# The Abelian Higgs Model without/with Chern-Simons term in 2+1dimensions at strong coupling

by

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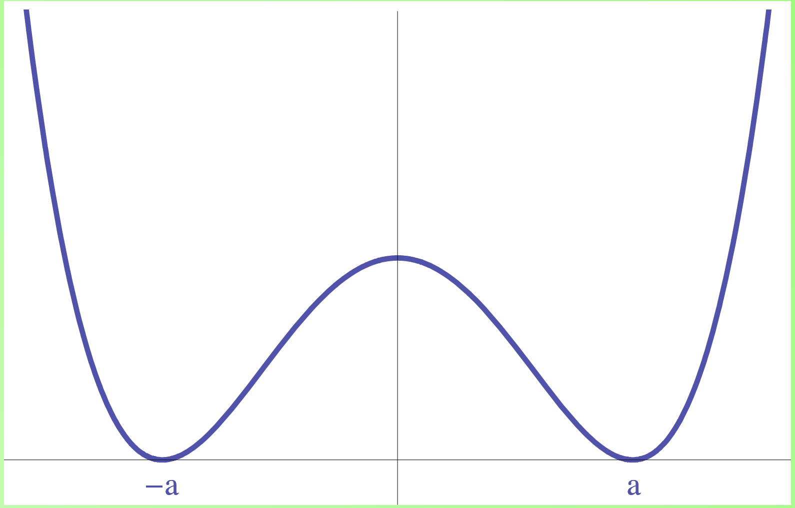
The Abelian Higgs model is also called scalar electrodynamics usually with a symmetry breaking potential:

$$V(\phi) = \lambda(|\phi|^2 - a^2)^2$$



# The Lagrangian:

$$\mathcal{L} = D_\mu \phi^* D^\mu \phi - \lambda(|\phi|^2 - a^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$$

has a spectrum of a massive vector boson, a massive neutral scalar and massive vortices

$$M = ea \qquad\qquad m = \sqrt{8\lambda} a$$

# Strong Coupling limit

- we take $\lambda \to \infty \quad e \to \infty$
- keeping the ratio fixed $\dfrac{\lambda}{e^2}$
- this decouples the massive vector gauge bosons and the massive neutral scalar, leaving just the vortices with finite mass

$$m_{vortex} = a^2 \times f(8\lambda/e^2)$$

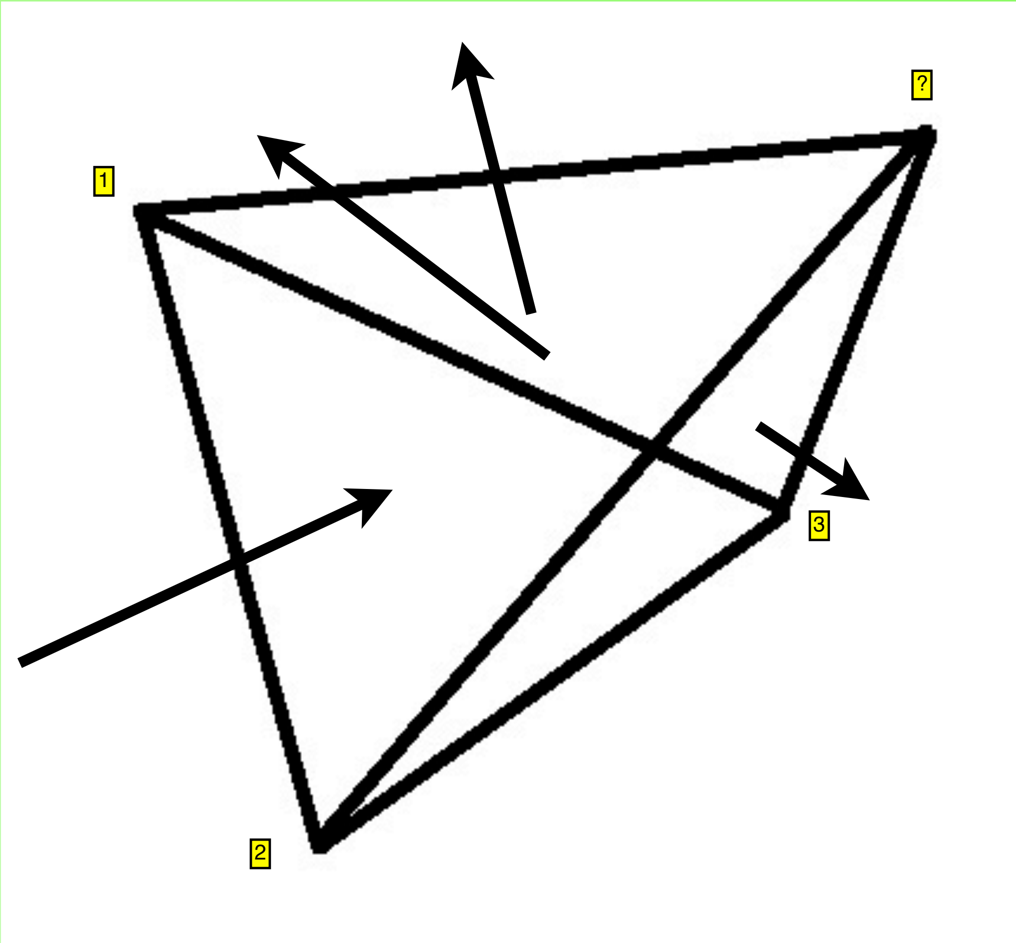
# Effective Theory of Free Massive Vortices

- Vortices have quantized flux
- Interactions fall off exponentially
- Thus as long as they are dilute, we can forget their mutual interactions
- With the Chern-Simons term, they do have long range interactions, but only due to Aharonov-Bohm interactions
- The vacuum to vacuum amplitude will be saturated by configurations of closed vortex loops, with Boltzmann weight proportional to the mass per unit length times the total length.

# Generating closed loops on the lattice

- we fill up space with a tetrahedral tessellation (jmol)
- we start with a BCC lattice, and adding the lines that split the cubic sides along the diagonals
- then we distribute the cube roots of unity on the vertices of the lattice
- if the phase change around a triangular face is $2\pi$, then we say that a unit of flux has pierced through that face

- topologically, the flux has no choice but to exit the tetrahedron that it has entered



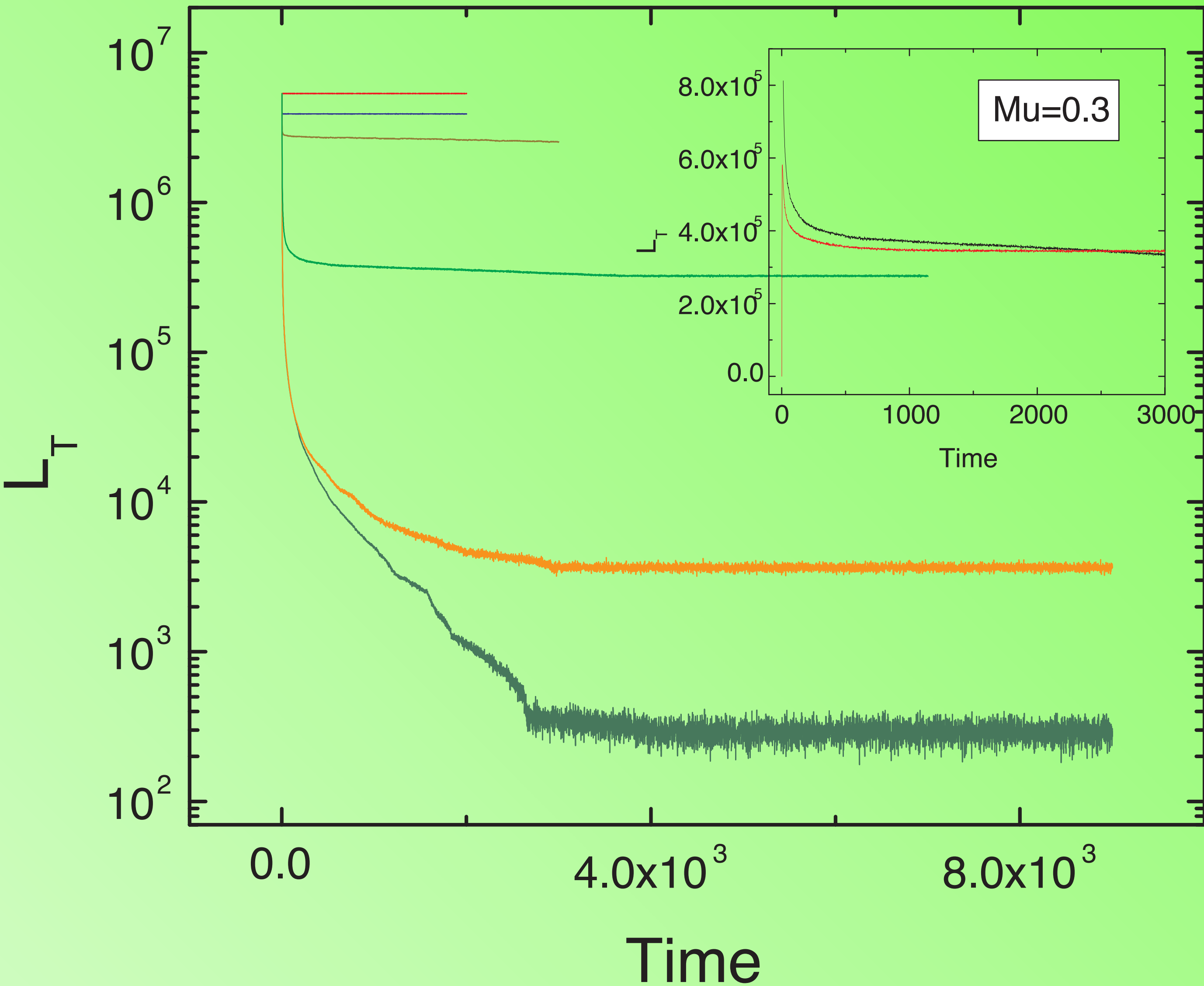
- but then it enters into the neighbouring tetrahedron, and so on
- the flux has no choice but to fill the entire lattice, or close on itself and form a vortex loop
- this way we generate a configuration of a box full of closed vortex loops
- the loops cannot intersect
- such a description was used in the context of cosmic string configurations, but on a cubic lattice, which allows two flux tubes to enter a cube and two to leave, without resolving how they are attached
- our lattice resolves the paths inside the cube

# Lattice thermal equilibrium

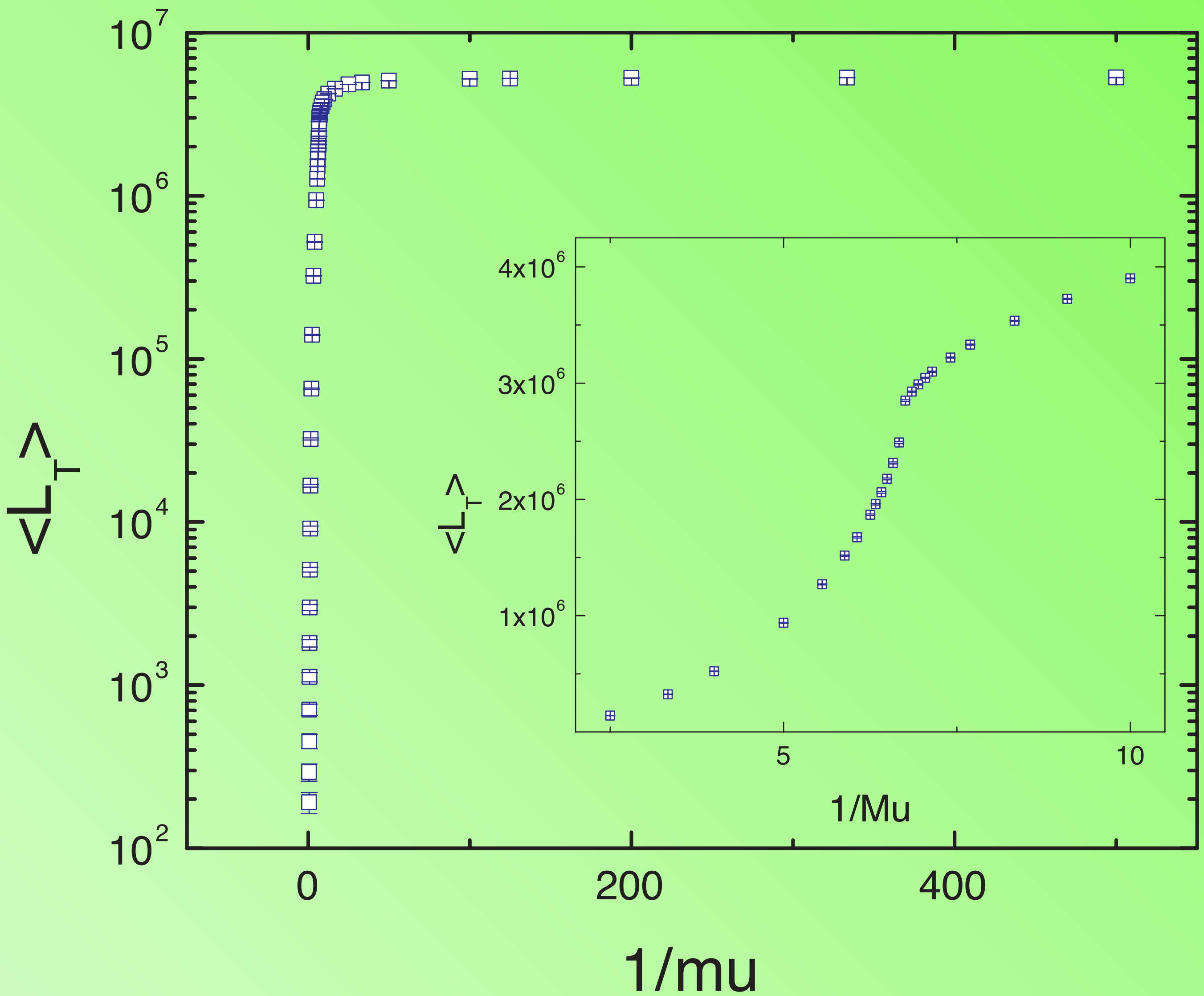
- we start with an arbitrary initial configuration and then update it with a Monte Carlo simulation to yield a system in thermal equilibrium
- the crudest approximation to the action is taken, simply the mass per unit length times the total length of the vortex loops
- the number of configurations a priori are $3^{2\times 10^6}$
- so the aim of the Monte Carlo is to extract a manageable set of configurations with the same statistical properties as the original set
- we end up with a set of 10,000 configurations

# Approach to equilibrium for various masses:

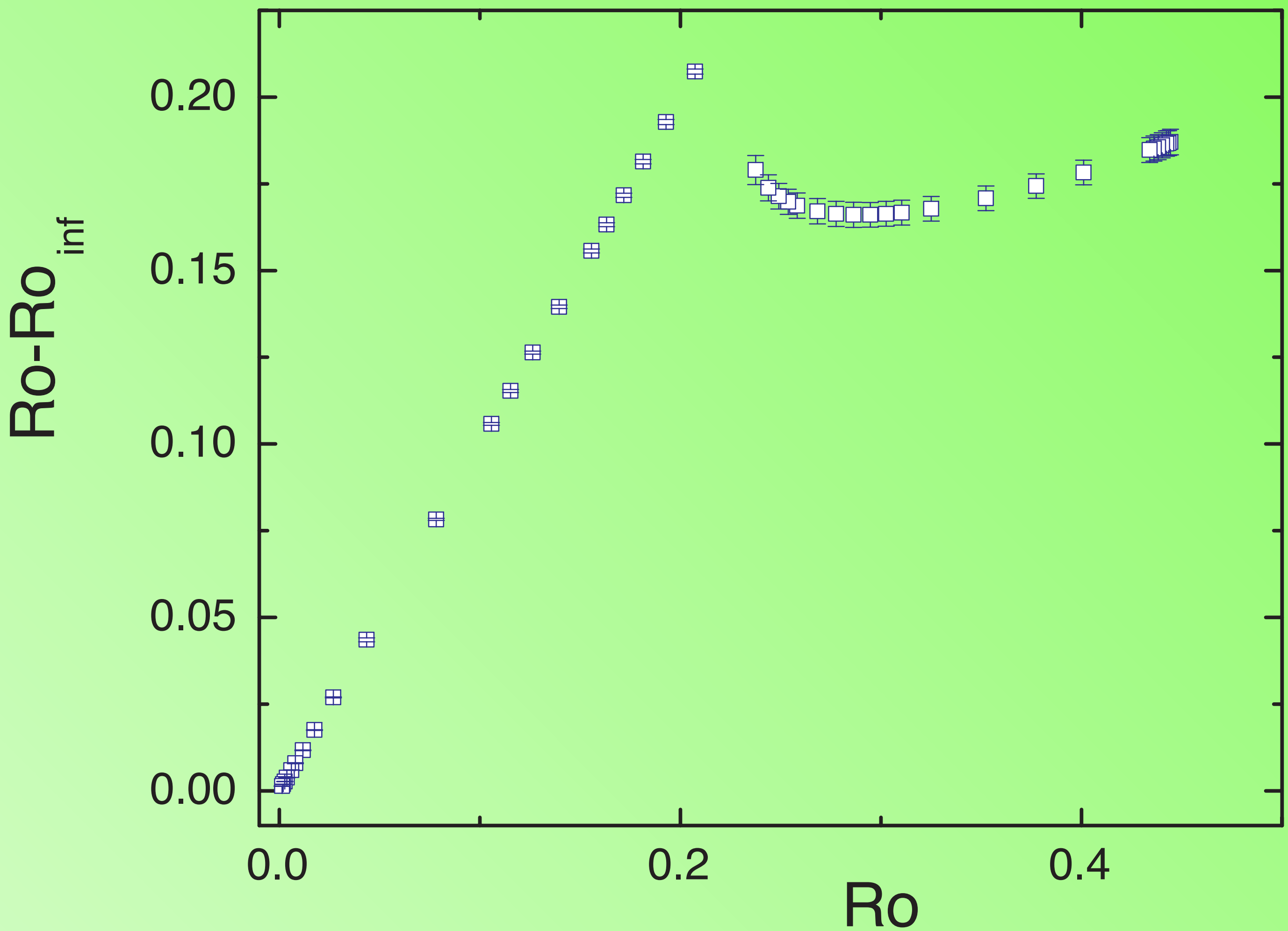
$$\mu = 0, 0.1, 0.15, 0.3, 0.9, 1.5$$



# Numerical evidence for a phase transition



- as a function of the density, we find a transition at $\mu \approx .152$

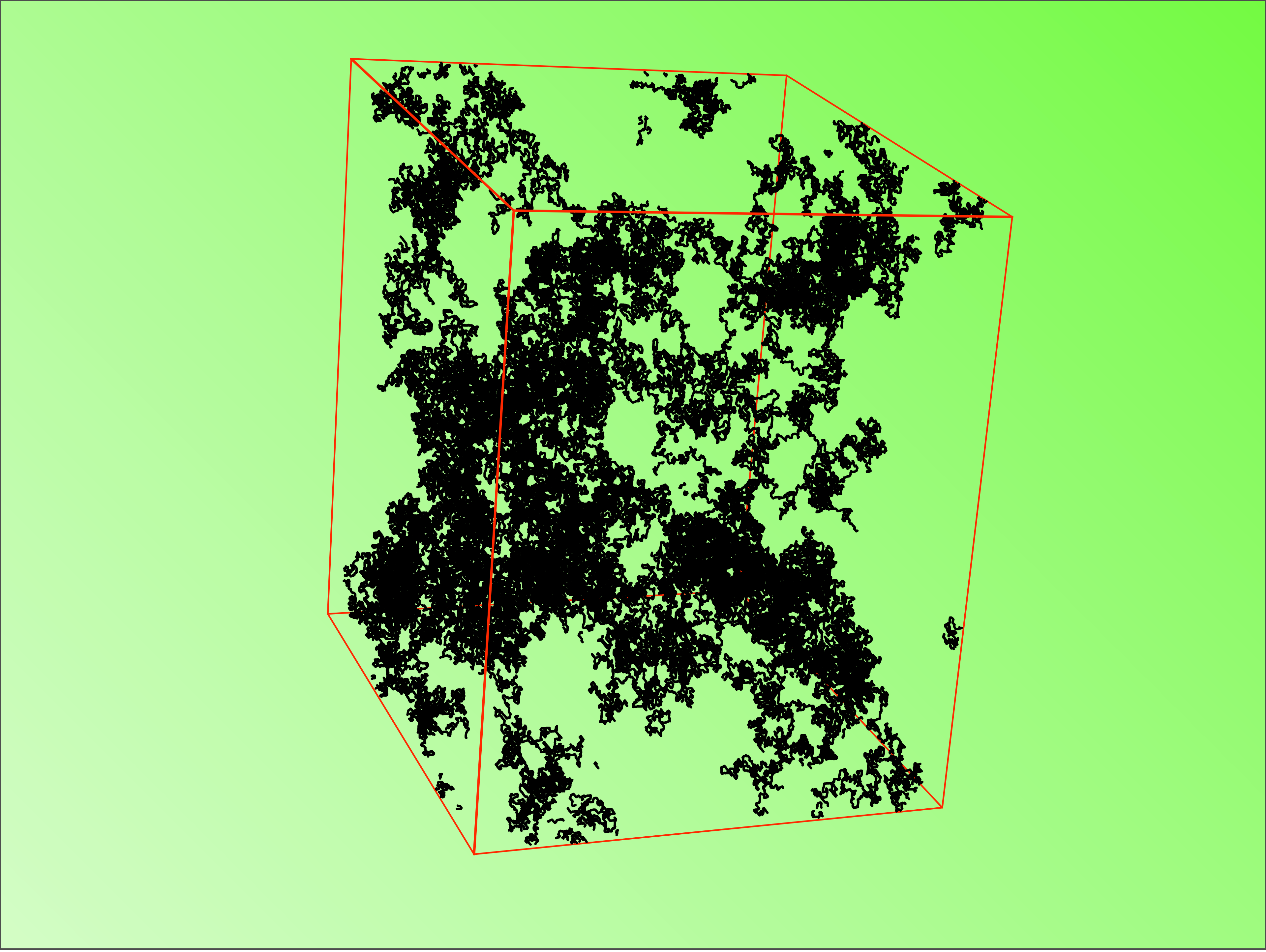


# Infinite loops

- if the length of a loop is much more than the natural maximum length of a loop in a given box, then we call it an infinite loop
- a loop corresponds to a 3-d random walk with one constraint (that it close)
- the average distance traversed behaves as

$$d \sim \sqrt{L}$$

- thus a loop considerably longer than 10000 should be considered as infinite, as $d = 100$.
- there is theoretical understanding why the system should have exactly one infinite loop and a thermal bath of finite loops, at high density.



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# Order parameters

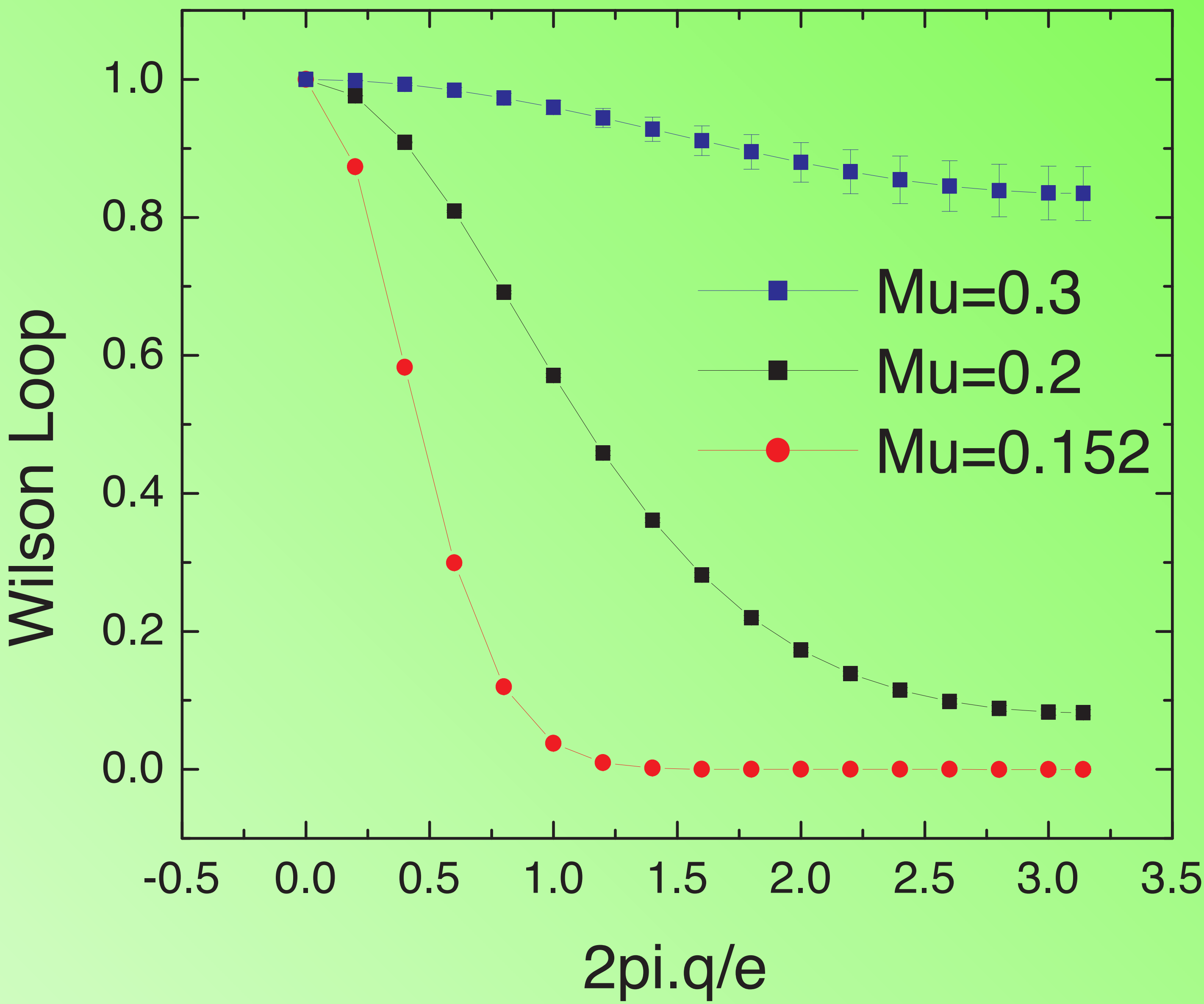
- we look at the Wilson loop and Polyakov loop $$W(L,T) = \left\langle \exp(-i\frac{q}{e}\oint A_\mu dx_\mu) \right\rangle,$$
- along a rectangular curve of dimensions $L \times T$
- as $T \to \infty$ we expect $W(L,T) \longrightarrow e^{-\Delta(L)\cdot T}$
- where the interpretation of $\Delta(L)$ is the energy shift obtained by the insertion of a quark/anti-quark pair into the vacuum, and moving them along the Wilson loop
- $\oint A_\mu dx_\mu$ is just the linking number of the Wilson loop with the other vortex loops

- thus we must compute

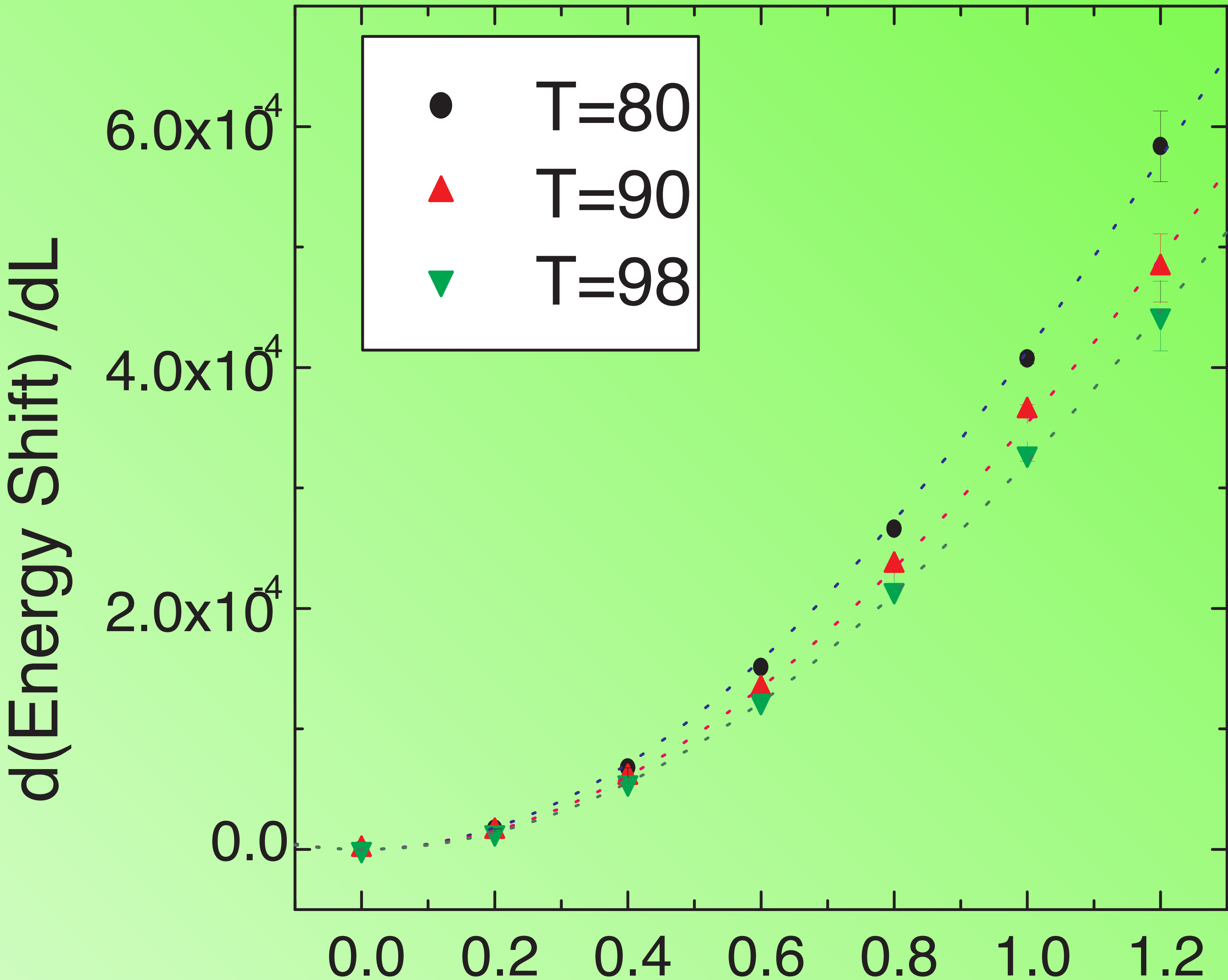
$$W(L,T) = \langle \exp(2\pi i q\nu/e) \rangle$$

- where $\nu$ is just the linking number
- if $\Delta(L) \sim L$ we have area law and linear confinement
- we compute $d\Delta(L)/dL$ and look for a constant part for $T \to \infty$
- so we compute it numerically at finite $T$ and extrapolate to $T \to \infty$

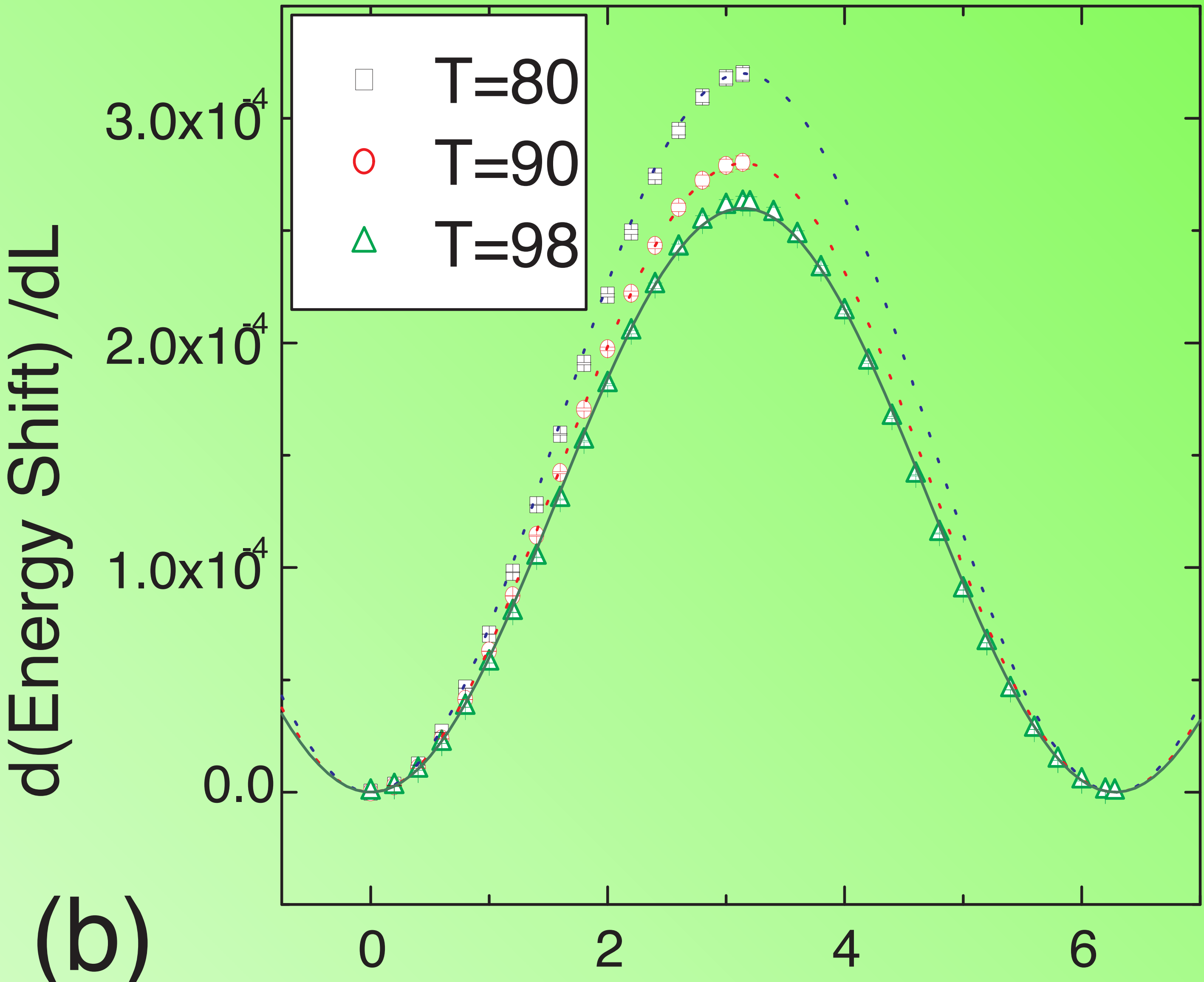
- Wilson loop for different values of $\mu$



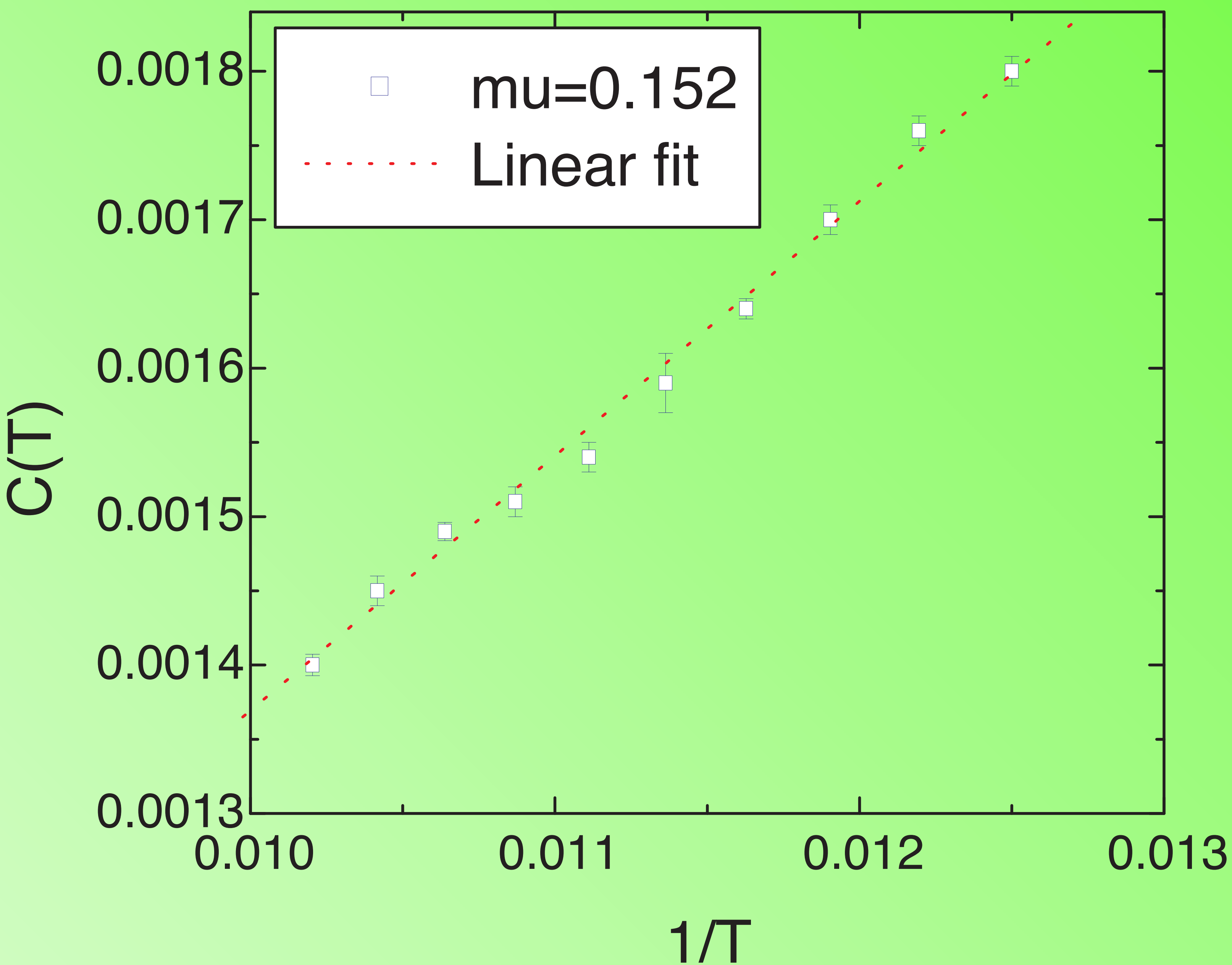
- $d\Delta/dL$ for different values of $T$ and $\mu = .152$



- $d\Delta/dL$ for different values of $T$ and $\mu = .2$
- the dotted lines are $C(T)\sin^2((2\pi q/e)/2)$



- *C(T)* as a function of *1/T*



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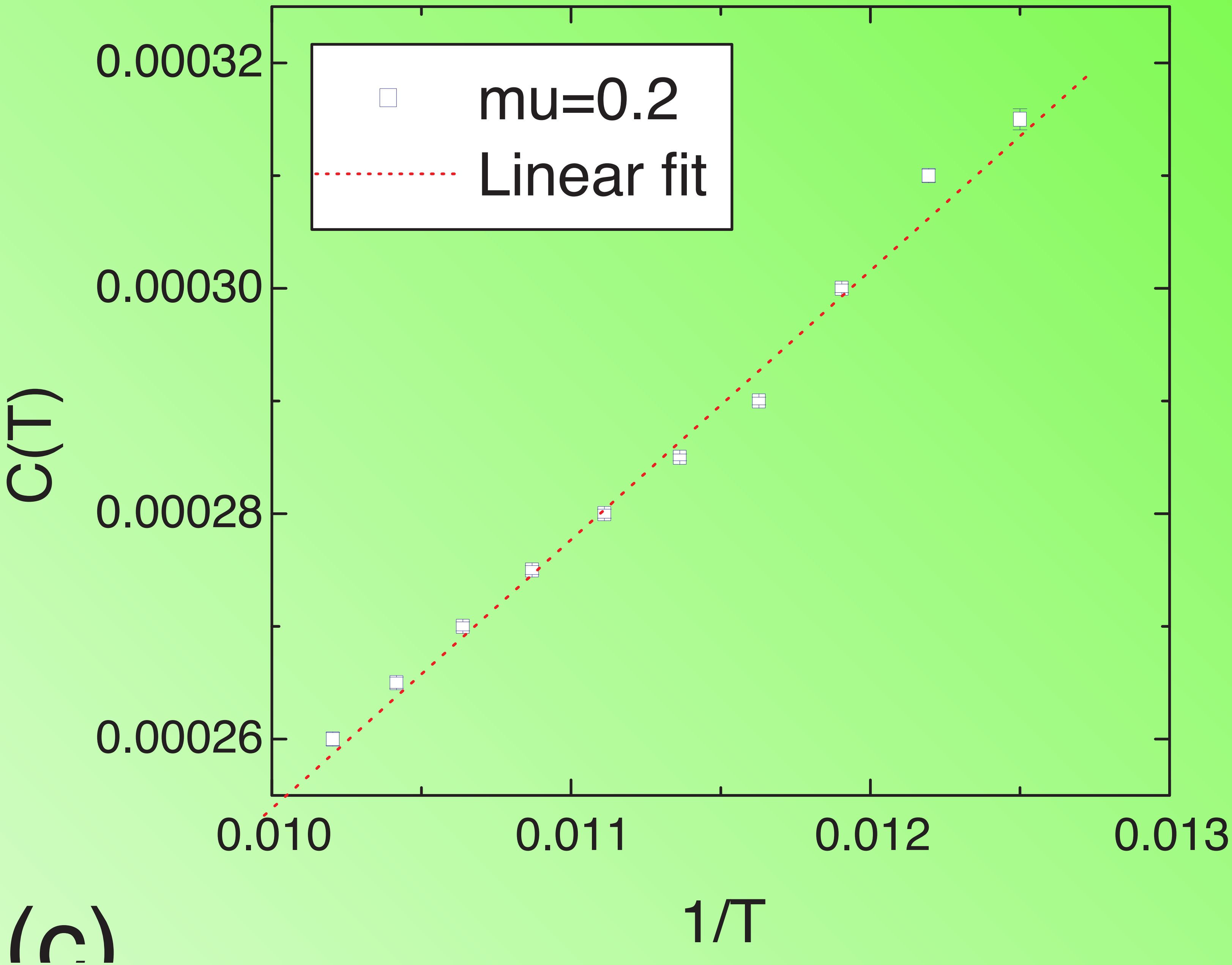
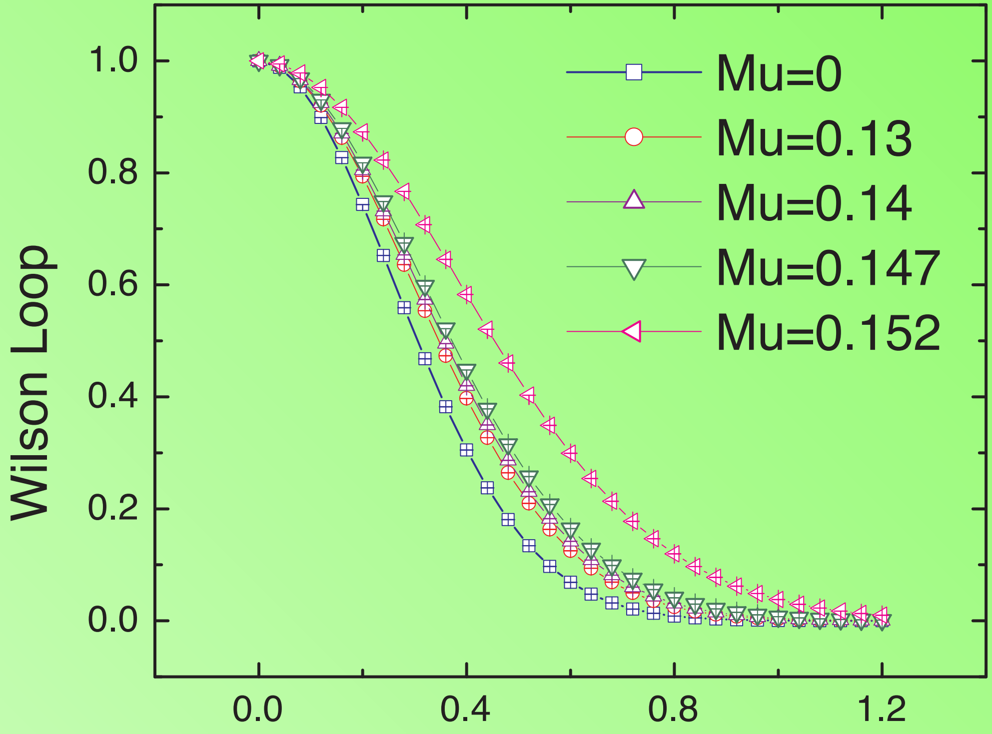


- unfortunately, it seems that $\lim_{T \to \infty} C(T) = 0$

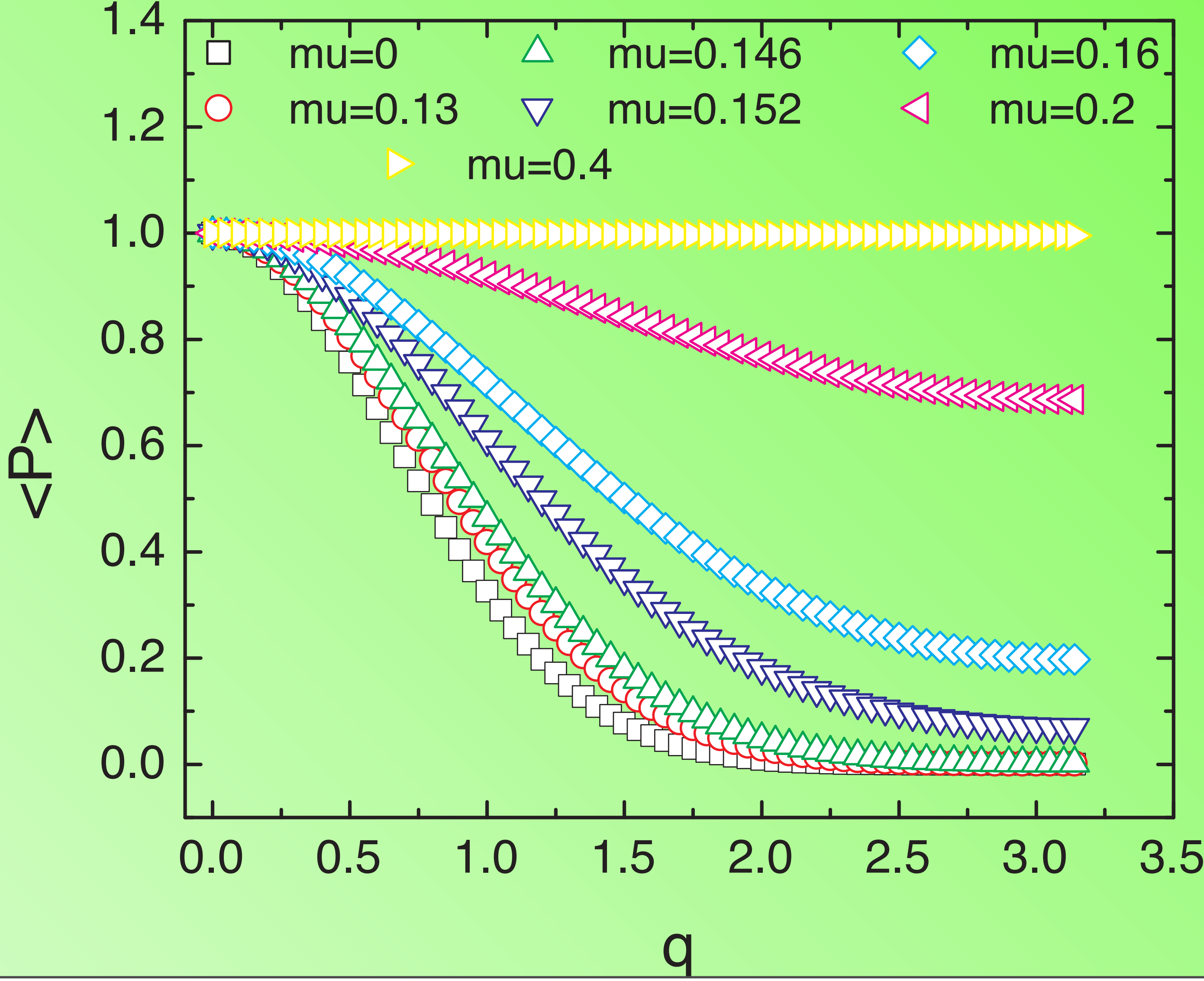
- this means that there is no area law
- however, there is clearly a cross over to a regime where the energy shift grows large enough so that we cannot resolve its behaviour:



# Polyakov loop

- this corresponds to a Wilson loop which wraps around the temporal direction for periodic boundary conditions. (This corresponds to finite temperature.)
- this order parameter shows no special behaviour, just like the Wilson loop

# Polyakov loop for different values of $\mu$ and for $T$=30



# Adding the Chern-Simons term

- the Chern-Simons term corresponds to the addition

$$\mathcal{L} = \mathcal{L}_1 + \frac{\kappa}{4}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho},$$

- there are fundamental differences including this term in the Lagrangian
- the Euclidean Lagrangian is no longer real, normally $iS_{Mink.} \rightarrow -S_E$ for

$$t \rightarrow -i\tau \qquad \qquad \partial_t \rightarrow i\partial_\tau$$

- the Chern-Simons term is linear in time derivatives

$$\int dt\partial_t \to \int d\tau\partial_\tau$$

- thus perturbation theory can be compromised, especially if the Euclidean critical points occur at complex field configurations
- worse, the Monte-Carlo simulation simply makes no sense, the complex integrand does not admit an interpretation as a probability density

$$e^{-S_E}e^{iS_c} \neq p$$

- but we can take only the real part of the Euclidean action to define the measure for the path integral

$$e^{-S_E} = p$$

- and then

$$e^{iS_c}$$

- is just a uni-modular phase that can be integrated against the measure defined by the real part

$$< \mathcal{O} > = \frac{\int \mathcal{D}(\phi, A)\mu(e^{-S_E})e^{iS_c}\mathcal{O}}{\int \mathcal{D}(\phi, A)\mu(e^{-S_E})e^{iS_c}}$$

- which can be calculated using Monte Carlo methods (this is ok for small imaginary part)

# Evaluating the Chern-Simons term

$$\mathcal{L} = \mathcal{L}_1 + \frac{\kappa}{4}\varepsilon^{\mu\nu\rho}A_\mu F_{\nu\rho},$$

- the Chern-Simons term for a network of vortex loops each carrying a unit of flux is simply (two times) the total signed linking number between all the vortex loops

$$\int d^3x \epsilon^{\mu\nu\rho} A_\mu F_{\mu\nu} = \oint dx^\mu A_\mu \int d^2 x_\perp B$$

$$= 2\pi \sum_{loops} \oint dx^\mu A_\mu = (2\pi)^2 \sum_{loops} n_{loop}$$

- to calculate this linking number along a dynamical vortex loop is not straightforward
- the loop passes through the middle of the triangular faces of the tetrahedra, there are no phases defined there.
- but we can imagine making a deformation of each loop to follow the edges and vertices
- then multiplication of the phases along the deformed loop will give the linking number with all the other loops
- but it can also add the “self” linking number of the original loop with the deformed loop

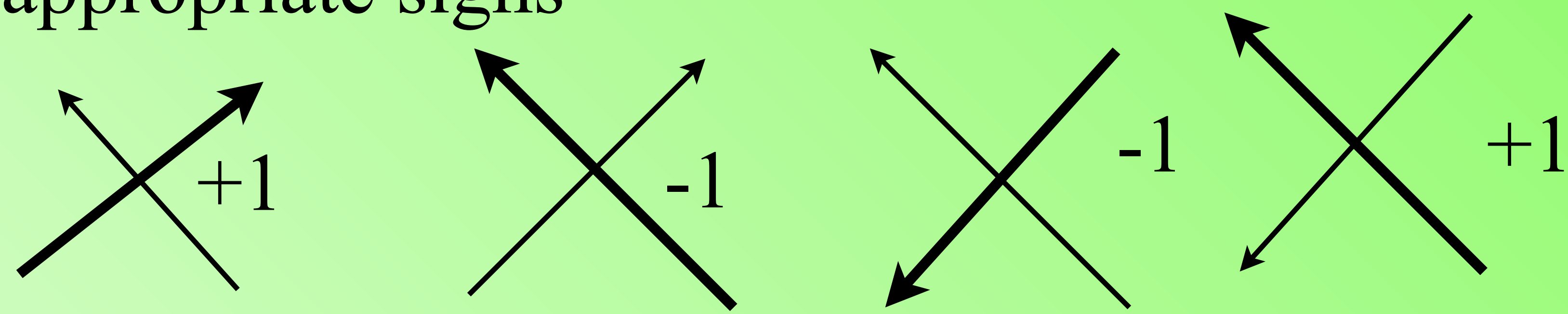
# Computing the self linking number

- the original loop and the deformed loop define a closed ribbon which twists and writhes
- the self linking number satisfies the relation Self linking number = Twist + Writhe
- twist is simply the usual notion of the twist along the ribbon (not an integer)
- writhe is a measure of how much the ribbon coils up (not an integer)

- the linking number between any two curves is actually just a double integral

$$N = \frac{1}{4\pi} \oint d\vec{x} \oint \cdot d\vec{y} \times \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3}$$

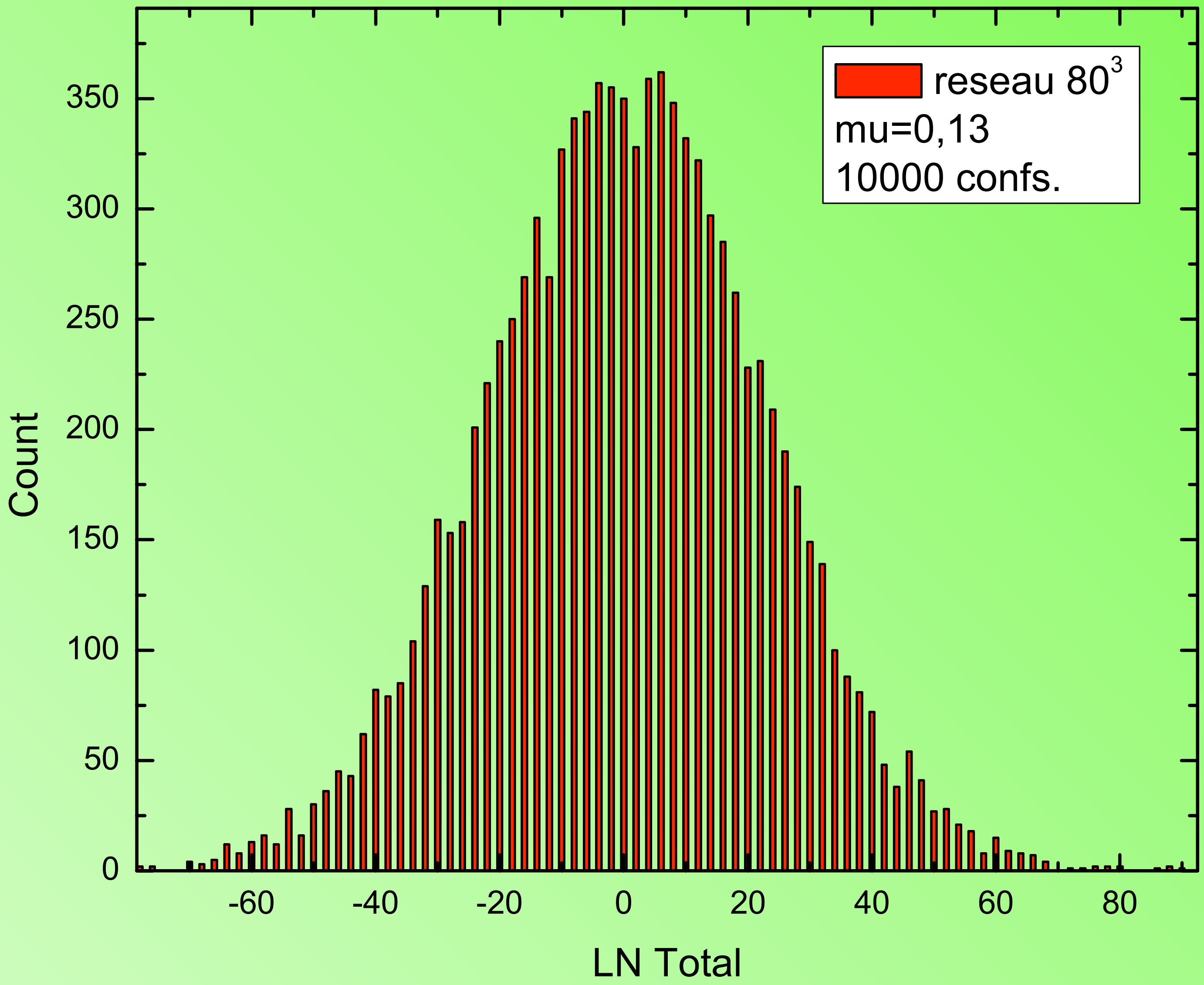
- but doing the integral along polygonal loops of length more than a million is not feasible
- but knot theorists tell us that all we have to do is project the loops on a two dimensional plane and count all the crossings with appropriate signs



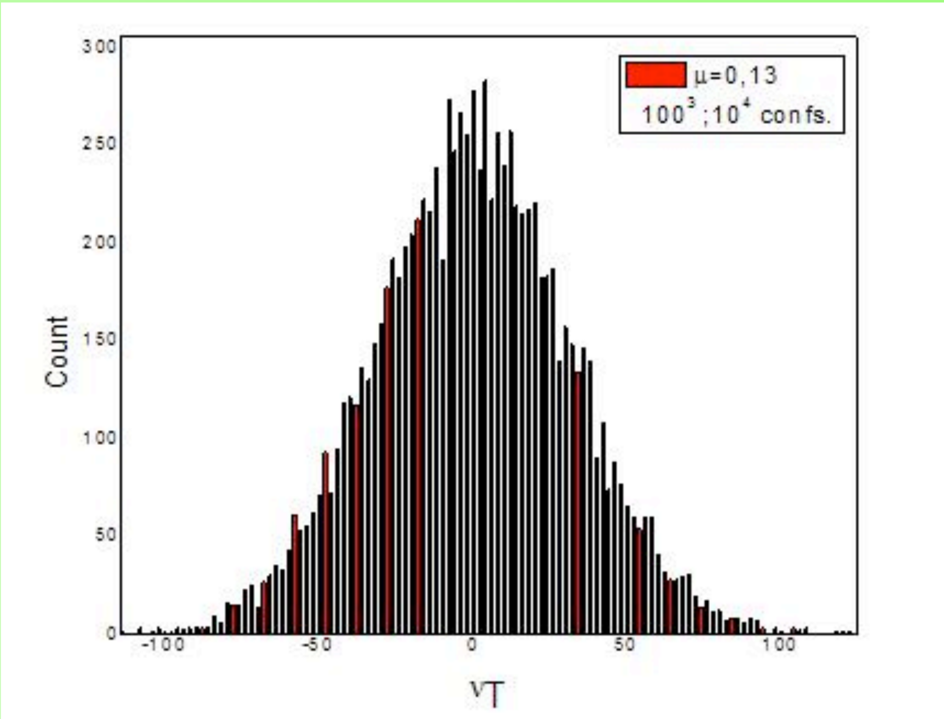
# Total linking number and Wilson loop

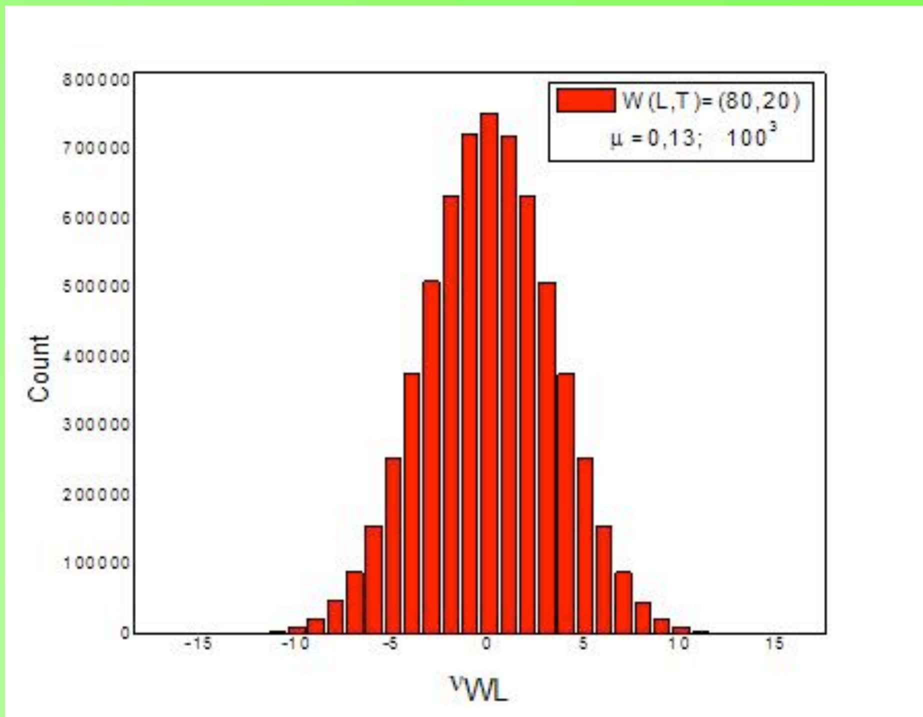
- the total linking number should behave much like the linking number of a Wilson loop
- we show the total linking number and the Wilson loop linking number for various values of the mass

# Total linking number distribution

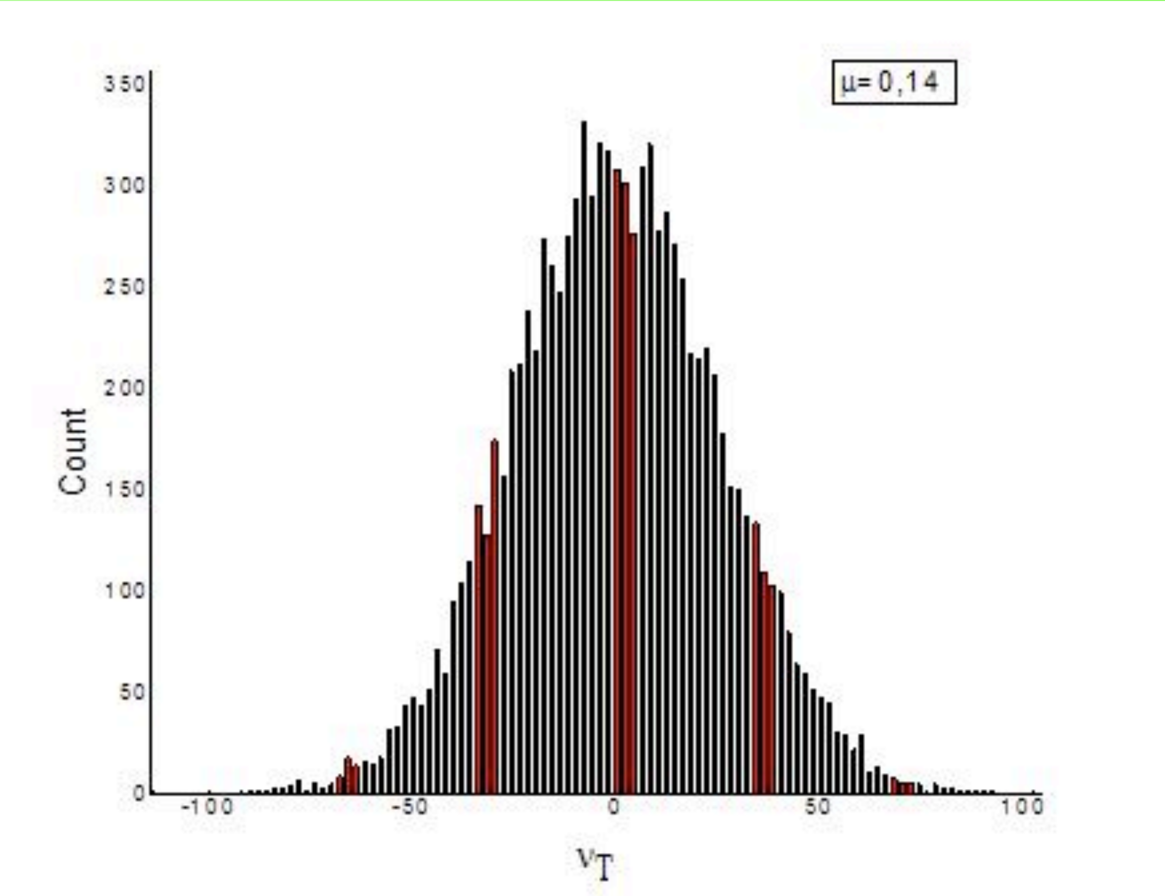


# Total linking number

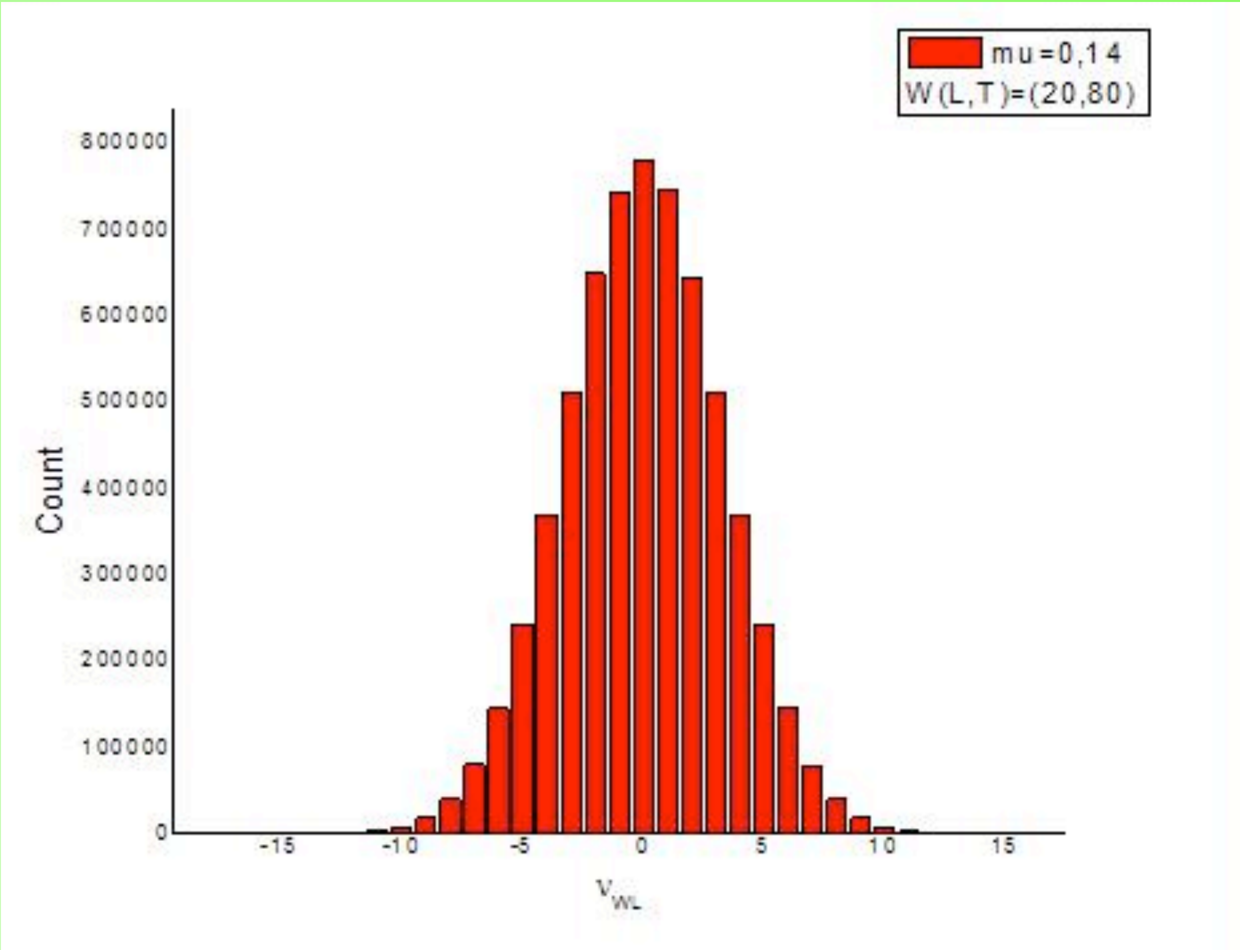


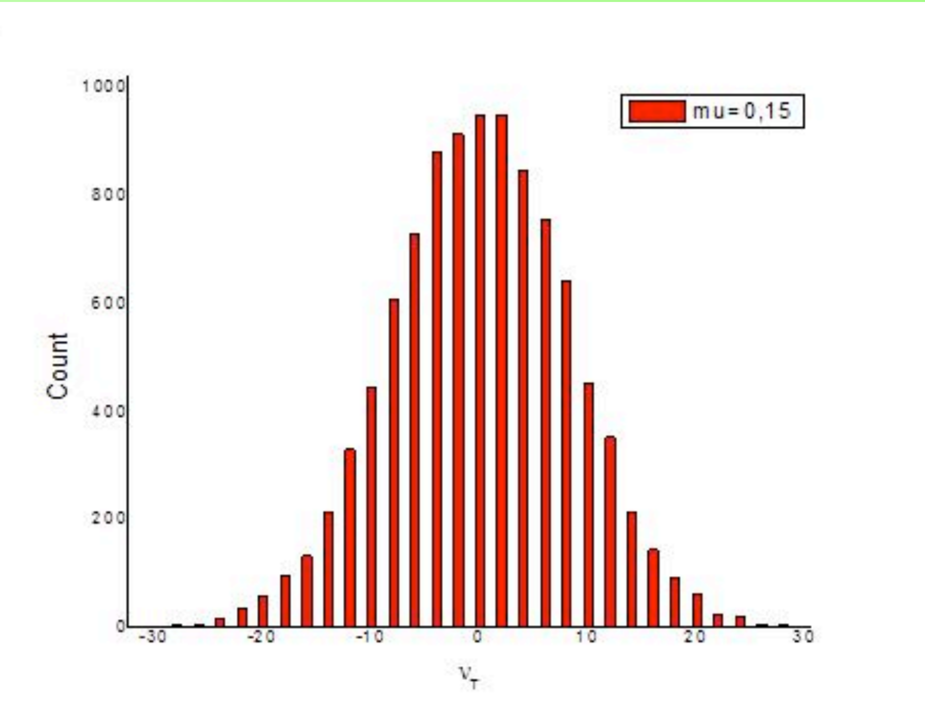
# Total linking number

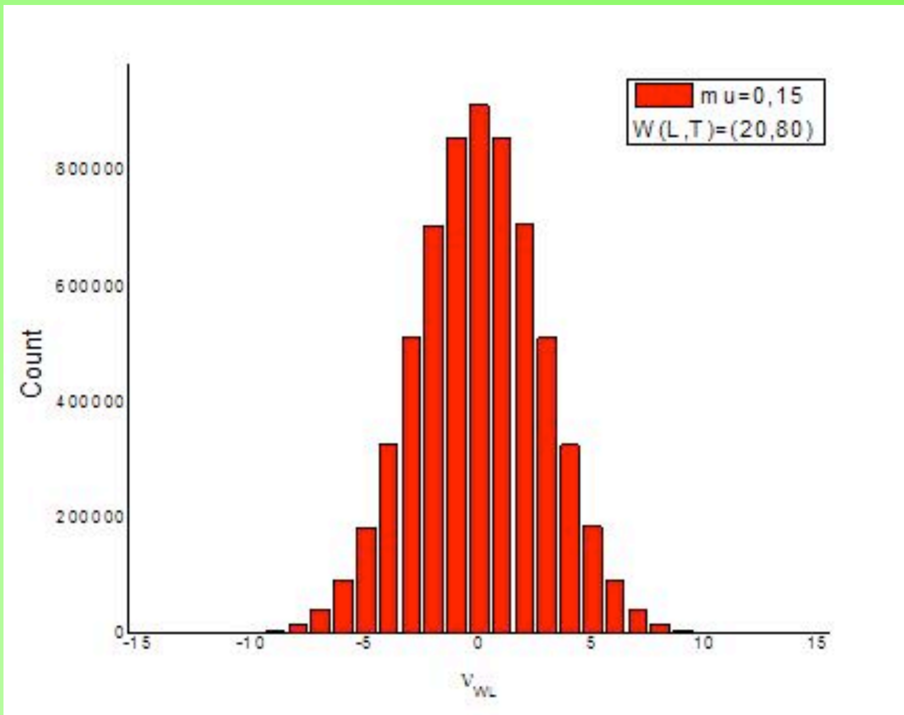


# Wilson loop linking

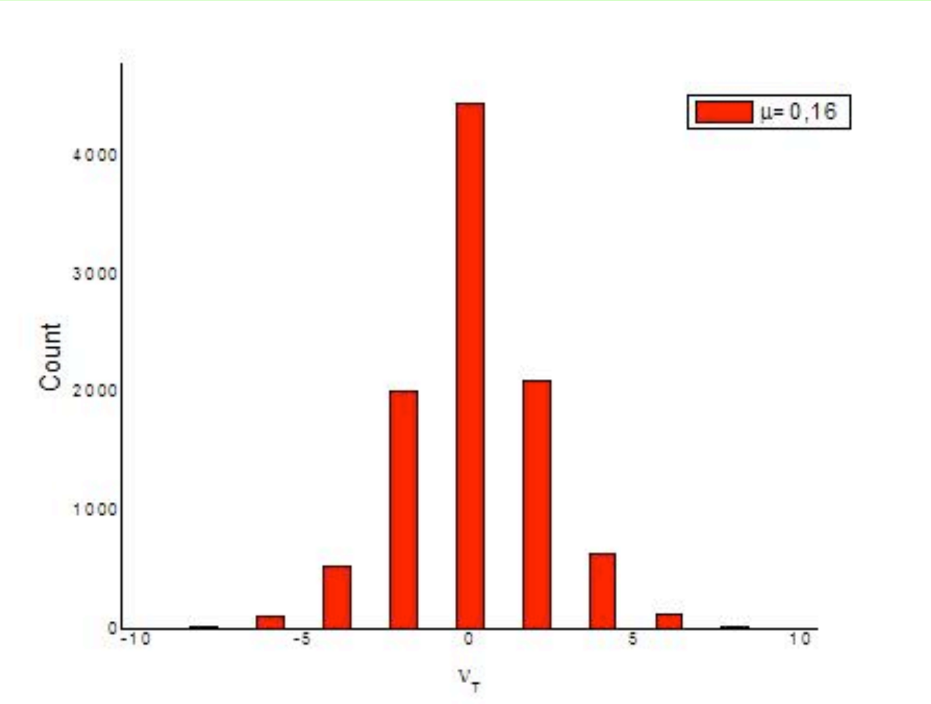


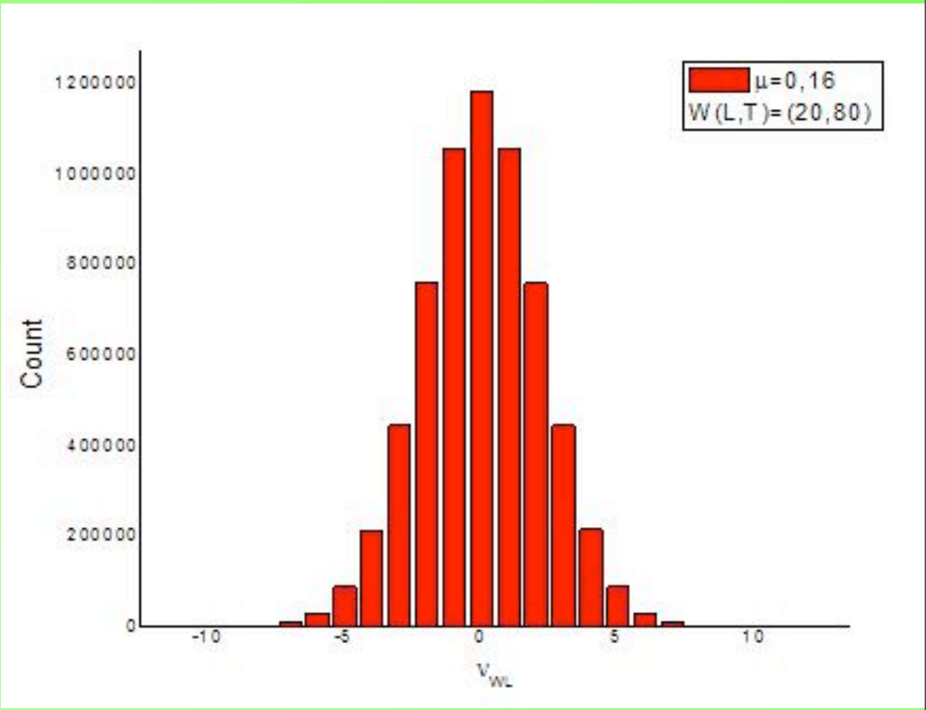
# Total linking number and Wilson loop linking number



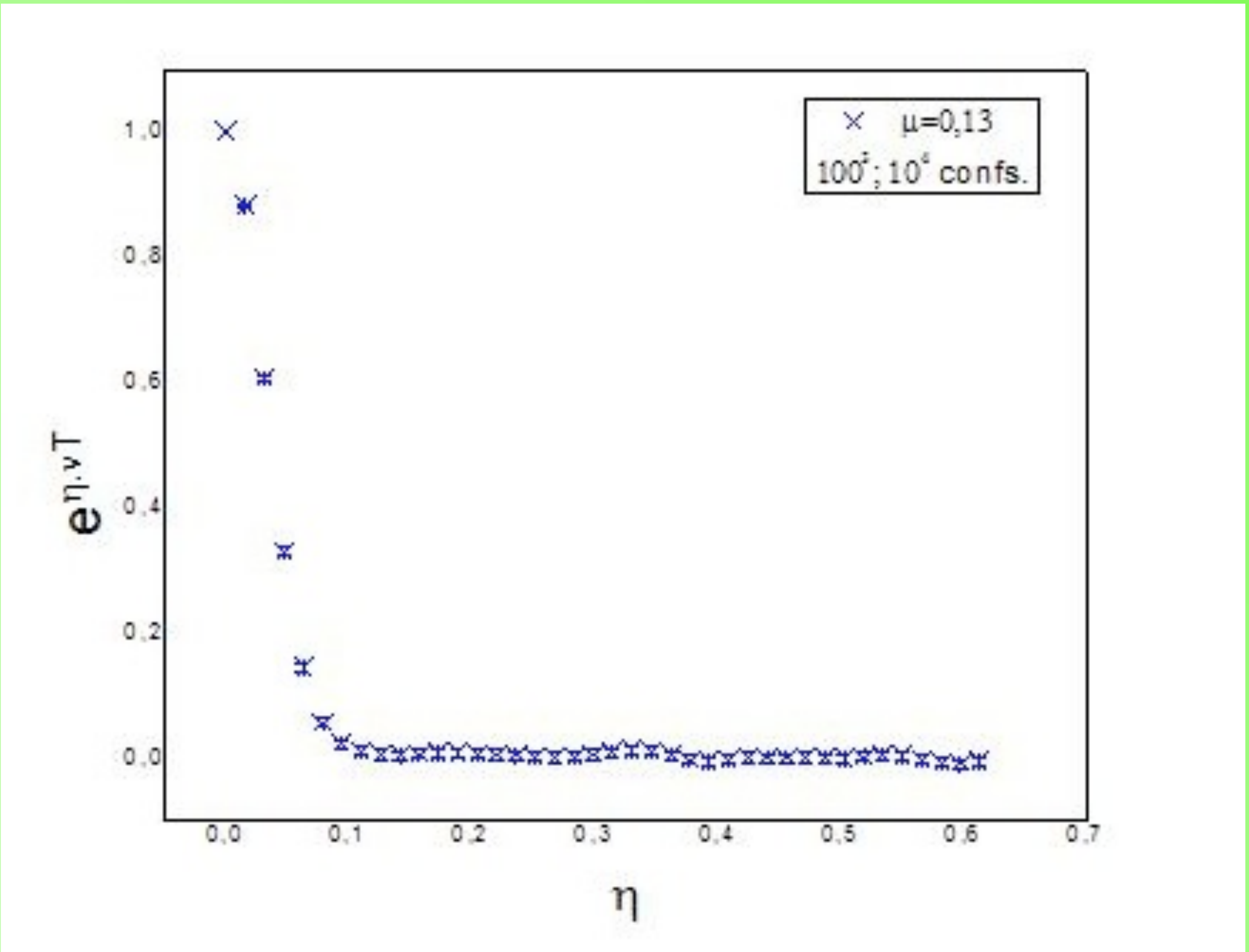



# Total linking number and Wilson loop linking number



# Chern-Simons term



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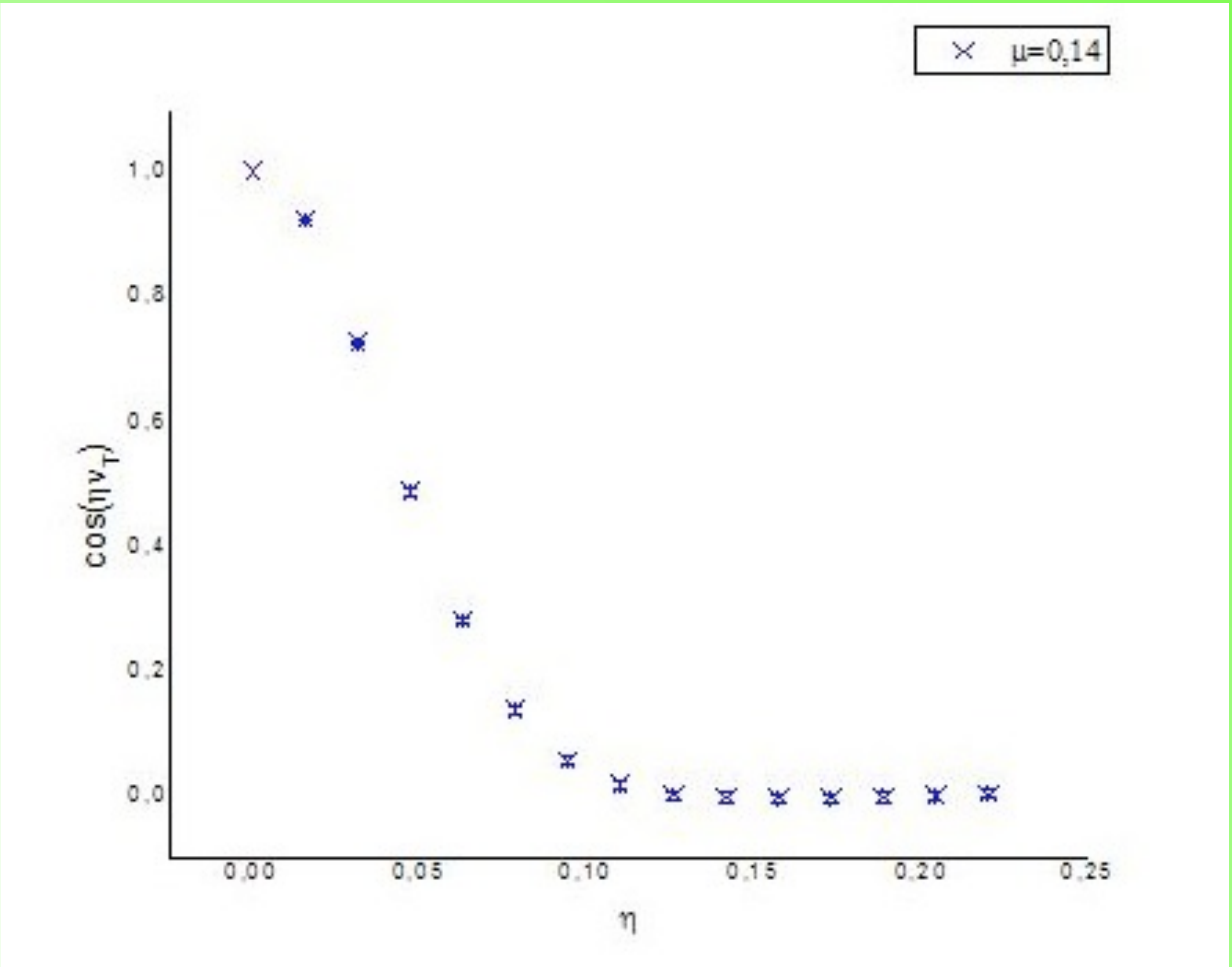

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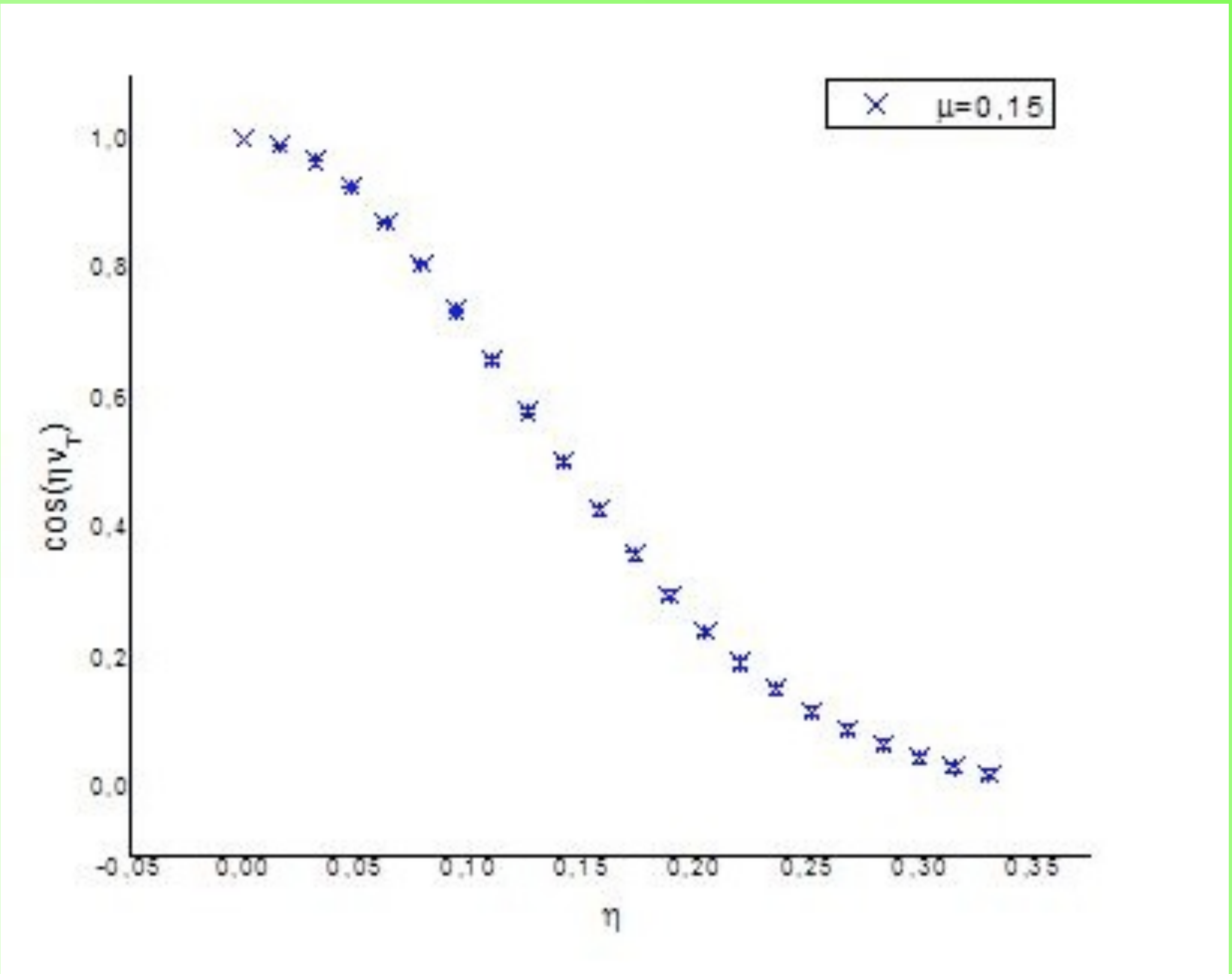

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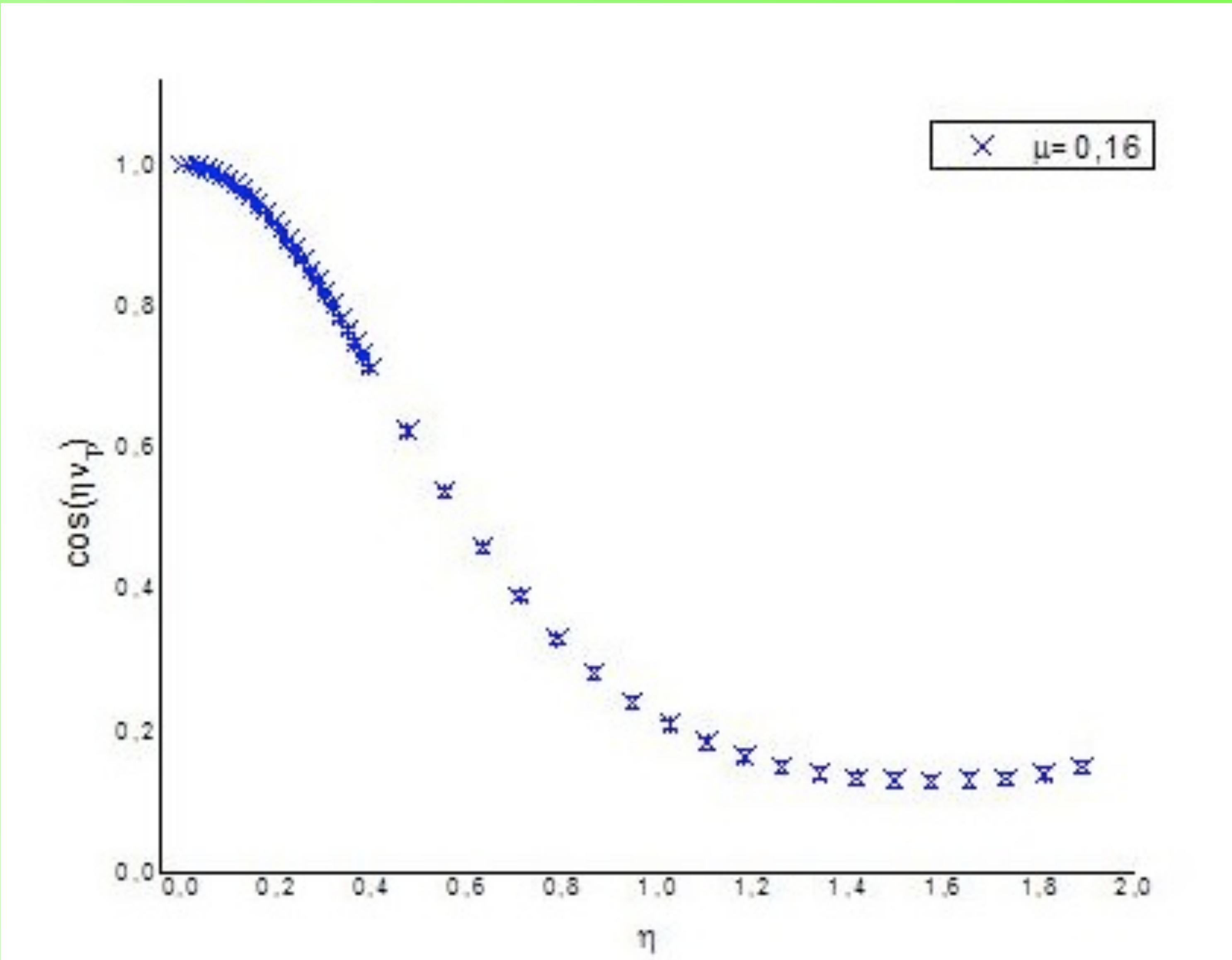

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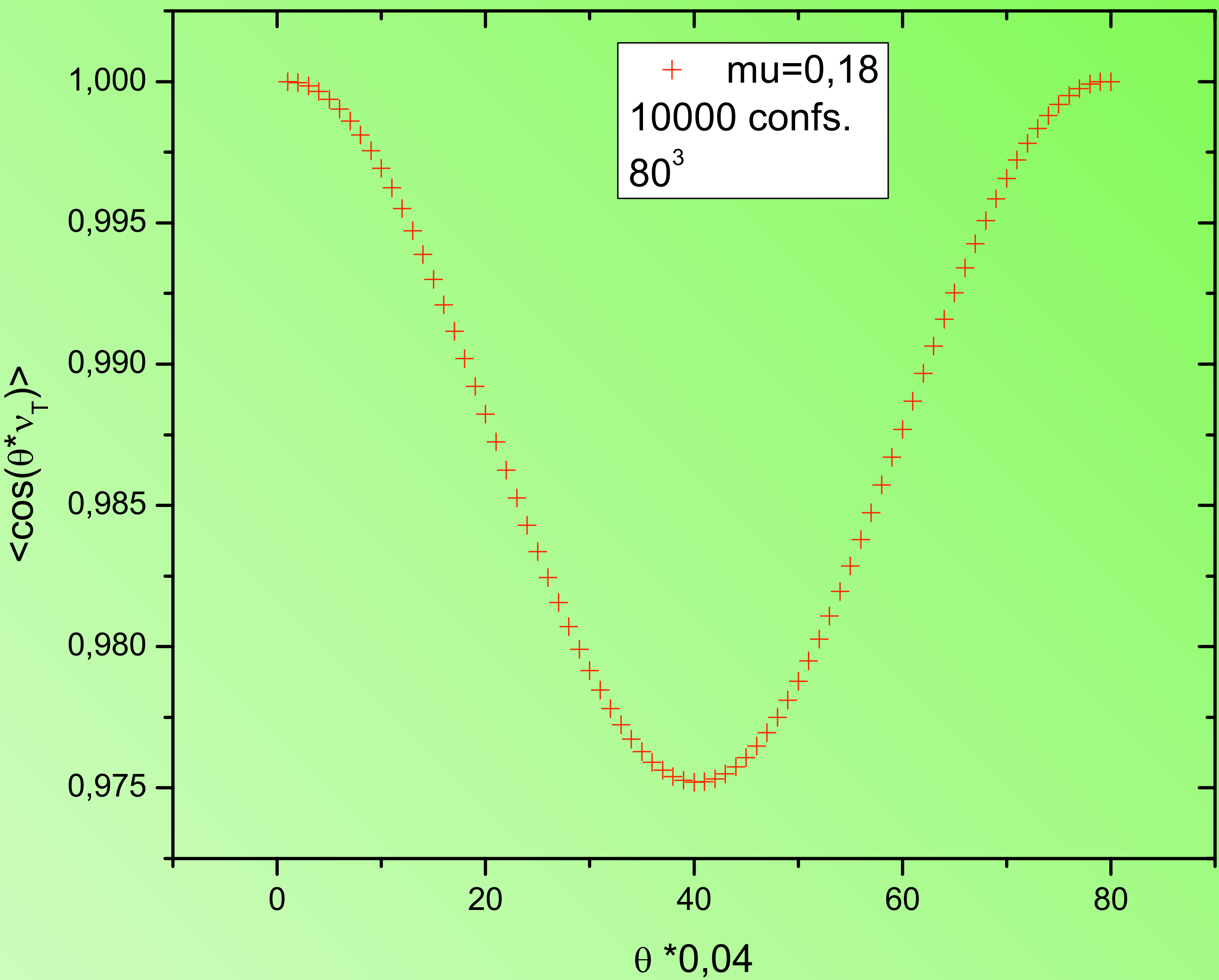

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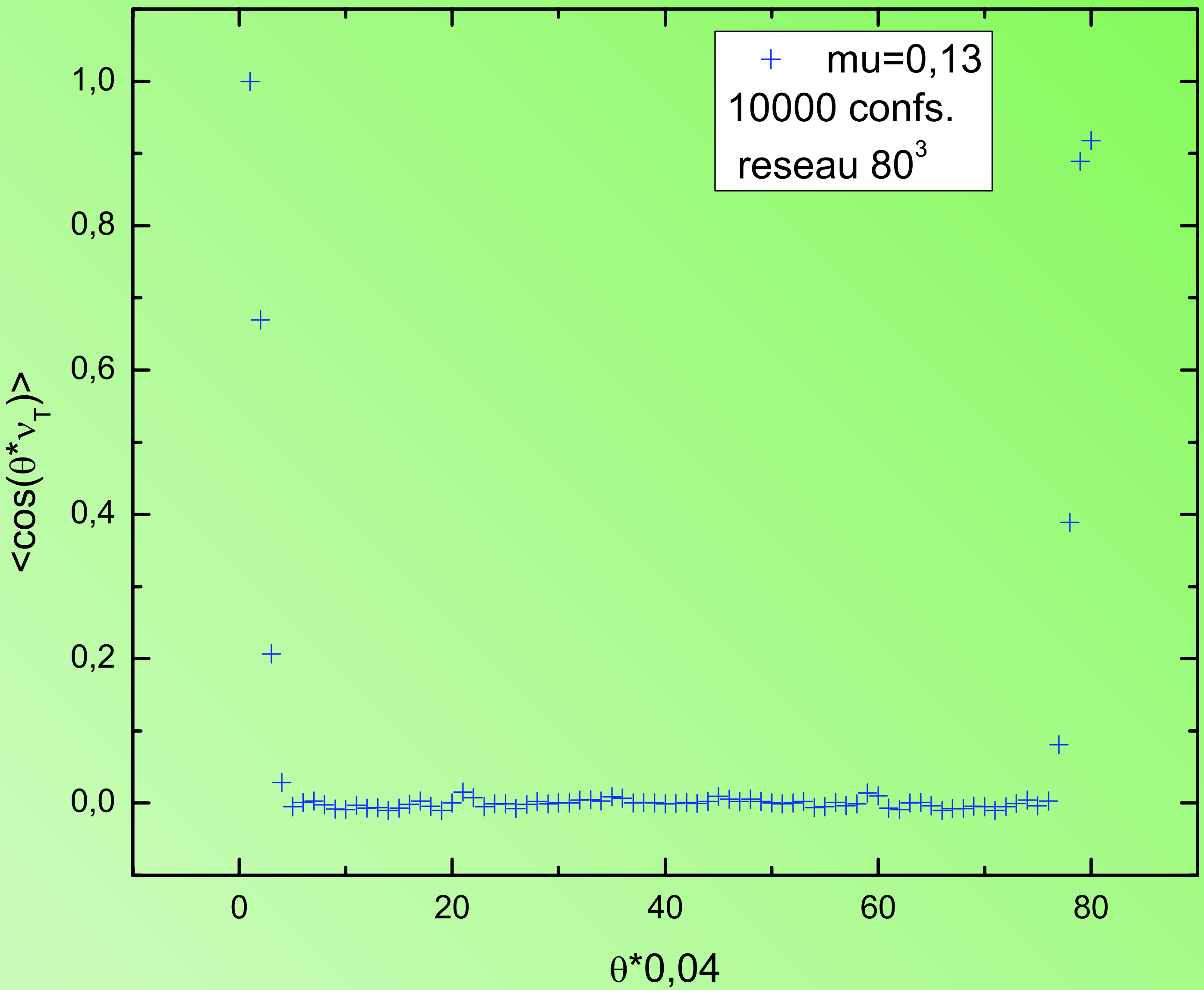
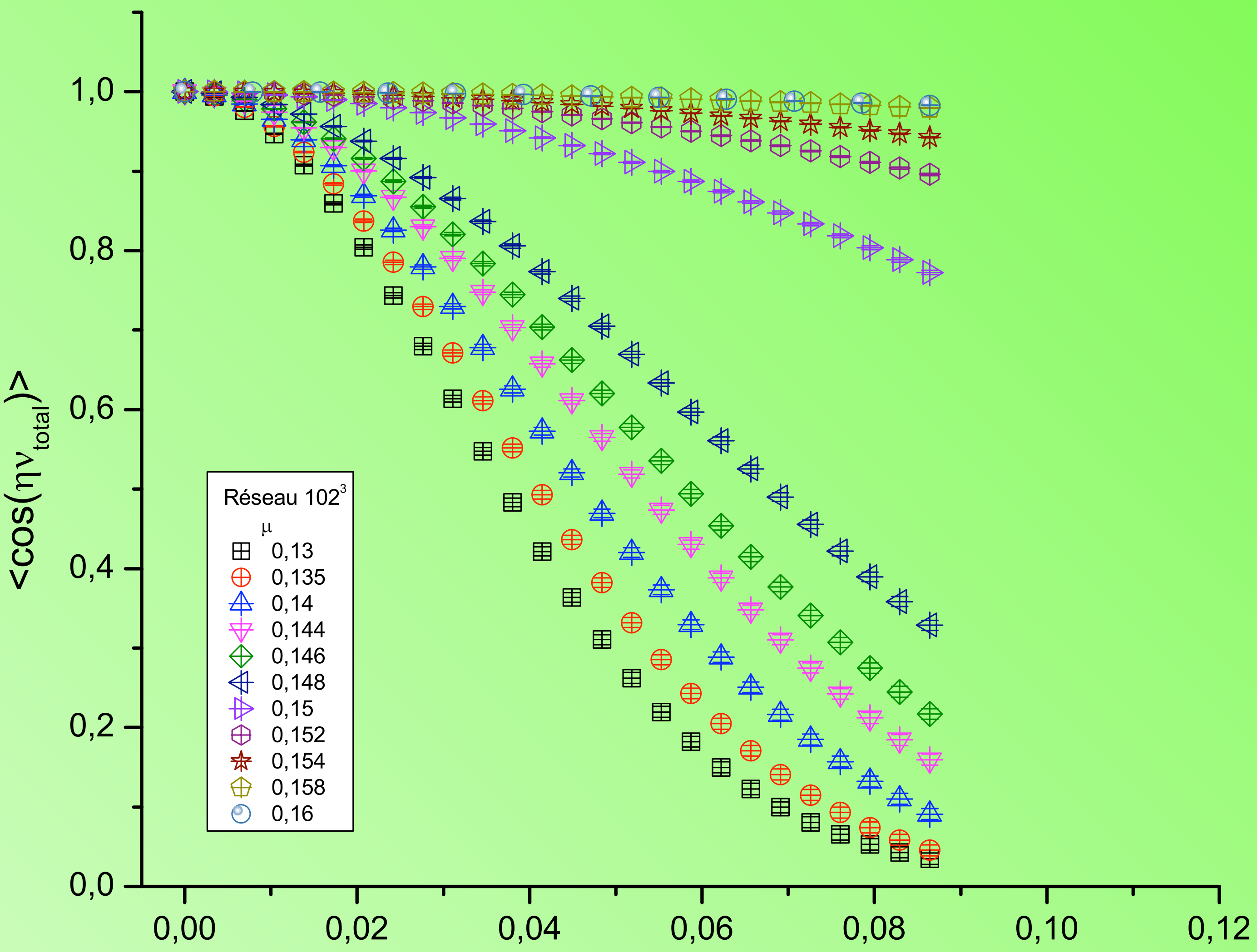
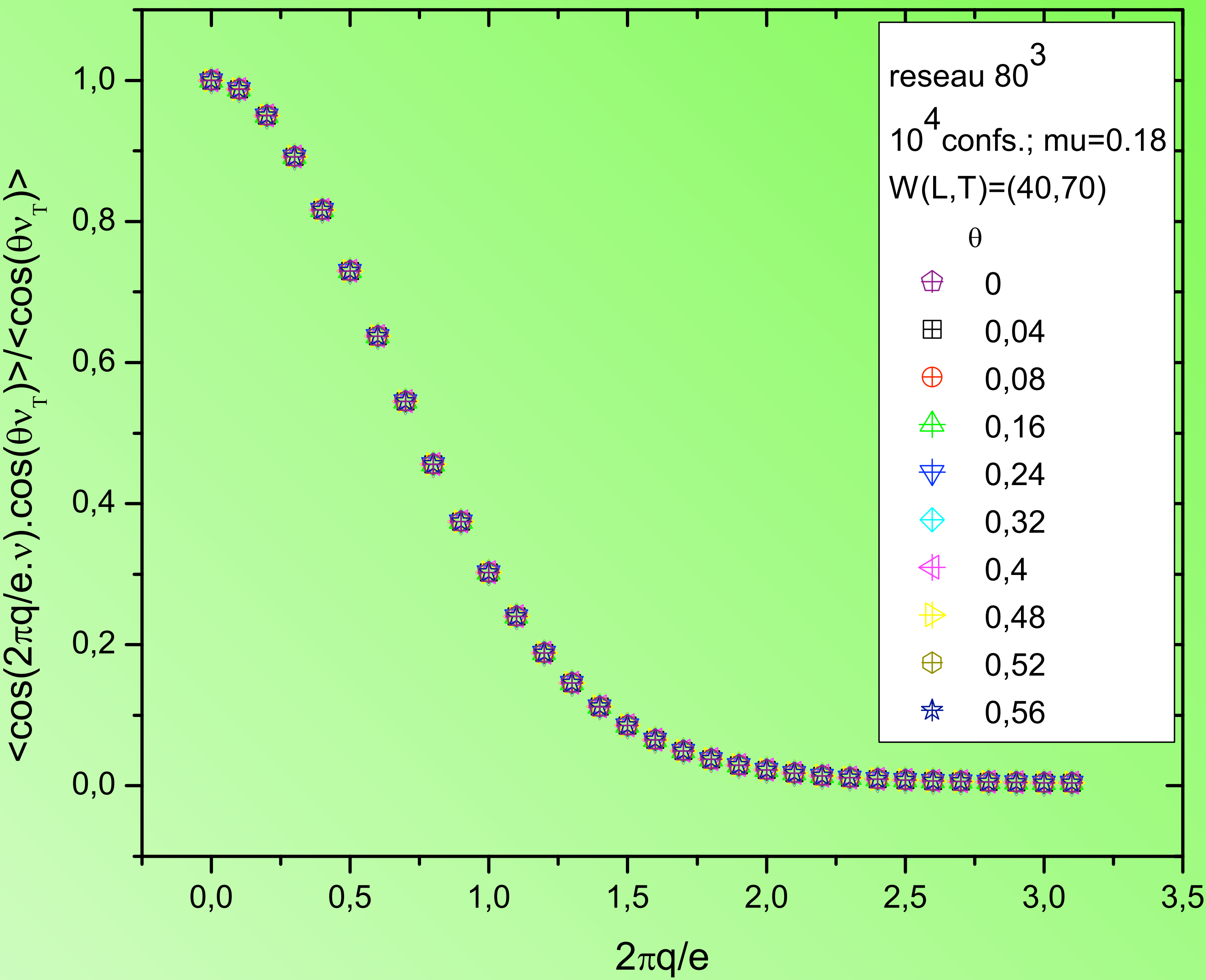


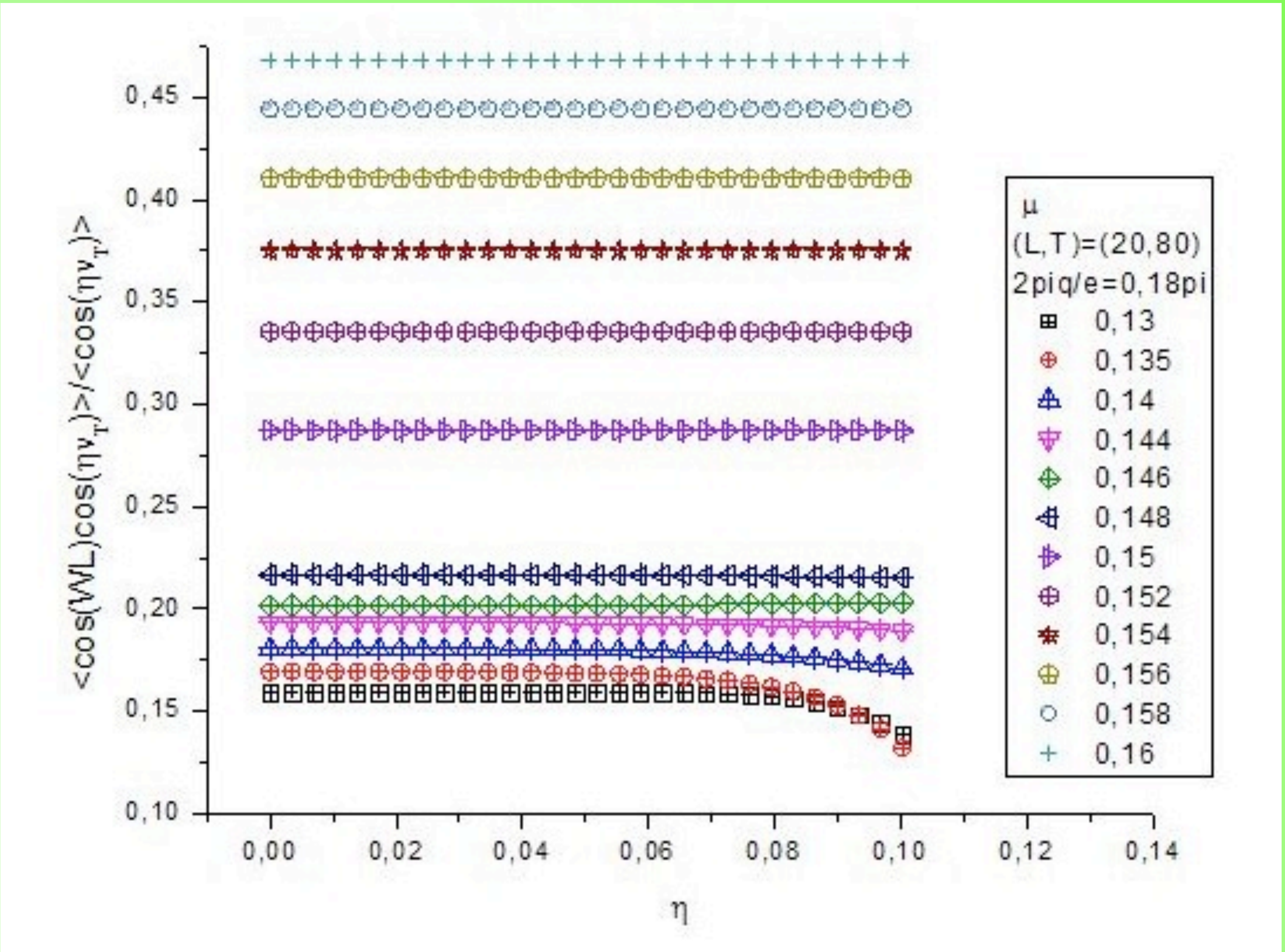
# Chern-Simons Term



# Wilson Loop with Chern Simons



# Wilson Loop with Chern-Simons Term

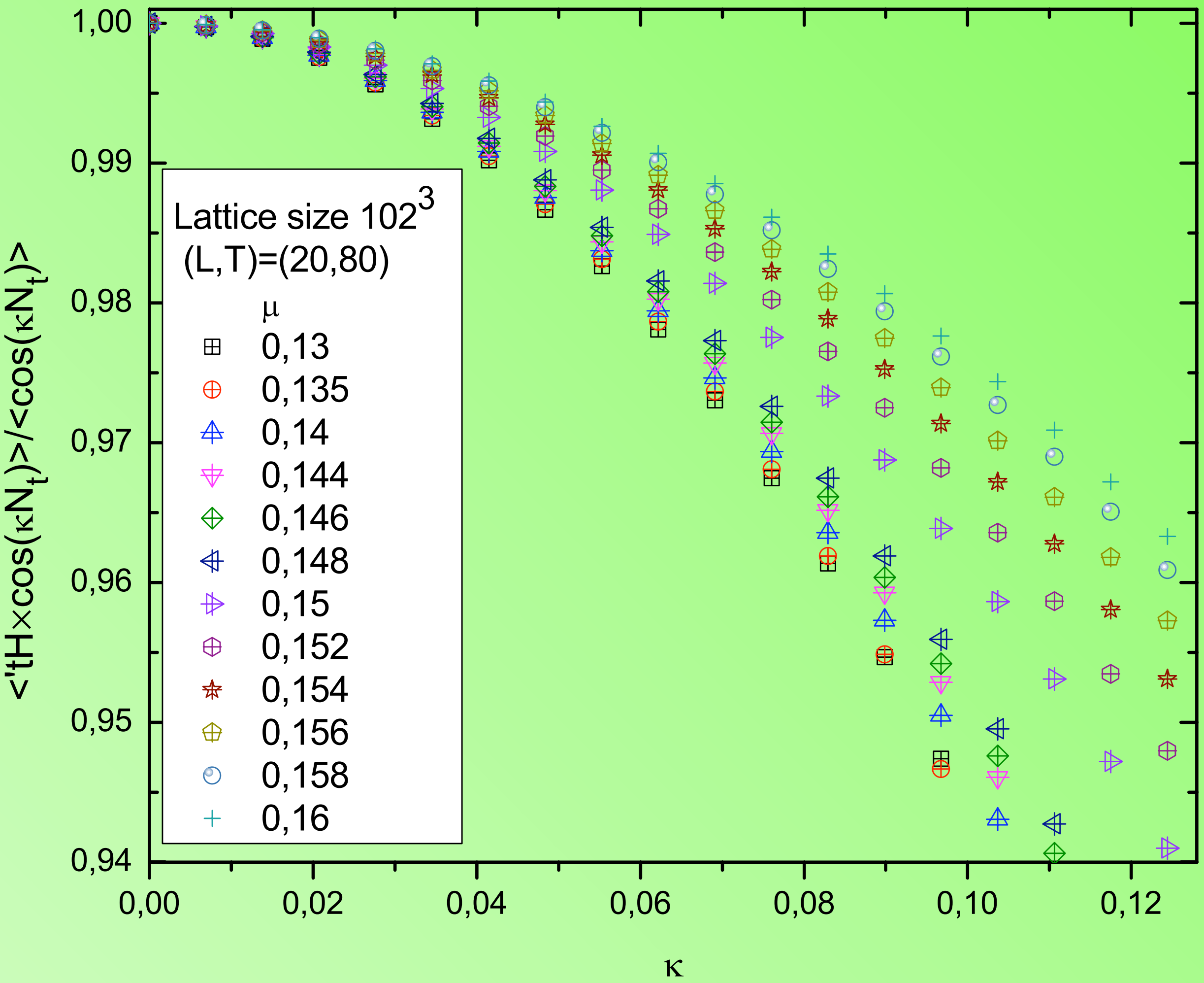


# ‘tHooft loop

- The ‘tHooft loop is defined as the expectation value obtained by performing the gauge field functional integral when a singular flux tube is inserted along a prescribed (usually) rectangular path
- in the absence of the Chern-Simons term this has no evident effect
- the Chern-Simons term however adds the linking number of the ‘tHooft loop with all the other dynamical loops, just as it does for the vortex loops

- thus the action is appended by twice the linking number of the ‘tHooft loop with all the other dynamical loops in each configuration, multiplied by *i* times the coefficient of the Chern-Simons term
- this is identical to what happens when we evaluate the Wilson loop, only that the charge of the inserted quarks is not $2\pi\, q/e$ but *i* times the coefficient of the Chern-Simons term
- hence in the theory without the Chern-Simons term, the ‘tHooft loop is just a constant equal to 1
- while in the theory with the Chern-Simons term, the ‘tHooft loop must behave in the same way as the Wilson loop
- thus the anyonic theory has a perimeter law for both the Wilson loop and the ‘tHooft loop even though there are no massless particles

# ‘tHooft Loop



Lattice size $102^3$
(L,T)=(20,80)
μ
0,13
0,135
0,14
0,144
0,146
0,148
0,15
0,152
0,154
0,156
0,158
0,16
<'tH×cos(κN_t)>/<cos(κN_t)>
κ
1,00
0,99
0,98
0,97
0,96
0,95
0,94
0,00
0,02
0,04
0,06
0,08
0,10
0,12

# Summary/Conclusions

- we have simulated the strong coupling limit of the 2+1 d Abelian Higgs model
- the effective description is one of non-intersecting closed vortex loops
- there exists a transition from small, finite vortex loops to one infinite loop in a thermal bath of small loops
- the Wilson loop seems to exhibit cross over behaviour, from perimeter law to some other potential, but no confinement

# continued

- the Chern-Simons term mesures the total linking number of the vortex configurations
- The distributions of total linking number is very similar to the distribution of Wilson loop linking number
- the question arises whether the average of the Wilson loop at fixed total linking number could be non-constant
- our results seem to show that in fact it is a constant
- we should search for another order parameter which would be sensitive to the total linking number

- it would be interesting to actually simulate the computation of the ‘t Hooft loop in the CS theory to confirm explicitly the analytical result that it must mimic the Wilson loop