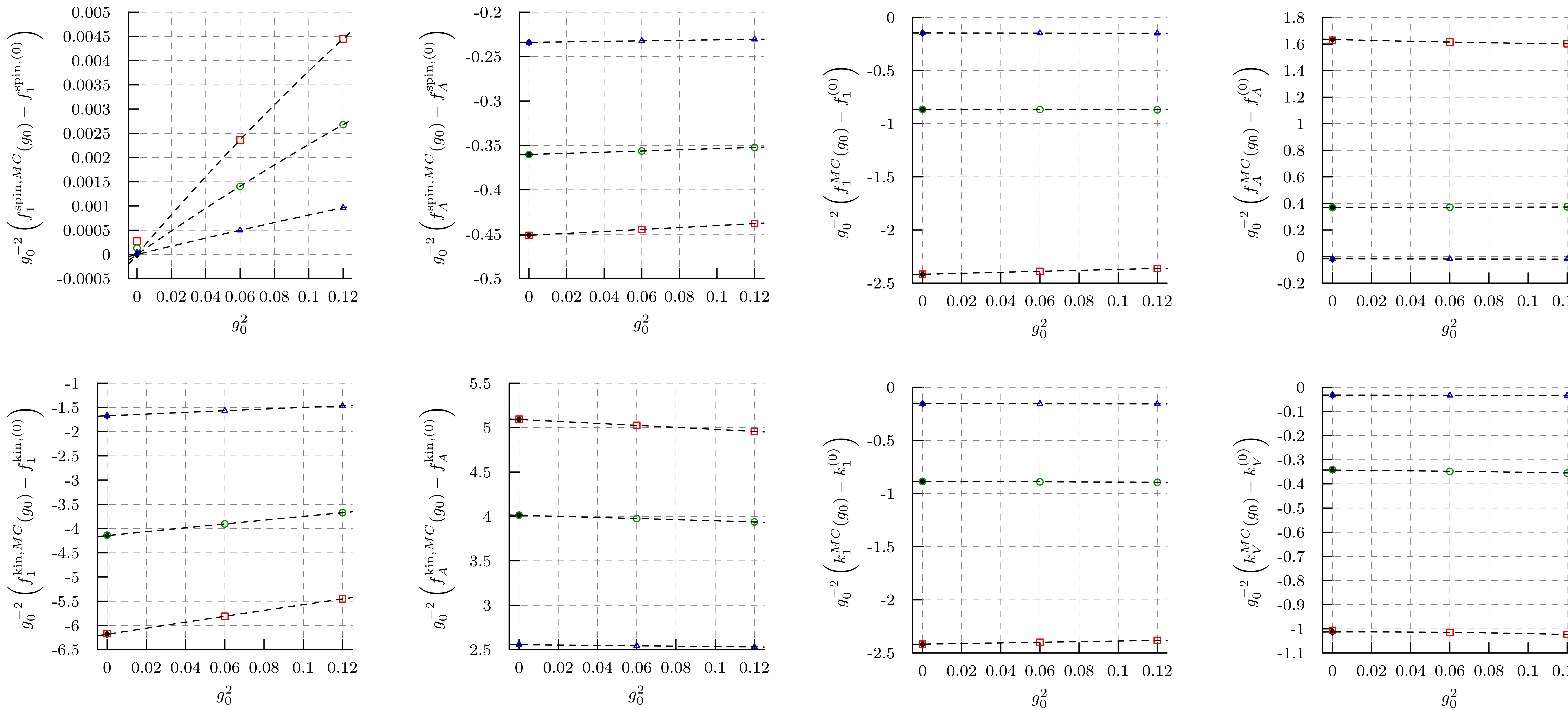
$$f(g_0) = f^{(0)} + g_0^2\, f^{(1)} + O(g_0^4)$$

We then extract an estimate for the one loop part from MC using

$$\tilde{f}^{(1)}(g_0) = g_0^{-2}\left(f^{MC}(g_0) - f^{(0)}\right)$$

and extrapolate linearly in g_0^2.

MC Cross Checks



Cross checks with MC data by Patrick Fritzsch.

Vertices - Details

The full result reads

$$\mathcal{U}^{(r)}(x,y) = \left(\frac{a}{L}\right)^{3r} \sum_{\mathbf{k}_1,a_1,\mu_1,t_1} \cdots \sum_{\mathbf{k}_r,a_r,\mu_r,t_r} q^{a_1}_{\mu_1}(\mathbf{k}_1;t_1)\,\ldots\,q^{a_r}_{\mu_r}(\mathbf{k}_r;t_r)$$
$$\times\, I_{a_r}\ldots I_{a_1} \sum_{0<u_1\leq\ldots\leq u_r\leq l}$$
$$\times\, \frac{r!}{\alpha_1!\,\ldots\,\alpha_l!} \prod_{j=1}^{r} s[u_j]\,\delta_{t[u_j],t_j}\,\delta_{\mu[u_j],\mu_j}\,e^{i\,\mathbf{k}_j\tilde{\mathbf{x}}[u_j]}.$$

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$$\times I_{a_r} \ldots I_{a_1} \sum_{0<u_1\leq\ldots\leq u_r\leq l} V(0)^{A^{\{u\}}} e^{i\mathcal{E}B^{\{u\}}} e^{i/2\left(C^{\{u\}}\cdot\Phi'+D^{\{u\}}\cdot\Phi\right)}$$

$$\times \frac{r!}{\alpha_1!\,\ldots\,\alpha_l!} \prod_{j=1}^{r} s[u_j]\,\delta_{t[u_j],t_j}\,\delta_{\mu[u_j],\mu_j}\,e^{i\,\mathbf{k}_j\tilde{\mathbf{x}}[u_j]}.$$

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$$\times I_{a_r} \ldots I_{a_1} \sum_{0<u_1\leq\ldots\leq u_r\leq l} V(0)^{A^{\{u\}}} e^{i\mathcal{E}B^{\{u\}}} e^{i/2\left(C^{\{u\}}\cdot\Phi' + D^{\{u\}}\cdot\Phi\right)}$$

$$\times \frac{r!}{\alpha_1!\,\ldots\,\alpha_l!} \prod_{j=1}^{r} s[u_j]\, \delta_{t[u_j],t_j}\, \delta_{\mu[u_j],\mu_j}\, e^{i\,\mathbf{k}_j\tilde{\mathbf{x}}[u_j]}.$$

This looks horrible, but can be constructed link by link defining an order-by-order multiplication

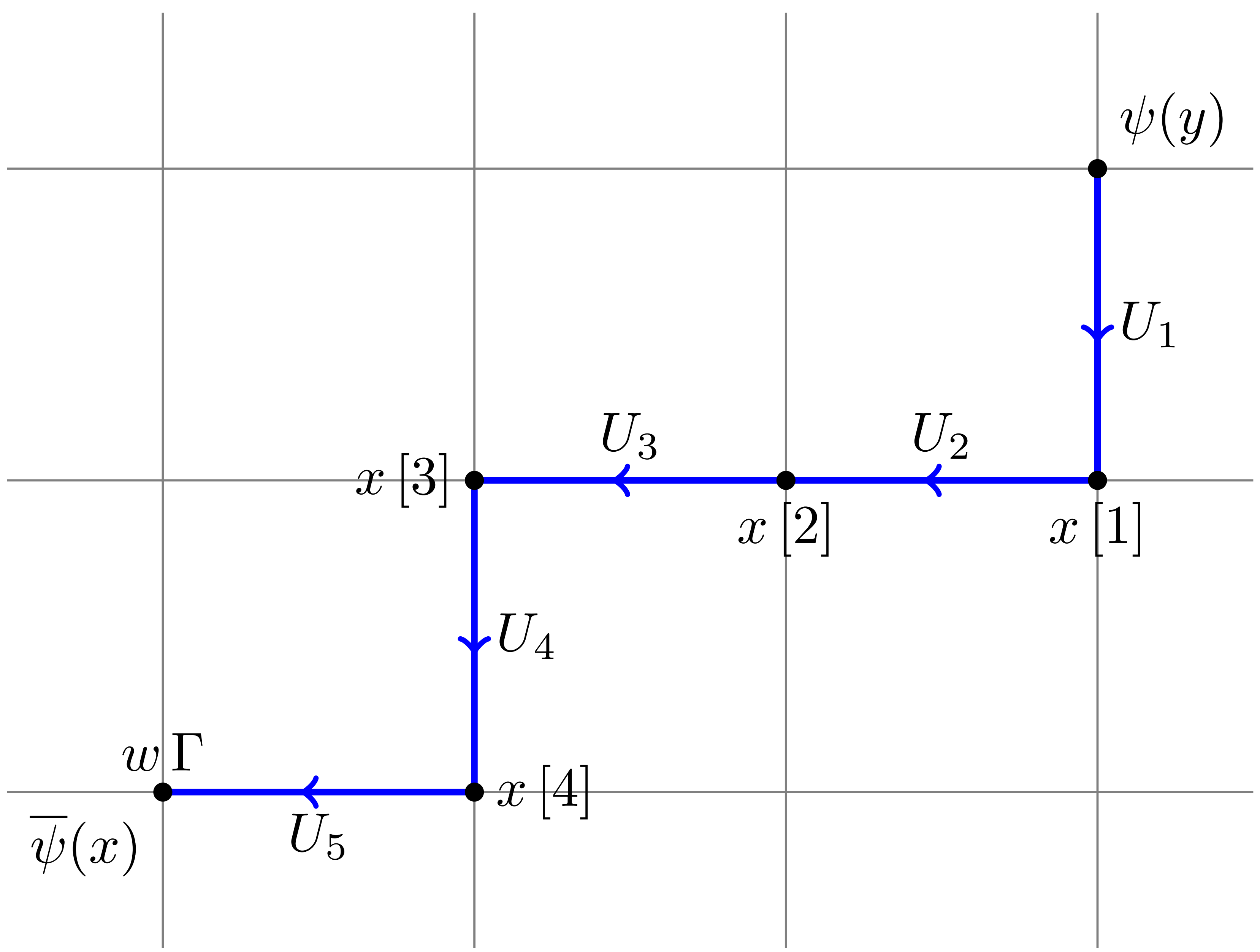
$$[U_\mu(x)\mathcal{U}(x,y)]^{(r)} = \sum_{s=0}^{r} U_\mu^{(s)}(x) \times \mathcal{U}^{(r-s)}(x,y).$$

The XML Input

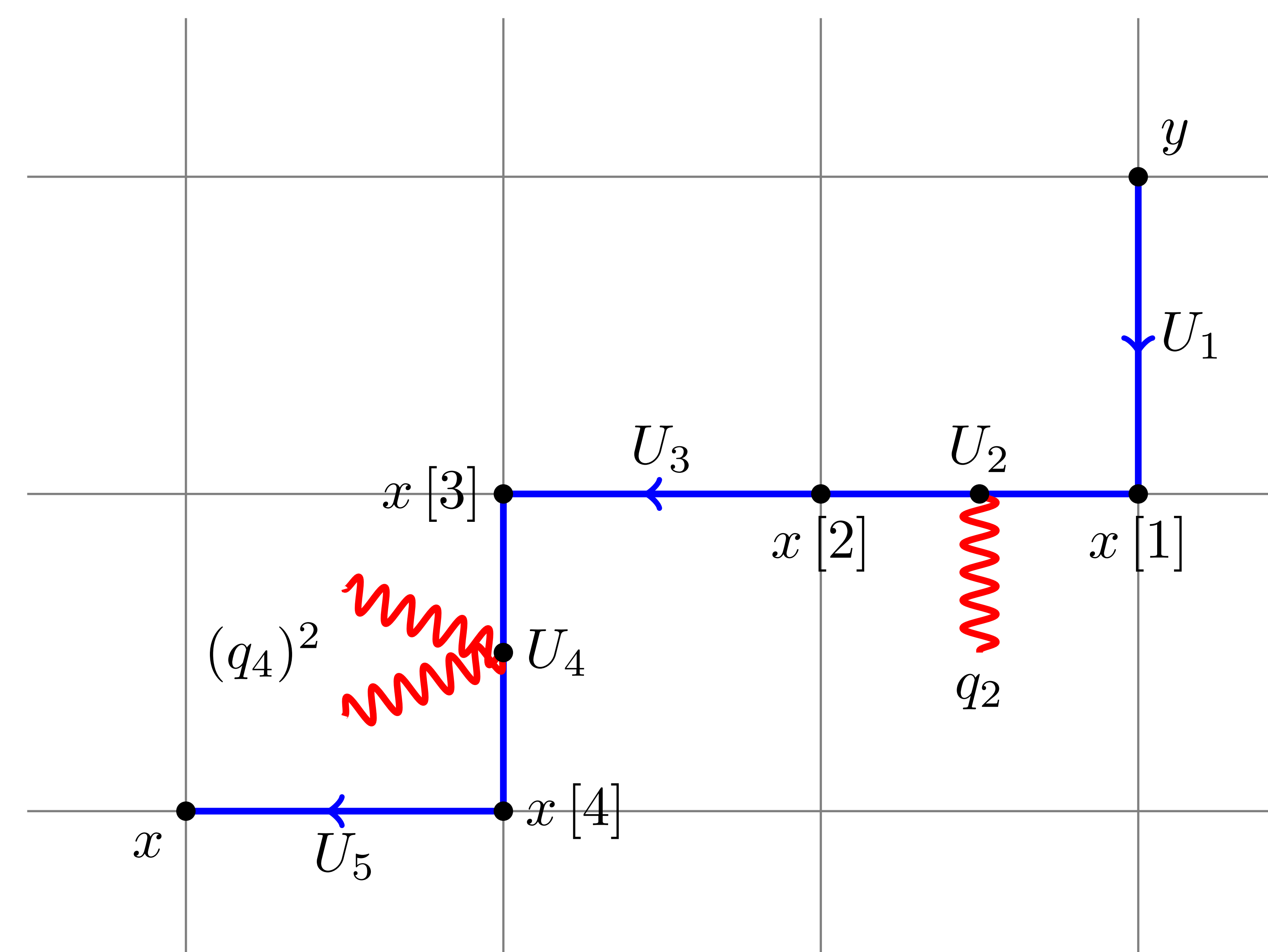
```
<boundary>
  <where>0</where> <!--x_0 = 0-->
  <spin>dirac</spin>
  <spins>15 -14</spins>
  <!-- 2 * P_+ * gamma_5 * P_- = -->
  <!-- 2 * gamma_5 * P_- = gamma_5 - gamma_5 * gamma_0 -->
</boundary>
<propagator>
  <thetax>theta</thetax> <!-- user parameter: theta-->
  <thetay>theta</thetay>
  <thetaz>theta</thetaz>
  <spin>dirac</spin>
  <action>HQET_stat</action>
  <!--from and to are optional tags, helping pastor to
      generate more efficient code -->
  <from>1</from>
  <to>x0</to>
</propagator>
```

Construction of a Vertex in Pictures

A parallel transporter like this:



Will yield $O\left(g_0^3\right)$ terms like this one:



Parse (and Compile)

GNU autotools make your life easy!

```
hal9000 | example_project $ ~/pastor-build/codegen/parse.py xml/phi_v0.xml

[...]

hal9000 | example_project $ cd source
hal9000 | source $ ./configure

[...]

hal9000 | source $ make
```

Run the Programs

f1_small_L.get file contents:

```
# Base path for the executables
BasePath /Users/dirk/tmp/pastor_build/codegen/example_project/source/
# Path for output and log files
WorkDir /Users/dirk/tmp/pastor_build/codegen/example_project/run/
# Bool (yes|no) if the propagators should be written
# to hard disk and their location
Propagators no -
# subdirectories and names for the observables
SubDir f1/.    f1_loop
SubDir f1/tree f1_tree
SubDir f1/db   f1_db
SubDir f1/dm   f1_dm
# Parameters can be given in various ways ...
# 4 to 8 in steps of 2
Parameter L 4:8:2
# formulae
Parameter T = L
Parameter x0 = T/2
# arrays
Parameter theta [0.0, 0.5, 1.0]
Parameter z [0, 1]
```

```
hal9000 | example_project $ ~/pastor-build/codegen/run.py xml/f1_small_L.get
```

Perturbative Expansion

Now, expand

$$S(x,y) = S^{(0)}(x,y) + g_0 S^{(1)}(x,y) + \dots$$
$$K(x,\mathbf{y}) = K^{(0)}(x,\mathbf{y}) + g_0 K^{(1)}(x,\mathbf{y}) + \dots$$

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and perform the gluon Wick contractions.

We may obtain $S^{(i)}(x,y)$ by solving

$$\Big(D^{(0)} + m + g_0 D^{(1)} + \ldots\Big)\Big(S^{(0)}(x,y) + g_0 S^{(1)}(x,y) + \ldots\Big) = \delta_{xy}$$

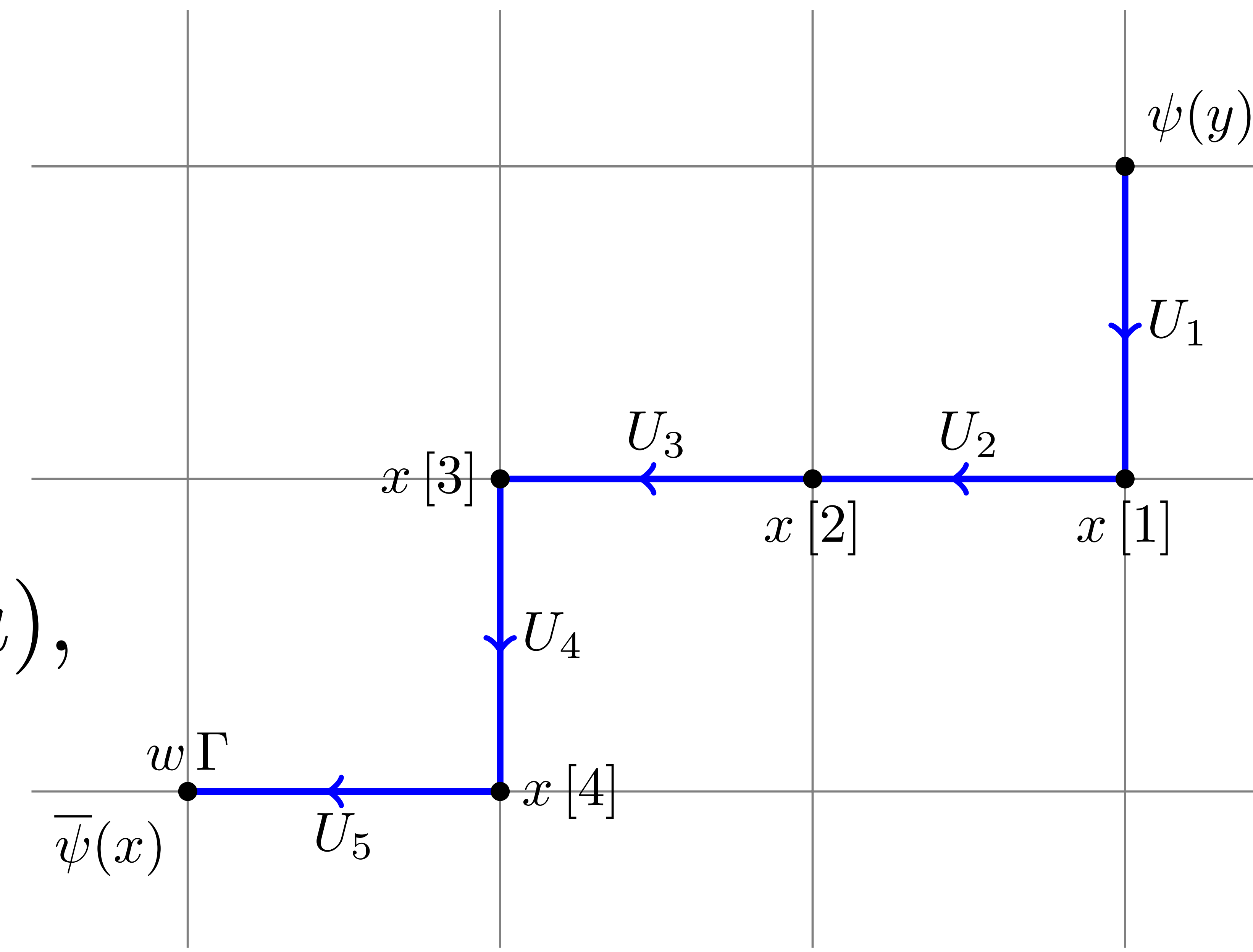
order by order.

M. Lüscher, P. Weisz, 1996.

Fermion Actions

We assume that the fermion action looks like this

$$
\begin{aligned}
S_F &= \sum_i c_i \, \overline{\psi}(x_i) \, \Gamma_i \mathcal{U}_i(x_i, y_i) \, \psi(y_i), \\
\mathcal{U}(x,y) &= U_l U_{l-1} \ldots U_1, \\
U_i &= U_{s[i]\mu[i]}(x[i]), \quad U_{-\mu}(x) = U_\mu^\dagger(x - \hat{\mu}), \\
s[i]\,\hat{\mu}[i] &= x[i-1] - x[i], \quad x[0] = y.
\end{aligned}
$$



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the points $x[i]$ define a path $y \to x$.

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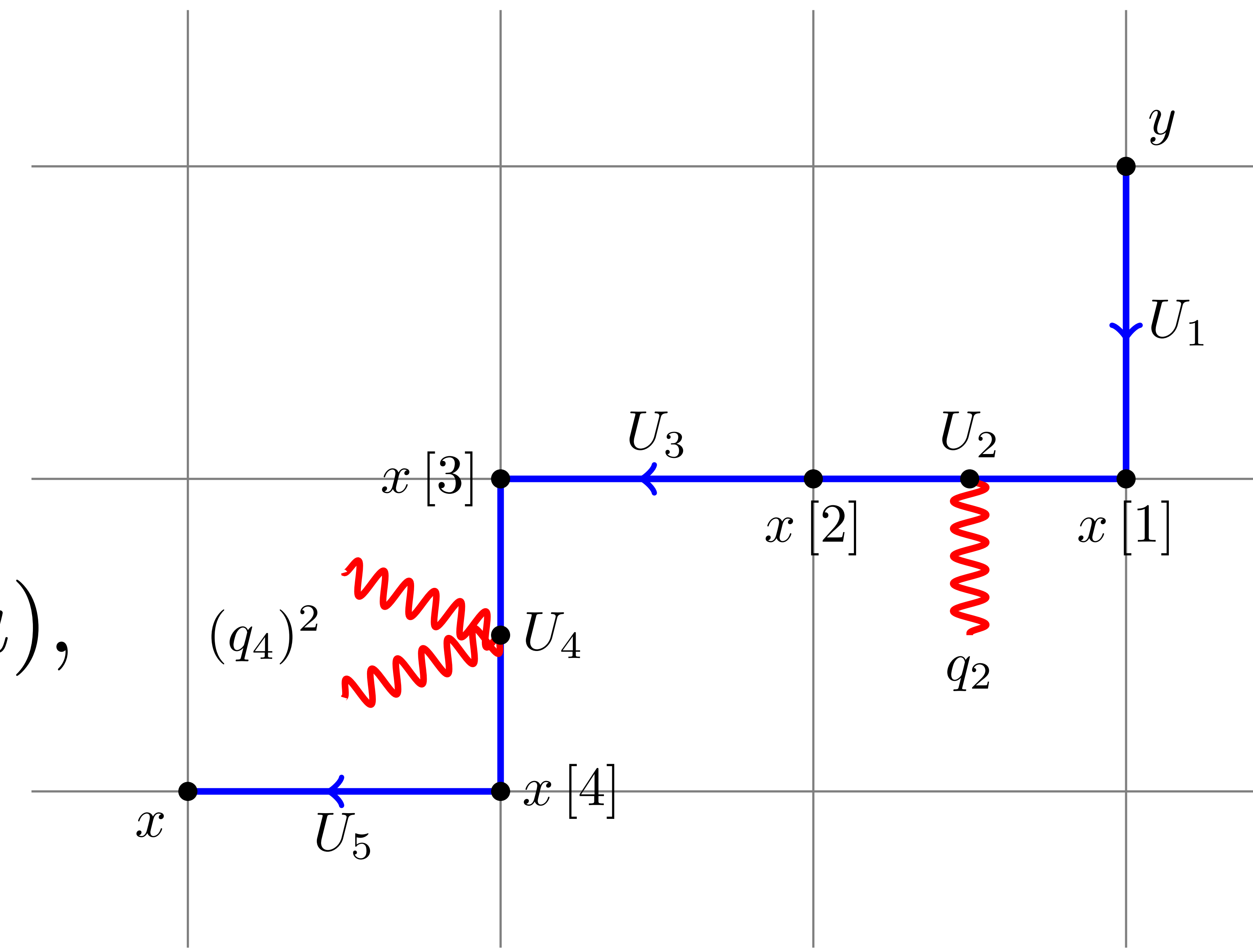

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Parallel Transporters

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General strategy:

Bring the contributions to a standard form.
Define a multiplication rule and construct $\mathcal{U}^{(r)}$ link by link.

M. Lüscher, P. Weisz, 1986. A. Hart, G.M. von Hippel, et. al., 2009. S. Takeda, U. Wolff, 2007.

Data Analysis

An observable with at most a logarithmic divergence looks like this

$$f(I) = \sum_{n=0}^{\infty} \frac{a_n + b_n \log I}{I^n} \qquad I = L/a$$

M. Lüscher, P. Weisz, 1996.

We extract the coefficients using successive fits.

A. Bode, P. Weisz, U. Wolff, 2000.

Round-off errors can be estimated using `long double` precision.