$N \to \infty$ Limit of the Principal Chiral Model (and applications to Gauge Theories)

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1. What is the Motivation?

Lattice gauge theories can be thought of many copies of the (integrable) principal chiral sigma model, coupled together.

Bardeen, Pearson (1976) Rabinovici (1980) (light-cone gauge).

Durhuus, Fröhlich (1980) (no gauge fixing).

Griffin (1991) (light-cone gauge).

P.O. (2005-) (axial gauge). The sigma models decouple in an anisotropic weak-coupling limit. Then the exact S-matrix and/or exact form factors may be used to study what happens as the isotropy is gradually restored.

3D gauge theory with two *small* couplings, g_0 and g_0',

$$S = -\sum_{\text{sites}} \left(\frac{1}{{g_0'}^2} \, \mathfrak{Re} \ \mathrm{Tr} \ U_0 U_1 U_0^\dagger U_1^\dagger + \frac{1}{{g_0}^2} \, \mathfrak{Re} \ \mathrm{Tr} \ U_0 U_2 U_0^\dagger U_2^\dagger + \frac{1}{{g_0}^2} \, \mathfrak{Re} \ \mathrm{Tr} \ U_1 U_2 U_1^\dagger U_2^\dagger \right)$$

As $g_0' \to 0$, the field strength in the $0,1$-planes becomes zero. The system reduces to principal chiral models, one for each x^2, in the $0,1$-layers (between the planes). We can study the theory as g_0' is increased.

Solvable model with confinement. High-energy/eikonal (large s, small t) limit. Verlinde+Verlinde('93), McLerran+Venugopalan('94). The bare couplings in two space and one time dimension are all WEAK.

2. What is calculable?

Corrections to the $q\bar{q}$-potential and the **glueball spectrum** for $g_0' > 0$. The techniques use the exact S-matrix and/or exact form factors of the principal chiral model.

The **integrabilty** of the quantized principal chiral sigma model is essential.

4. What are the problems?

A. Couplings are not all weak in 3+1. The 1+1 field theory in the high-energy/eikonal approach is probably more complicated than the sigma model. Anisotropic R.G.: P.O.+J. Xiao (2008), P.O. and A. Cortés Cubero (2011, 2012).

B. In $2+1$ dimensions, the scaling limit is not the standard one (CROSSOVER). This is because we need

$$g_0'^{\,2} \ll g_0^{-1} \exp -4\pi/(g_0^2 N) \; .$$

The only way I know to beat this problem is a real-space R.G. in the 2-direction. Konik and Adamov (2009) did this for the closely related 3D Ising model.

C. No way to study potentials for $N > 2$ (until now).

5. How does it work?

Hamiltonian Formulation

Lattice spacing $\to 0$ in the $0,1$-directions, solve Gauss' law in the axial gauge $A_1 = 0$ (Mandelstam (1977)). Remaining field:

Lattice Gauge Field:

$$U(x) = U_2(x) = \exp i \int_{x^2}^{x^2+a} dy^2 A_2(x^0, x^1, y^2) \ .$$

Currents:

$$j^{\rm L}_\mu(x)_b = {\rm i Tr}\, t_b\, \partial_\mu U(x)\, U(x)^\dagger \ , \ \ j^{\rm R}_\mu(x)_b = {\rm i Tr}\, t_b\, U(x)^\dagger \partial_\mu U(x) \ , \ \ \mu = 0, 1.$$

$$H = H_0 + H_1,$$

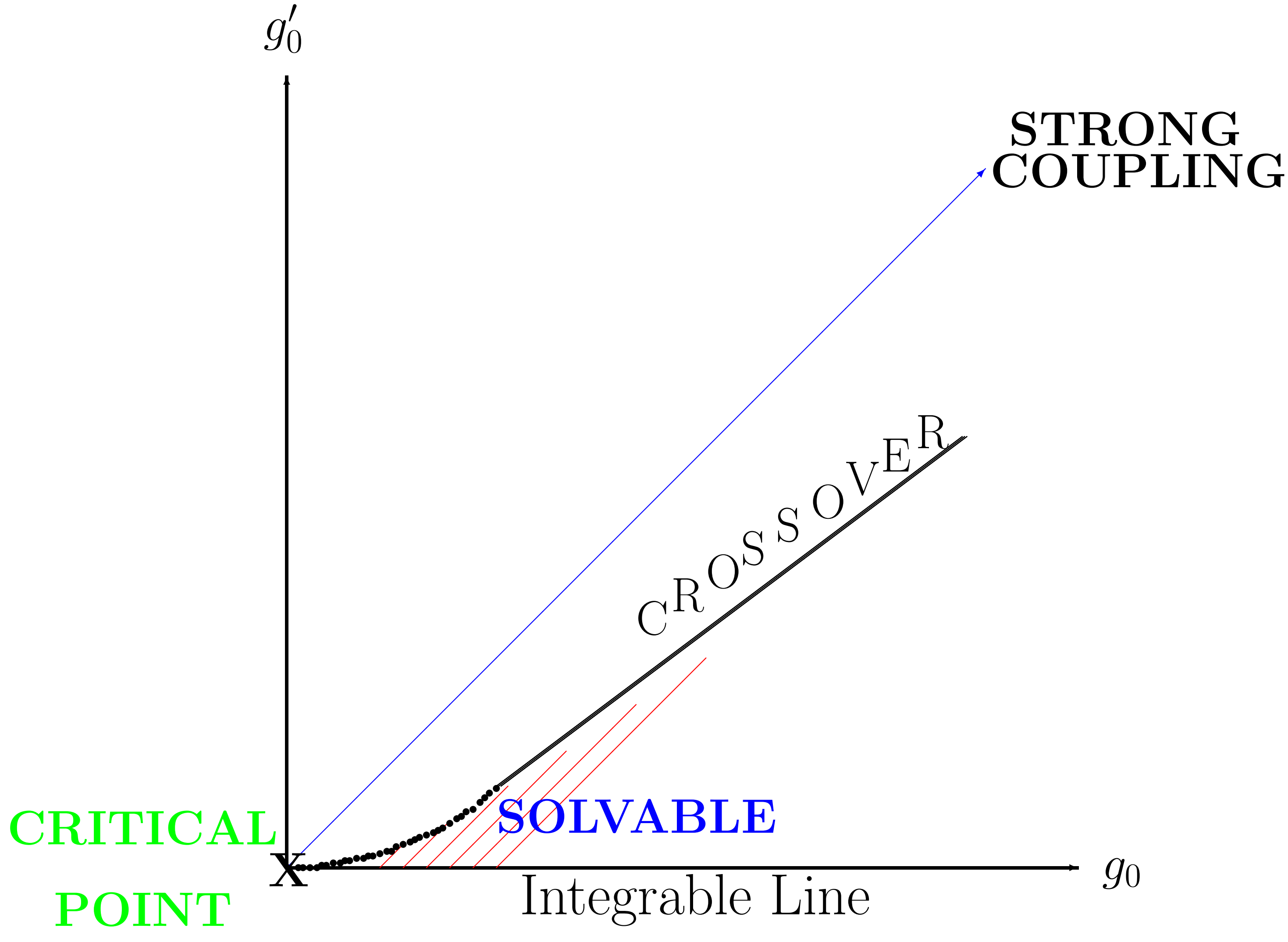
$$H_0 = \sum_{x^2} H_{PCM} = \sum_{x^2} \int dx^1 \frac{1}{2g_0^2} \{ [\, j_0^{\mathrm{L}}(x^1, x^2)_b \,]^2 + [\, j_1^{\mathrm{L}}(x^1, x^2)_b \,]^2 \} \ ,$$

$$H_1 = -\sum_{x^2} \int dx^1 \int dy^1 \frac{(g_0')^2}{4g_0^4 a^2} \, |x^1 - y^1| \, \left[j_0^{\mathrm{L}}(x^1, x^2)_b - j_0^{\mathrm{R}}(x^1, x^2 - a)_b \right]$$
$$\times \ \left[j_0^{\mathrm{L}}(y^1, x^2)_b - j_0^{\mathrm{R}}(y^1, x^2 - a)_b \right],$$

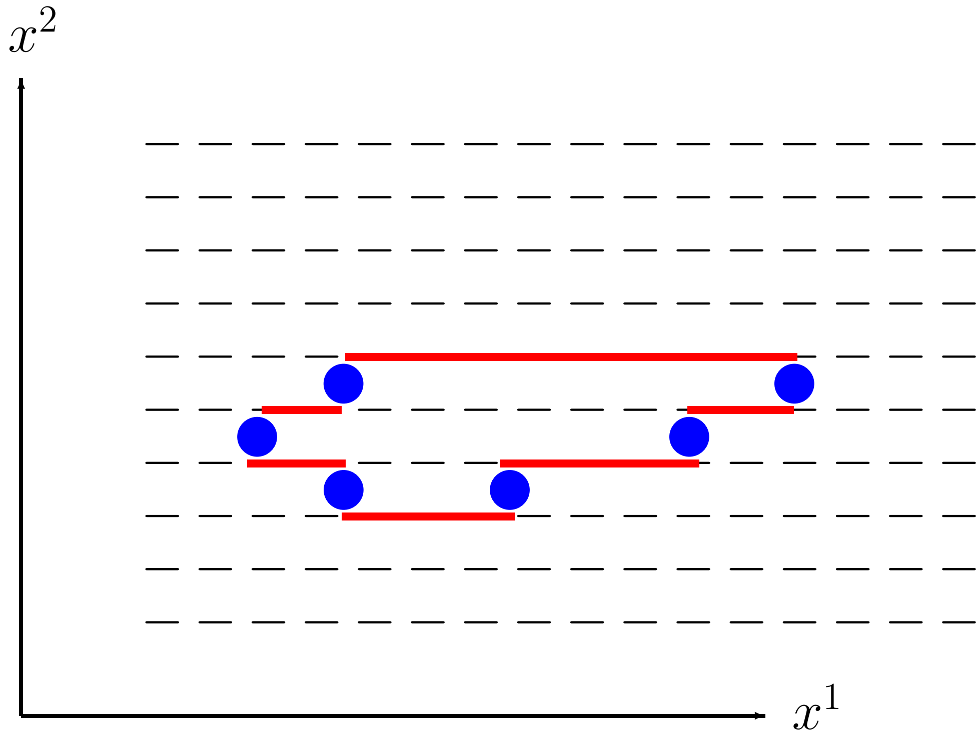
Residual Gauss' law at each x^2, on physical states:

$$\int dx^1 \left[j_0^L(x^1, x^2)_b - j_0^R(x^1, x^2 - a)_b \right] \Psi = 0 \ .$$

d=2+1 PHASE DIAGRAM



Glueball for **SU(2)**



Bullets are massive "solitons" of transverse electric flux. Red lines are longitudinal electric flux. Flux can terminate on quarks.

6. What is new?

Progress on the $\mathrm{SU}(N)_{\mathrm{Left}} \times \mathrm{SU}(N)_{\mathrm{Right}}$ sigma model:

$$\mathcal{L} = \frac{1}{2g_0^2}\int d^2x \,\mathrm{Tr}\, \partial^\mu U^\dagger \partial_\mu U, \quad U \in \mathrm{SU}(N),\ \mu = 0, 1,$$

in the 't Hooft limit, $N \to \infty$, with fixed $g_0\sqrt{N}$. Feynman diagrams are planar. Correlation functions of

$$\Phi(x) \sim Z^{-1/2} U(x)$$

are not those of a free field. Note $\sim$, instead of $=$ (equality only for v.e.v.'s.)

Exact $1/N$-expansion of form factors:

P.O., "Summing Planar Diagrams by an Integrable Bootstrap", Phys. Rev. D84, 105005 (2011). Field form factors of 1, 3 excitations.

P.O., "Summing Planar Diagrams by an Integrable Bootstrap II", arXiv:1205.1763 [hep-th] (2012). All the field form factors $\Longrightarrow$ exact correlation functions.

A. Cortés Cubero, "Multiparticle Form Factors of the Principal Chiral Model at Large N", arXiv: 1205.2069 [hep-th] (2012). Current form factors of 2, 4 excitations.

S Matrix of PCSM

Polyakov+Wiegmann (1983), Abdalla[2]+Lima Santos (1984), Wiegmann (1984)

Spectrum: $m_r = m_1 \frac{\sin(\pi r/N)}{\sin(\pi/N)},\ r = 1, \ldots, N-1$.

Elementary color dipoles $r = 1$ $(q\bar{q})$, and bound states $r > 1$. Elementary antiparticle: $r = N-1$. $\theta = \theta_{12} = \theta_1 - \theta_2$ **relative rapidity,** $m \cosh\theta_j = E_j,\ m \sinh\theta_j = p_j$.

(r=1) by (r=1) S-matrix, sans kinematic factors:

$$S_{11}(\theta) = \frac{\sin(\theta/2 - \pi \mathrm{i}/N)}{\sin(\theta/2 + \pi \mathrm{i}/N)}\, S_{\mathrm{CGN}}(\theta) \otimes S_{\mathrm{CGN}}(\theta),$$

$$S_{\mathrm{CGN}}(\theta) = \frac{\Gamma(\mathrm{i}\theta/2\pi + 1)\Gamma(-\mathrm{i}\theta/2\pi - 1/N)}{\Gamma(\mathrm{i}\theta/2\pi + 1 - 1/N)\Gamma(-\mathrm{i}\theta/2\pi)}\left(\mathbb{1} - \frac{2\pi \mathrm{i}}{N\theta} P\right).$$

Crossing $(\theta \to \pi\mathrm{i} - \theta)$, **fusion, give full S matrix.**

$1/N$-Expansion of the(Generalized) S Matrix

Forget these expressions (if you like)!

't Hooft limit is $N \to \infty$, holding $m = m_1$ fixed.

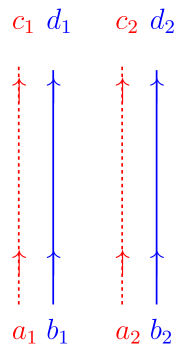
The binding energy vanishes, unless $r = N - 1$, *i.e.* an antiparticle. Nothing else survives.

$$S_{11}(\theta) = \left[1 + O(1/N^2)\right]\left[1 - \frac{2\pi \mathrm{i}}{N\theta}(P \otimes 1 + 1 \otimes P) - \frac{4\pi^2}{N^2\theta^2} P \otimes P\right].$$

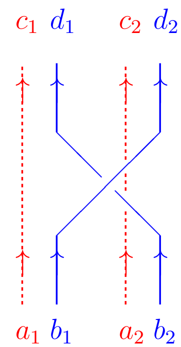
(See the figure on the next slide!)

$$S_{PP}(\theta)^{a_1 b_1; a_2 b_2}_{ c_2 d_2; c_1 d_1} = \left[1 + O\left(\frac{1}{N^2}\right)\right]$$

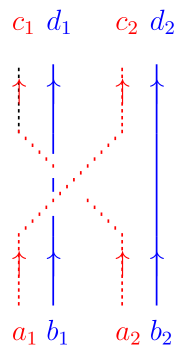
$\times \Bigg[$



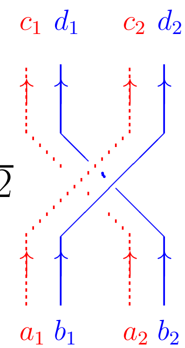
$-\dfrac{2\pi \mathrm{i}}{N\theta}$



$-\dfrac{2\pi \mathrm{i}}{N\theta}$



$-\dfrac{4\pi^2}{N^2\theta^2}$



$\Bigg]$

Zamolodchikov Algebra of Generalized Creation Operators

$$\mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\,\mathfrak{A}^\dagger_P(\theta_2)_{a_2b_2} \;=\; S_{PP}(\theta_{12})^{c_2d_2;c_1d_1}_{a_1b_1;a_2b_2}\,\mathfrak{A}^\dagger_P(\theta_2)_{c_2d_2}\,\mathfrak{A}^\dagger_P(\theta_1)_{c_1d_1}$$

$$\mathfrak{A}^\dagger_A(\theta_1)_{b_1a_1}\,\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2} \;=\; S_{AA}(\theta_{12})^{d_2c_2;d_1c_1}_{b_1a_1;b_2a_2}\,\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\,\mathfrak{A}^\dagger_A(\theta_1)_{d_1c_1}$$

$$\mathfrak{A}^\dagger_P(\theta_1)_{a_1b_1}\,\mathfrak{A}^\dagger_A(\theta_2)_{b_2a_2} \;=\; S_{PA}(\theta_{12})^{d_2c_2;c_1d_1}_{a_1b_1;b_2a_2}\,\mathfrak{A}^\dagger_A(\theta_2)_{d_2c_2}\,\mathfrak{A}^\dagger_P(\theta_1)_{c_1d_2}\,,$$

$$P = \text{Particle}\,, \quad A = \text{Antiparticle.}$$

Associativity implies Yang-Baxter equation.

As $N \to \infty$, $\mathfrak{A}^\dagger$'s commute. The master field is built from $\mathfrak{A}^\dagger$'s, $\mathfrak{A}$'s.

Smirnov's form-factor axioms:

These follow from the LSZ formulas and integrability.

1. ***Scattering Axiom*** **(Watson's theorem):**

$$\langle 0|\Phi(0)_{b_0a_0}\,\mathfrak{A}^{\dagger}_{I_1}(\theta_1)_{C_1}\cdots\mathfrak{A}^{\dagger}_{I_j}(\theta_j)_{C_j}\mathfrak{A}^{\dagger}_{I_{j+1}}(\theta_{j+1})_{C_{j+1}}\cdots\cdots\mathfrak{A}^{\dagger}_{I_M}(\theta_M)_{C_M}|0\rangle$$

$$= S_{I_jI_{j+1}}(\theta_{j\;j+1})^{C'_{j+1}C'_j}_{\;\;C_jC_{j+1}}\langle 0|\Phi(0)_{b_0a_0}\,\mathfrak{A}^{\dagger}_{I_1}(\theta_1)_{C_1}$$

$$\times\;\cdots\times\mathfrak{A}^{\dagger}_{I'_{j+1}}(\theta_{j+1})_{C'_{j+1}}\mathfrak{A}^{\dagger}_{I'_j}(\theta_j)_{C'_j}\cdots\mathfrak{A}^{\dagger}_{I_M}(\theta_M)_{C_M}|0\rangle,$$

where I_k, $k=1,\dots,M$ is P or A (particle or antiparticle) and C_k denotes a pair of indices (*e.g.* a_kb_k, for $C_k=P$ and b_ka_k, for $C_k=A$).

2. ***Periodicity Axiom*** **(Generalized Crossing):**

$$\langle 0|\Phi(0)_{b_0 a_0}\, \mathfrak{A}^{\dagger}_{I_1}(\theta_1)_{C_1}\mathfrak{A}^{\dagger}_{I_2}(\theta_2)_{C_2}\cdots\mathfrak{A}^{\dagger}_{I_n}(\theta_n)_{C_n}|0\rangle$$
$$= \langle 0|\Phi(0)_{b_0 a_0}\, \mathfrak{A}^{\dagger}_{I_n}(\theta_n - 2\pi \mathrm{i})_{C_n}\, \mathfrak{A}^{\dagger}_{I_1}(\theta_1)_{C_1}\cdots\mathfrak{A}^{\dagger}_{I_{n-1}}(\theta_{M-1})_{C_{n-1}}|0\rangle.$$

3. ***Annihilation-Pole Axiom*****:**

$$\mathrm{Res}\,|_{\theta_{1n}=-\pi \mathrm{i}}\, \langle\, 0\,|\,\Phi(0)_{b_0 a_0}\mathfrak{A}^{\dagger}_{I_1}(\theta_1)_{C_1}\mathfrak{A}^{\dagger}_{I_2}(\theta_2)_{C_2}\cdots\mathfrak{A}^{\dagger}_{I_n}(\theta_n)_{C_n}|0\rangle$$

$$= -2\mathrm{i}\langle\, 0\,|\,\Phi(0)_{b_0 a_0}\,|\mathfrak{A}^{\dagger}_{I_2}(\theta_2)_{C_2'}\mathfrak{A}^{\dagger}_{I_3}(\theta_2)_{C_3'}\cdots\mathfrak{A}^{\dagger}_{I_{n-1}}(\theta_{n-1})_{C_{n-1}'}|0\rangle$$
$$\times\ \Big[S_{I_1 I_2}(\theta_{12})^{C_2' D_1}_{\ \ C_1 C_2} S_{I_1 I_3}(\theta_{13})^{C_3' D_2}_{\ \ D_1 C_3}\cdots S_{I_1 I_{n-1}}(\theta_{1\; n-1})^{C_n C_{n-1}'}_{\ \ D_{n-2} C_{n-1}}$$
$$-\ \delta^{C_n}_{\ C_1}\delta^{C_2'}_{\ C_2}\delta^{C_3'}_{\ C_3}\cdots\delta^{C_{n-1}'}_{\ C_{n-1}}\Big].$$

4. ***Lorentz-Invariance Axiom*: For the scalar operator Φ, and boost $\Delta\theta$,**

$$\langle 0|\Phi(0)_{b_0a_0}\,\mathfrak{A}^{\dagger}_{I_1}(\theta_1+\Delta\theta)_{C_1}\cdots\mathfrak{A}^{\dagger}_{I_M}(\theta_M+\Delta\theta)_{C_M}|0\rangle$$
$$=\langle 0|\Phi(0)_{b_0a_0}\,\mathfrak{A}^{\dagger}_{I_1}(\theta_1)_{C_1}\cdots\mathfrak{A}^{\dagger}_{I_M}(\theta_M)_{C_M}|0\rangle,$$

5. ***Bound-State Axiom*: There are poles on the imaginary axis of rapidity differences θ_{jk}, due to bound states.**

6. ***Minimality Axiom*. Form factors are holomorphic, except possibly for bound-state poles or annihilation poles, for rapidity differences θ_{jk}, in the complex strip $0<\mathfrak{Im}\,\theta_{jk}<2\pi$.**

Renormalized-Field Form Factors

$$\langle 0 | \Phi(0)_{b_0 a_0} \\ | A, \theta_1, b_1, a_1; \cdots; A, \theta_{M-1}, b_{M-1}, a_{M-1}; P, \theta_M, a_M, b_M; \cdots; P, \theta_{2M-1}, a_{2M-1}, b_{2M-1} \rangle \\ = \frac{1}{N^{M-1/2}} \sum_{\sigma,\tau \in S_M} F_{\sigma\tau}(\theta_1, \theta_2, \dots, \theta_{2M-1}) \prod_{j=0}^{M-1} \delta_{a_j \, a_{\sigma(j)+M}} \delta_{b_j \, b_{\tau(j)+M}} ,$$

where $F = F^0 + O(1/N)$,

$$F^0_{\sigma\tau}(\theta_1, \theta_2, \dots, \theta_{2M-1}) = \frac{(-4\pi)^{M-1} K_{\sigma\tau}}{\prod_{j=1}^{M-1} [\theta_j - \theta_{\sigma(j)+M} + \pi \mathrm{i}][\theta_j - \theta_{\tau(j)+M} + \pi \mathrm{i}]} ,$$

and where

$$K_{\sigma\tau} = \begin{cases} 1 , & \sigma(j) \neq \tau(j), \text{ for all } j \\ 0 , & \text{otherwise} \end{cases} .$$

2-Pt. Wightman Function to Order $1/N^0$

$$
\begin{aligned}
Z^{-1}\ \frac{1}{N}\langle 0|\mathrm{Tr}\ U(x)U(0)^{\dagger}|0\rangle &= \int \frac{d\theta}{4\pi}\ \exp[\mathrm{i}m(x^{-}e^{\theta}+x^{+}e^{-\theta})] \\
&+ \ \frac{1}{4\pi}\sum_{l=1}^{\infty}\int d\theta_1\cdots d\theta_{2l+1}\ \exp\left[\mathrm{i}\sum_{j=1}^{2l+1} m(x^{-}e^{\theta_j}+x^{+}e^{-\theta_j})\right] \\
&\times \ \prod_{j=1}^{2l}\frac{1}{(\theta_j-\theta_{j+1})^2+\pi^2}\ +\ O\left(\frac{1}{N}\right),
\end{aligned}
$$

where $x^{\pm}=(x^0\pm x^1)/2.$

The renormalization factor Z **is determined by** $U(0)U(0)^{\dagger}=1.$

A. Cortés Cubero's current form factors.

Two-excitation form factor:

$$\langle\, 0\, | j^L_\mu(0)_{a_0c_0} | A, \theta_1, b_1, a_1; P, \theta_2, b_2, a_2 \rangle$$
$$= \;(p_1 - p_2)_\mu \,\frac{2\pi \mathrm{i}}{\theta_{12} + \pi \mathrm{i}} \left(\delta_{a_0a_2}\delta_{c_0a_1} - \frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_2} \right) \delta_{b_1b_2} + O\left(\frac{1}{N^2}\right).$$

Four-excitation form factor:

$$
\begin{aligned}
&\langle\, 0\, | j^L_\mu(0)_{a_0c_0} | A,\theta_1,b_1,a_1; A,\theta_2,b_2,a_2; P,\theta_3,a_3,b_3; P,\theta_4,a_4,b_4\rangle \\
&= \frac{8\pi^2 \mathrm{i}}{N}[p_1+p_2-p_3-p_4]_\mu \\
&\times \left[\frac{1}{(\theta_{14}+\pi\mathrm{i})(\theta_{23}+\pi\mathrm{i})(\theta_{24}+\pi\mathrm{i})}\left(\delta_{a_0a_3}\delta_{a_1c_0}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_3}\right)\delta_{a_2a_4}\delta_{b_1b_4}\delta_{b_2b_3}\right. \\
&+ \frac{1}{(\theta_{13}+\pi\mathrm{i})(\theta_{23}+\pi\mathrm{i})(\theta_{24}+\pi\mathrm{i})}\left(\delta_{a_0a_4}\delta_{a_1c_0}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_1a_4}\right)\delta_{a_2a_3}\delta_{b_1b_3}\delta_{b_2b_4} \\
&+ \frac{1}{(\theta_{14}+\pi\mathrm{i})(\theta_{13}+\pi\mathrm{i})(\theta_{24}+\pi\mathrm{i})}\left(\delta_{a_0a_3}\delta_{a_2c_0}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_3}\right)\delta_{a_1a_4}\delta_{b_1b_3}\delta_{b_2b_4} \\
&+ \left.\frac{1}{(\theta_{14}+\pi\mathrm{i})(\theta_{13}+\pi\mathrm{i})(\theta_{23}+\pi\mathrm{i})}\left(\delta_{a_0a_4}\delta_{a_2c_0}-\frac{1}{N}\delta_{a_0c_0}\delta_{a_2a_4}\right)\delta_{a_1a_3}\delta_{b_1b_4}\delta_{b_2b_3}\right] \\
&+ O\left(\frac{1}{N^3}\right). \qquad (1)
\end{aligned}
$$

7. What comes next?

- Explicit calculation of renormalization constants. Comparison with perturbation theory.
- Exact correlation functions of currents.
- Comparison of field and current form factor correlations. O.P.E.'s.
- Application to (2+1)-dimensional gauge theories in the 't Hooft limit.
- Attack crossover with real-space RG.

THANK YOU!